International Journal of Mathematics and Soft Computing
Vol.2, No.2 (2012), 35 - 42.

ISSN 2249 - 3328

Continuous Block Hybrid-Predictor-Corrector method for the solution of $y'' = f(x, y, y')$

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Abstract

A method of collocation and interpolation of the power series approximate solution at some grid and off-grid points is considered to generate a continuous linear multistep method for the solution of general second order initial value problems at constant step size. We use continuous block method to generate independent solutions which serves as predictors at selected points within the interval of integration. The efficiency of the proposed method was tested and was found to compete favorably with the existing methods.

Keywords: Collocation, interpolation, power series approximant, grid points, off grid points, continuous block method.
AMS Subject Classification(2010): 65L05, 65L06, 65D30.

1 Introduction

This paper considers the solution to general second order initial value problem of the form

$$y'' = f\left(x, y(x), y'(x)\right), y(x_0) = y_0, y'(x_0) = y'_0 \tag{1}$$

Awoyemi [2, 3, 5], Kayode [8, 9], Adesanya, Anake and Udoh [1] have studied the direct solution to (1). They proposed continuous linear multistep methods which were implemented in predictor-corrector mode. Continuous methods have the advantage of evaluating at all points within the integration interval, thus reduces the computational burden when evaluation is required at more than one point within the grid. They developed a separate reducing order of accuracy predictors and adopted Taylor series expansion to provide the starting values in order to implement the corrector.

Jator [6], Jator and Li [7], Omar and Sulaiman [10], Awoyemi, Adebile, Adesanya and Anake [4], Zarina, Mohammed and Iskanla [11] have proposed discrete block method of the form

$$A^{(0)}\mathbf{Y}_m = \mathbf{e}y_n + h^{\mu}\mathbf{d}\mathbf{F}(\mathbf{y}_n) + h^{\mu}\mathbf{b}\mathbf{F}(\mathbf{Y}_m) \tag{2}$$

to cater for the setback of the predictor corrector method.

In this paper, we propose a continuous block formular which has an advantage of evaluating at all the points within the interval of integration. It has the same properties as the continuous linear multistep method which is extensively discussed by Awoyemi [2, 3, 5]. The continuous block method is evaluated at selected grid points to generate the discrete block (2) which serve as a predictor for the hybrid linear multistep method which is the corrector.

2 Methodology

2.1 Development of continuous hybrid linear multistep

Consider a monomial power series as the approximate solution in the form

$$y(x) = \sum_{j=0}^{(r+s-1)} a_j x^j \tag{3}$$

The second derivative of (3) gives

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2} \tag{4}$$

where s and r are the number of interpolating points. Substituting (4) in (1) we get

$$f\left(x, y(x), y'(x)\right) = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2} \tag{5}$$

$x \in [a, b]$, a_j's are the parameters to be determined. The step size (h) is given as

$$h = x_{n+i} - x_n, n = 0(1)N.$$

Interpolating (3) at $x = x_{n+s}, s = 0(\frac{1}{2})\frac{3}{2}$ and collocating (5) at $x = x_{n+r}, r = 0(\frac{1}{2})2$ gives a system of equations

$$AX = U \tag{6}$$

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 & x_{n+\frac{1}{2}}^7 & x_{n+\frac{1}{2}}^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+\frac{3}{2}} & x_{n+\frac{3}{2}}^2 & x_{n+\frac{3}{2}}^3 & x_{n+\frac{3}{2}}^4 & x_{n+\frac{3}{2}}^5 & x_{n+\frac{3}{2}}^6 & x_{n+\frac{3}{2}}^7 & x_{n+\frac{3}{2}}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 & 30x_{n+\frac{1}{2}}^4 & 42x_{n+\frac{1}{2}}^5 & 56x_{n+\frac{1}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x_{n+\frac{3}{2}}^2 & 20x_{n+\frac{3}{2}}^3 & 30x_{n+\frac{3}{2}}^4 & 42x_{n+\frac{3}{2}}^5 & 56x_{n+\frac{3}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \end{bmatrix}$$

$$X = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_7 & a_8 \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_n & y_{n+\frac{1}{2}} & y_{n+1} & y_{n+\frac{3}{2}} & f_n & f_{n+\frac{1}{2}} & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2} \end{bmatrix}^T$$

Solving (6)for a_j's using Gaussian elimination method and substituting in (3) gives a continuous hybrid linear multistep method in the form

$$\begin{aligned} y(t) &= \sum_{j=0}^{1} \alpha_j(t) y_{n+j} + \alpha_{\frac{1}{2}}(t) y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}}(t) y_{n+\frac{3}{2}} + h^2 \sum_{j=0}^{2} \beta_j(t) f_{n+j} \\ &\quad + \beta_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}}(t) f_{n+\frac{3}{2}} \end{aligned} \tag{7}$$

where $t = \frac{x - x_n}{h}$

$$\begin{aligned}
\alpha_0 &= \frac{1}{21}\left(128t^7 - 896t^6 + 2352t^5 - 2800t^4 + 1344t^3 - 149t + 21\right) \\
\alpha_{\frac{1}{2}} &= \frac{1}{217}\left(2688t^8 - 20096t^7 + 59136t^6 - 87024t^5 + 66416t^4 - 2248t^3 + 1728t\right) \\
\alpha_1 &= -\frac{1}{217}(5376t^8 - 36224t^7 + 90496t^6 - 101136t^5 + 46032t^4 - 4032t^3 - 729t) \\
\alpha_{\frac{3}{2}} &= \frac{1}{651}(8064t^8 - 52352t^7 + 121856t^6 - 115248t^5 + 25648t^4 + 14784t^3 - 2752t) \\
\beta_0 &= \frac{1}{78120}(1008t^8 - 15968t^7 + 82936t^6 - 200592t^5 + 247331t^4 - 151354t^3 + 39060t^2 - 2421t) \\
\beta_{\frac{1}{2}} &= -\frac{1}{9765}(3024t^8 - 6984t^7 - 41972t^6 + 183330t^5 - 252952t^4 + 125328t^3 - 9774t) \\
\beta_1 &= -\frac{1}{4340}(10864t^8 - 70144t^7 + 160888t^6 - 144704t^5 + 19243t^4 + 28308t^3 + 4455t) \\
\beta_{\frac{3}{2}} &= -\frac{1}{9765}(3024t^8 - 19384t^7 + 44828t^6 - 43218t^5 + 11788t^4 + 3808t^3 - 846t) \\
\beta_2 &= \frac{1}{11160}(144t^8 - 864t^7 + 1928t^6 - 1872t^5 + 613t^4 + 78t^3 - 27t)
\end{aligned}$$

Evaluating (7) at $t = 4$, we have

$$\begin{aligned} y_{n+2} &= -y_n - \frac{128}{31} y_{n+\frac{1}{2}} + \frac{318}{31} y_{n+1} - \frac{128}{31} y_{n+\frac{3}{2}} \\ &\quad + \frac{h^2}{13020}\left(23f_n + 688f_{n+\frac{1}{2}} + 2358f_{n+1} + 688f_{n+\frac{3}{2}} + 23f_{n+2}\right) \end{aligned} \tag{8}$$

2.2 Development of continuous block predictor

Interpolating (3) at $x = x_{n+s}, s = 0, 1$ and collocating (5) at $x = x_{n+r}, r = 0(\frac{1}{2})2$ gives a system of equations in the form of (5) where

$$A = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\
0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 & 30x_{n+\frac{1}{2}}^4 \\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\
0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x_{n+\frac{3}{2}}^2 & 20x_{n+\frac{3}{2}}^3 & 30x_{n+\frac{3}{2}}^4 \\
0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4
\end{bmatrix}$$

$$X = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}^T$$
$$U = \begin{bmatrix} y_n & y_{n+1} & f_n & f_{n+\frac{1}{2}} & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2} \end{bmatrix}^T$$

Solving for $a_j's$ gives a continuous hybrid linear multistep method in the form

$$y(t) = \sum_{j=0}^{1} \alpha_j(t)\, y_{n+j} + h^2 \sum_{j=0}^{2} \beta_j(t)\, f_{n+j} + \beta_{\frac{1}{2}}(t)\, f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}}(t)\, f_{n+\frac{3}{2}} \tag{9}$$

where

$$\begin{aligned}
t &= \frac{x - x_n}{h} \\
\alpha_0 &= 1 - t, \quad \alpha_1 = t \\
\beta_0 &= \frac{1}{360}\left(8t^6 - 60t^5 + 175t^4 - 250t^3 + 180t^2 - 53t\right) \\
\beta_{\frac{1}{2}} &= -\frac{1}{45}\left(4t^6 - 27t^5 + 65t^4 - 60t^3 + 18t\right) \\
\beta_1 &= \frac{1}{60}\left(8t^6 - 48t^5 + 95t^4 - 60t^3 + 5t\right) \\
\beta_{\frac{3}{2}} &= -\frac{1}{45}\left(4t^6 - 21t^5 + 35t^4 - 20t^3 + 2t\right) \\
\beta_2 &= \frac{1}{360} 8t^6 - 36t^5 + 55t^4 - 30t^3 + 3t).
\end{aligned}$$

Solving for the independent solution $y_{n+r}, r = 1\left(\frac{1}{2}\right)2$, gives a continuous hybrid block

$$\mathbf{Y}^{(m)}(t) = \sum_{j=0}^{2} \frac{(jh)^m}{m!} \mathbf{y}_n^m + h^\mu \sum_{j=0}^{2} \sigma_j(t)\, f_{n+j} + h^\mu \sigma_{\frac{1}{2}}(t)\, f_{n+\frac{1}{2}} + h^\mu \sigma_{\frac{3}{2}}(t)\, f_{n+\frac{3}{2}} \tag{10}$$

where the coefficient of f_{n+j} are given by

$$\begin{aligned}
\sigma_0 &= \frac{1}{360}\left(8t^6 - 27t^5 + 175t^4 - 256t^3 + 180t^2\right) \\
\sigma_{\frac{1}{2}} &= -\frac{1}{45}\left(4t^6 - 27t^5 + 65t^4 - 60t^3\right) \\
\sigma_1 &= \frac{1}{60}\left(8t^6 - 48t^5 + 95t^4 - 60t^3\right) \\
\sigma_{\frac{3}{2}} &= -\frac{1}{45}\left(4t^6 - 21t^5 + 35t^4 - 20t^3\right) \\
\sigma_2 &= \frac{1}{360} 8t^6 - 36t^5 + 55t^4 - 30t^3).
\end{aligned}$$

Evaluating (10) at $t = 0(\frac{1}{2})2$ gives a discrete block method in the form of equation (2),

$$\mathbf{d} = \begin{bmatrix} \frac{367}{5760} & \frac{53}{360} & \frac{147}{640} & \frac{14}{45} & \frac{251}{1440} & \frac{29}{180} & \frac{27}{160} & \frac{7}{45} \end{bmatrix}^T$$

$$\mathbf{b} = \begin{bmatrix} \frac{3}{32} & \frac{2}{5} & \frac{117}{160} & \frac{16}{15} & \frac{323}{720} & \frac{31}{45} & \frac{51}{80} & \frac{32}{45} \\ -\frac{47}{960} & -\frac{1}{12} & \frac{27}{320} & \frac{4}{15} & -\frac{11}{60} & \frac{2}{15} & \frac{9}{20} & \frac{4}{15} \\ \frac{29}{1440} & \frac{2}{45} & \frac{3}{32} & \frac{16}{45} & \frac{53}{720} & \frac{1}{45} & \frac{21}{80} & \frac{32}{45} \\ -\frac{7}{1920} & -\frac{1}{120} & -\frac{9}{640} & 0 & -\frac{19}{1440} & -\frac{1}{180} & -\frac{3}{160} & \frac{7}{45} \end{bmatrix}^T$$

$$\mathbf{e} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$A^0 = 8 \times 8$ identity matrix.

3 Analysis of the basic properties of the block

3.1 Order of the method

We define a linear operator on (7) to give

$$\mathbf{L}\{y(x) : h\} = y(x) - \begin{bmatrix} \sum_{j=0}^{1} \alpha_j(t)\, y_{n+j} + \alpha_{\frac{1}{2}}(t)\, y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}}(t)\, y_{n+\frac{3}{2}} + \\ h^2 \sum_{j=0}^{2} \beta_j(t)\, f_{n+j} + \beta_{\frac{1}{2}}(t)\, f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}}(t)\, f_{n+\frac{3}{2}} \end{bmatrix} \tag{11}$$

Expanding y_{n+j} and f_{n+j} in Taylor series expansion and comparing the coefficient of h gives

$$\begin{aligned} \mathbf{L}\{y(x) \; : \; h\} &= C_0 y(x) + C_1 h y'(x) + ... + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) \\ &\quad + C_{p+2} h^{p+2} y^{p+2}(x) + ... \end{aligned} \tag{12}$$

Definition 3.1. *The difference operator* $\mathbf{L}$ *and the associated continuous linear multistep method (9) are said to be of order* p *if* $C_0 = C_1 = ... = C_p = C_{p+1} = 0$ *and* C_{p+2} *is called the error constant and implies that the local truncation error is given by* $t_{n+k} = C_{p+2} h^{(p+2)} y^{(p+2)}(x) + O\left(h^{p+3}\right)$.

The order of the proposed discrete scheme is 8, with error constant $C_{p+2} = \frac{-79}{599961600}$.

3.2 Consistency

A linear multistep method (7) is said to be consistent if it has order $p \geq 1$ and if $\rho(1) = \rho'(1) = 0$ and $\rho''(1) = 2!\sigma(1)$ where $\rho(r)$ is the first characteristic polynomial and $\sigma(r)$ is the second characteristic polynomial.

For the proposed method,

$\rho(r) = r^2 + \frac{128}{31} r^{\frac{3}{2}} - \frac{318}{31} r + \frac{128}{31} r^{\frac{1}{2}} + 1$ and

$\sigma(r) = \frac{1}{465}\left(23r^2 + 688r^{\frac{3}{2}} + 2358r + 688r^{\frac{1}{2}} + 23\right)$.
Clearly $\rho(1) = \rho'(1) = 0$ and $\rho''(1) = 2!\sigma(1)$.

Hence the proposed method is consistent.

3.3 Zero stability

A linear multistep method is said to be zero stable, if the zeros of the first characteristic polynomial $\rho(r)$ satisfies $\mid r \mid \leq 1$ and is simple for $\mid r \mid = 1$.

For the proposed method, one of the roots of the first characteristic polynomial is 25.93. Hence the proposed method is not zero stable.

3.4 Stability interval

The method (7) is said to be absolute stable if for a given h, all roots z_s of the characteristic polynomial $\pi(z,h) = \rho(z) + h^2\sigma(z) = 0$, satisfies $\mid z_s \mid < 1, s = 1, 2, ..., n.$ where $h = -\lambda^2 h^2$ and $\lambda = \frac{\partial f}{\partial y.}$.

The boundary locus method is adopted to determine the region of absolute stability. Substituting the test equation $y'' = -\lambda^2 h^2$ in (7) and substituting $r = \cos\theta + i\sin\theta$ gives the stability region $[-10.143, 0]$ after evaluating $h(\theta)$ at an interval of 30^0 in $\left[0, 180^0\right]$.

4 Numerical Experiments

4.1 Test Problems

We test the proposed method with second order initial value problems.

Problem 1: Consider the non-linear initial value problem (I.V.P)

$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}.h = 0.05$

Exact solution: $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$.

This problem was solved by Awoyemi [5] where a method of order 8 is proposed and it is implemented in predictor-corrector mode with $h = 1/320$. Jator [6] also solved this problem in block method where a block of order 6 and step-length of 5 is proposed with $h = 0.05$. We compared the result with these two results as shown in Table 1.

Problem 2: We consider the non-linear initial value problem (I.V.P)

$y'' = \frac{(y')^2}{2y} - 2y, y(\frac{\pi}{6}) = \frac{1}{4}, y'(\frac{\pi}{6}) = \frac{\surd 3}{2}$.

Exact solution:$(\sin x)^2$.

This problem was solved by Awoyemi [5] where a method of order 8 is proposed and it is implemented in predictor-corrector mode with $h = 1/320$. Jator [6] also solved this problem in block method where a block of order 6 and step-length of 5 is proposed with $h = 0.049213$. We compared the result with these two results as shown in Table 2.

Table 1 for Problem 1

x	Exact result	Computed result	Error	Error in [4]	Error in [6]
0.1	1.050041729278	1.050041729277	7.5028(-13)	6.6391(-14)	7.1629(-12)
0.2	1.100335347731	1.100335347721	9.7410(-12)	2.0012(-10)	1.5091(-11)
0.3	1.151140435936	1.151140435898	3.7638(-11)	1.7200(-09)	4.5286(-11)
0.4	1.202732554054	1.202732553956	9.7765(-11)	5.8946(-09)	1.0808(-10)
0.5	1.255412811882	1.255412811674	2.0825(-10)	1.4434(-08)	1.7818(-10)
0.6	1.309519604203	1.309519603807	3.9604(-10)	4.1866(-08)	4.4434(-10)
0.7	1.365443754271	1.365443753566	7.0460(-10)	5.3109(-08)	7.4446(-10)
0.8	1.423648930193	1.423648928984	1.2095(-09)	9.1131(-08)	1.5009(-09)
0.9	1.484700278594	1.484700276542	2.0511(-09)	1.4924(-07)	3.7579(-09)
1.0	1.549306144334	1.549306140827	3.5066(-09)	2.3718(-07)	4.7410(-09)

Table 2 for Problem 2

x	Exact result	Computed result	Error	Error in [5]	Error in [9]
1.1035988	0.797152508880	0.797152508692	1.8811(-10)	4.1632(-07)	2.8047(-10)
1.2035988	0.871118127112	0.871118126867	2.4539(-10)	4.5866(-07)	2.7950(-10)
1.3035988	0.930288436746	0.930288436443	3.0306(-10)	4.0928(-07)	2.1490(-10)
1.4035988	0.972304504262	0.972304503903	3.5819(-10)	2.6295(-07)	5.4975(-11)
1.5035988	0.995491281635	0.995491281226	4.0838(-10)	4.5539(-08)	1.1545(-10)
1.6035988	0.998924385222	0.998924384771	4.5128(-10)	4.8054(-07)	4.4825(-10)
1.7035988	0.982466948018	0.982466947533	4.8473(-10)	1.0322(-06)	7.7969(-10)
1.8035988	0.946775076109	0.946775075602	5.0696(-10)	1.6785(-06)	1.1840(-09)
1.9035988	0.893271691795	0.893271691278	5.1697(-10)	2.3857(-06)	1.6318(-09)
2.0035988	0.824089806171	0.824089805657	5.1381(-10)	3.1108(-06)	2.0567(-09)

5 Conclusion

In this paper we have proposed a two-step two-point hybrid method. Continuous block method which has the properties of evaluation at all points with the interval of integration is adopted to give the independent solution at non overlapping intervals as the predictor to an order eight corrector. This new method forms a bridge between the predictor-corrector method and block method. Hence it shares the properties of both the methods. The new method performed better than the block method and the predictor corrector method. Therefore we recommend this method in developing numerical methods for the solutions of initial value problems of ordinary differential equations.

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