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$(r, 2, r(r-1))$-regular graphs

N. R. Santhi Maheswari
Department of Mathematics,
G.Venkataswamy Naidu College, Kovilpatti, INDIA.
E-mail: nrsmaths@yahoo.com

C. Sekar
Department of Mathematics,
Aditanar College of Arts and Science,
Tiruchendur-628216, INDIA.
E-mail: sekar.acas@gmail.com

Abstract

A graph G is called $(r, 2, r(r-1))$-regular if each vertex in the graph G is at a distance one away from exactly r number of vertices and at a distance two away from exactly $r(r-1)$ number of vertices. That is, $d(v) = r$ and $d_2(v) = r(r-1)$, for all v in G. In this paper, we prove that for any $r > 0$, r-regular graph with girth at least five is $(r, 2, r(r-1))$-regular and vice versa and also suggest a method to construct $(r, 2, r(r-1))$-regular graph on $n \times 2^{r-2}$ vertices.

Keywords: Distance degree regular graph, (d, k)-regular graph, girth, diameter, semiregular, $(r, 2, k)$-regular.
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1 Introduction

In this paper, we consider only finite, simple, connected graphs. We follow graph theoretic terminology proposed in the works of J. A. Bondy and U.S.R. Murty [4] and Harary [6]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$ respectively. The degree of a vertex v is the number of edges incident at v. A graph G is regular if all its vertices have the same degree.

For a connected graph G, the distance $d(u, v)$ between two vertices u and v is the length of a shortest (u, v) path. Therefore, the degree of a vertex v is the number of vertices at a distance 1 from v, and it is denoted by $d(v)$. The set of all verticees at a distance one from v is called the neighbourhood of v and is denoted by $N(v)$. This observation suggests a generalization of degree. That is, $d_d(v)$ is defined as the number of vertices at a distance d from v. Hence $d_1(v) = d(v)$ and $N_d(v)$ denote the set of all vertices that are at a distance d away from v in a graph G. Hence $N_1(v) = N(v)$. Girth of a graph is the smallest cycle in the graph and the diameter of graph $G = max\{d(u, v)/u, v \in V(G)\}$. For any $S \subseteq V(G)$, $G(S)$ is the subgraph of G induced by the vertices of S.

A graph G is called distance d-regular[4] if every vertex of G has the same number of vertices at a distance d from it. We call this graph by (d, k)-regular if every vertex of G has k number of vertices at a distance d from it. That is, a graph G is said to be (d, k) - regular graph if $d_d(v) = k$, for all v in G. The $(1, k)$-regular graphs are nothing but the k-regular graphs. A graph G is $(2, k)$ - regular if $d_2(v) = k$, for all $v \in G$.

The concept of the semiregular graph was introduced and studied by Alison Northup [2]. A graph G is said to be k-semiregular graph if each vertex of G is at a distance two away from exactly k vertices of G. We observe that $(2, k)$-regular graphs are k-semiregular graphs. Note that a $(2, k)$-regular graph may be regular or non-regular. Among the four $(2, k)$-regular graphs given in Figure 1, (i) and (ii) are regular whereas (iii) and (iv) are non-regular.

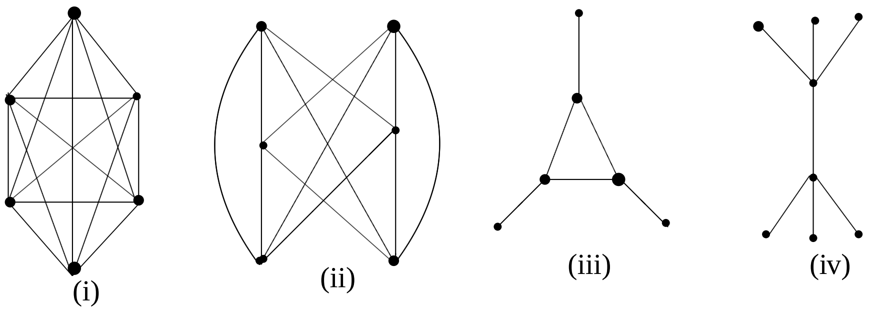


Figure 1.

In this paper, we consider only regular graphs with $(2, k)$-regular. We call r-regular graph with $(2, k)$-regular by $(r, 2, k)$-regular graph. A graph is called $(r, 2, k)$-regular if $d(v) = r$ and $d_2(v) = k$, for all v in G. If a graph G is $(r, 2, k)$-regular, then $0 \leq k \leq r(r-1)$. We have shown that an r-regular graph with girth at least five is $(r, 2, r(r-1))$-regular and vice versa. This result motivates to suggest some methods to construct r-regular graphs for any $r > 1$ with girth at least five.

2 $(r, 2, r(r-1))$-regular graphs

Definition 2.1. *A graph G is called $(r, 2, r(r-1))$-regular if each vertex in the graph G is at a distance one away from exactly r number of vertices and each vertex in the graph G is at a distance two away from exactly $r(r-1)$ number of vertices. That is, $d(v) = r$ and $d_2(v) = r(r-1)$, for all $v \in V(G)$.*

Example 2.2. *(i) K_2 is a $(1, 2, 1(1-1))$-regular graph.*

(ii) Any cycle C_n $(n \geq 5)$ is $(2, 2, 2(2-1))$-regular graph.

(iii) $(3, 2, 3(3-1))$-regular graphs are shown in Figure 2.

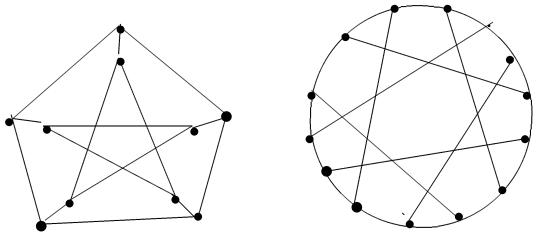


Figure 2.

3 $(r, 2, r(r-1))$-regular graphs related with girth

Theorem 3.1. *For any $r > 1$, a graph G is $(r, 2, r(r-1))$-regular if and only if G is r-regular with girth at least five.*

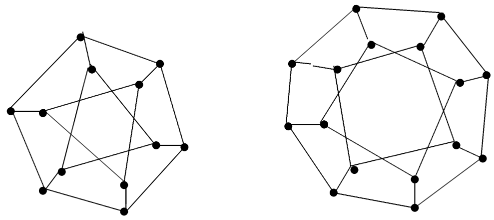
Proof. For any $r > 1$, let G be a $(r, 2, r(r-1))$-regular graph and $v \in V(G)$. Let $N(v) = \{v_1, v_2, v_3, \cdots, v_r\}$. Then $d(v) = r$ and $d_2(v) = r(r-1)$, for all $v \in V(G)$ if and only if G is r-regular graph and $N(v)$ has distinct neighbours if and only if G is r-regular graph with $G(N(v))$ has no edge and $N(v_i) \cap N(v_j) = \{v\}$, for $1 \leq i < j \leq r$ if and only if G is r-regular graph which does not contain triangle and four cycle if and only if G is an r-regular graph and has girth at least five. ■

Note that construction of $(r, 2, r(r-1))$-regular graph is equivalent to the construction of r-regular graphs with girth at least five. With this characterization, we can easily construct $(r, 2, r(r-1))$-regular graphs.

4 Construction of $(r, 2, r(r-1))$-regular graphs

Theorem 4.1. *For any $n \geq 5$, $(n \neq 6, 8)$ and any $r > 1$, there exists a $(r, 2, r(r-1))$-regular graph on $n \times 2^{r-2}$ vertices with girth five.*

Proof. Let us prove this result by induction on r. When $r = 2$, any n-cycle $(n \geq 5)$ is the required graph. Let G be the Peterson graph $P(n, 2)$ with the vertex set $V(G) = \{x_i^{(1)}, x_i^{(2)}/0 \leq i \leq n-1\}$ and the edge set $E(G) = \{x_i^{(1)}x_i^{(2)}, x_i^{(2)}x_{i+1}^{(2)}, x_i^{(1)}x_{i+2}^{(1)}/0 \leq i \leq n-1\}$ where the subscripts are taken modulo n. If $n = 6$, then G is a graph with girth three, and if $n = 8$, G is a graph with girth four as shown in Figure 3.



girth 3 girth 4

Figure 3.

Therefore, we cannot take $n = 6, 8$ for constructing a graph with girth five. For any $n \neq 6, 8$ we have for $0 \leq i \leq n-1$,

$$N_2(x_i^{(1)}) = \{x_{i+n-4}^{(1)}, x_{i+4}^{(1)}, x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+n-2}^{(2)}, x_{i+n-1}^{(2)}\} \text{ and } d_2(x_i^{(1)}) = 6 = 3(3-1).$$
$$N_2(x_i^{(2)}) = \{x_{i+2}^{(2)}, x_{i+n-2}^{(2)}, x_{i+1}^{(1)}, x_{i+2}^{(1)}, x_{i+n-2}^{(1)}, x_{i+n-1}^{(1)}\} \text{ and } d_2(x_i^{(2)}) = 6 = 3(3-1).$$

G is a $(3, 2, 3(3-1))$-regular graph on $n \times 2^{3-2}$ vertices with girth 5.

Step 1: Take another copy of G as G'. Let $V(G') = \{x_i^{(3)}, x_i^{(4)}/0 \leq i \leq n-1\}$ and $E(G') = \{x_i^{(3)}x_i^{(4)}, x_i^{(4)}x_{i+1}^{(4)}, x_i^{(3)}x_{i+2}^{(3)}/0 \leq i \leq n-1\}$, the subscripts being taken modulo n. The desired graph G_1 has the vertex set $V(G_1) = V(G) \cup V(G')$ and the edge set $E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)}x_{i+1}^{(4)}, x_i^{(2)}x_i^{(3)}/0 \leq i \leq n-1\}$. Now the resulting graph G_1 is a 4-regular graph having $n \times 2^{4-2}$ vertices with girth five.

Now consider the edges $x_i^{(1)}x_{i+1}^{(4)}$, for $0 \leq i \leq n-1$.

$$N(x_i^{(1)}) = \{x_{i+2}^{(1)}, x_{i+n-2}^{(1)}, x_i^{(2)}\} \text{ in } G \text{ and } |N(x_i^{(1)})| = 3 \text{ in } G,\ 0 \leq i \leq n-1\ .$$

$N(N(x_i^{(1)})) = \{x_{i+n-2}^{(4)}, x_{i+n-1}^{(4)}, x_i^{(3)}\}$ in G' and $|N(N(x_i^{(1)}))| = 3$ in G', $0 \leq i \leq n-1$.

$N(x_{i+1}^{(4)}) = \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+1}^{(3)}\}$ in G' and $|N(x_{i+1}^{(4)})| = 3$ in G', $0 \leq i \leq n-1$.

$N(N(x_{i+1}^{(4)})) = \{x_{i+n-1}^{(1)}, x_{i+1}^{(1)}, x_{i+1}^{(2)}\}$ in G and $|N(N(x_{i+1}^{(4)}))| = 3$ in G, $0 \leq i \leq n-1$.

d_2 of each vertex in $C^{(1)}$, where $C^{(1)}$ is the cycle induced by the vertices $\{x_i^{(1)}/0 \leq i \leq n-1\}$ in G_1.

$d_2(x_i^{(1)})$ in G_1 = $d_2(x_i^{(1)})$ in G + $|N(x_{i+1}^{(4)})|$ in G' + $|N(N(x_i^{(1)}))|$ in G' =$6+3+3 = 12 = 4(4-1)$, $(0 \leq i \leq n-1)$.
d_2 of each vertex in $C^{(4)}$, where $C^{(4)}$ is the cycle induced by the vertices $\{x_i^{(4)}/0 \leq i \leq n-1\}$ in G_1.

$d_2(x_{i+1}^{(4)})$ in G_1 = $(d_2(x_{i+1}^{(4)}))$ in G' + $|N(x_i^{(1)})|$ in G + $|N(N(x_{i+1}^{(4)}))|$ in $G = 6+3+3 = 12 = 4(4-1)$, $0 \leq i \leq n-1$.
Next consider the edges $x_i^{(2)}x_i^{(3)}$, for $0 \leq i \leq n-1$.

$N(x_i^{(2)}) = \{x_{i+1}^{(2)}, x_{i+n-1}^{(2)}, x_i^{(1)}\}$ in G and $|N(x_i^{(2)})| = 3$ in G, $0 \leq i \leq n-1$.

$N(N(x_i^{(2)})) = \{x_{i+1}^{(3)}, x_{i+n-1}^{(3)}, x_{i+1}^{(4)}\}$ in G' and $|N(N(x_i^{(2)}))| = 3$ in G', $0 \leq i \leq n-1$.

$N(x_i^{(3)}) = \{x_{i+2}^{(3)}, x_{i+n-2}^{(3)}, x_i^{(4)}\}$ in G' and $|N(x_i^{(3)})| = 3$ in G', $0 \leq i \leq n-1$.

$N(N(x_i^{(3)})) = \{x_{i+2}^{(2)}, x_{i+n-2}^{(2)}, x_{i+n-1}^{(1)}\}$ in G and $|N(N(x_i^{(3)})| = 3$ in G, $0 \leq i \leq n-1$.

d_2 of each vertex in $C^{(2)}$, where $C^{(2)}$ is the cycle induced by the vertices $\{x_i^{(2)}/0 \leq i \leq n-1\}$ in G_1.

$d_2(x_i^{(2)})$ in G_1 = $d_2(x_i^{(2)})$ in G + $|N(x_i^{(3)})|$ in G' + $|N(N(x_i^{(2)}))|$ in G' = $6+3+3 = 12 = 4(4-1)$, $0 \leq i \leq n-1$.
d_2 of each vertex in $C^{(3)}$, where $C^{(3)}$ is the cycle induced by the vertices $\{x_i^{(3)}/0 \leq i \leq n-1\}$ in G_1.

$d_2(x_i^{(3)})$ in G_1 = $d_2(x_i^{(3)})$ in G' + $|N(x_i^{(2)})|$ in G + $|N(N(x_i^{(3)})|$ in G = $6+3+3 = 12 = 4(4-1)$, $0 \leq i \leq n-1$.

In G_1, for $1 \leq t \leq 4$, $d_2(x_i^{(t)}) = 4(4-1)$, for $0 \leq i \leq n-1$. G_1 is $d(4, 4(4-1))$-regular on $n \times 2^{4-2}$ vertices with the vertex set. $V(G_1) = \{x_i^{(t)}/1 \leq t \leq 2^{4-2}, 0 \leq i \leq n-1\}$ and the edge set $E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)}x_{i+1}^{(4)}, x_i^{(2)}x_i^{(3)}/0 \leq i \leq n-1\}$ of girth 5.

That is, the result is true for $r = 4$.

Step 2: Take another copy of G_1 as G_1' with the vertex set $V(G_1') = \{x_i^{(t)}/2^{4-2}+1 \leq t \leq 2^{4-1},\ 0 \leq i \leq n-1\}$ and each $x_i^{(t)}$, $2^{4-2}+1 \leq t \leq 2^{4-1}$ corresponds to $x_i^{(t)}$, $1 \leq t \leq 2^{4-2}$ for $0 \leq i \leq n-1$. The desired graph G_2 has the vertex set $V(G_2) = V(G_1) \cup V(G_1')$ and the edge set $E(G_2) = E(G_1) \cup E(G_1') \cup \{x_i^{(1)}x_{i+1}^{(8)}, x_i^{(2)}x_i^{(7)}, x_i^{(3)}x_{i+1}^{(6)}, x_i^{(4)}x_i^{(5)}\}$. Now the resulting graph G_2 is a 5-regular graph having $n \times 2^{5-2}$ vertices with girth 5.
Consider the edges $x_i^{(1)}x_{i+1}^{(8)}$ for $0 \leq i \leq n-1$.

$N(x_i^{(1)}) = \{x_{i+2}^{(1)}, x_{i+n-1}^{(1)}, x_i^{(2)}, x_{i+1}^{(4)}\}$ in G_1 and $|N(x_i^{(1)})| = 4$ in G_1.

$N(N(x_i^{(1)})) = \{x_{i+n-2}^{(8)}, x_{i+n-1}^{(8)}, x_i^{(7)}, x_{i+1}^{(5)}\}$ in G_1' and $|N(N(x_i^{(1)}))| = 4$ in G_1'.
$N(x_{i+1}^{(8)}) = \{x_i^{(8)}, x_{i+2}^{(8)}, x_{i+1}^{(7)}, x_i^{(5)}\}$ in G_1' and $|N(x_{i+1}^{(8)})| = 4$ in G_1'.
$N(N(x_{i+1}^{(8)})) = \{x_{i+1}^{(1)}, x_{i+n-1}^{(1)}, x_{i+1}^{(2)}, x_i^{(4)}\}$ in G_1 and $|N(N(x_{i-1}^{(8)}))| = 4$ in G_1.
d_2 of each vertex in $C^{(1)}$, where $C^{(1)}$ is the cycle induced by the vertices $\{x_i^{(1)}/0 \leq i \leq n-1\}$ in G_2.

$d_2(x_i^{(1)})$ in G_2 = $d_2(x_i^{(1)})$ in G_1 + $|N(x_{i+1}^{(8)})|$ in G_1' + $|N(N(x_i^{(1)}))|$ in G_1' = $12+4+4 = 20 = 5(5-1)$, $0 \leq i \leq n-1$.
d_2 of each vertex in $C^{(8)}$, where $C^{(8)}$ is the cycle induced by the vertices $\{x_i^{(8)}/0 \leq i \leq n-1\}$ in G_2.

$d_2(x_{i+1}^{(8)})$ in G_2 = $(d_2(x_{i+1}^{(8)}$ in G_1' +$|N(x_i^{(1)})|$ in G_1 + $|(N(N(x_{i+1}^{(8)}))|$ in G_1 = $12+4+4 = 20 = 5(5-1)$, $0 \leq i \leq n-1$.
Next consider the edge $x_i^{(2)}x_i^{(7)}$, for $0 \leq i \leq n-1$.

$N(x_i^{(2)}) = \{x_{i+1}^{(2)}, x_{i+n-1}^{(2)}, x_i^{(1)}, x_i^{(3)}\}$ in G_1 and $|N(x_i^{(2)})| = 4$ in G_1.
$N(N(x_i^{(2)})) = \{x_{i+1}^{(7)}, x_{i+n-1}^{(7)}, x_{i+1}^{(8)}, x_{i+1}^{(6)}\}$ in G_1' and $|N(N(x_i^{(2)}))| = 4$ in G_1'.
$N(x_i^{(7)}) = \{x_{i+2}^{(7)}, x_{i+n-2}^{(7)}, x_i^{(8)}, x_i^{(6)}\}$ in G_1' and $|N(x_i^{(7)})| = 4$ in G_1'.
$N(N(x_i^{(7)})) = \{x_{i+2}^{(2)}, x_{i+n-2}^{(2)}, x_{i+n-1}^{(1)}, x_{i+n-1}^{(3)}\}$ in G_1 and $|N(N(x_i^{(7)}))| = 4$ in G_1.
d_2 of each vertex in $C^{(2)}$, where $C^{(2)}$ is the cycle induced by the vertices $\{x_i^{(2)}/0 \leq i \leq n-1\}$ in G_2.

$d_2(x_i^{(2)})$ in G_2 = $d_2(x_i^{(2)})$ in G_1 + $|N(x_i^{(7)})|$ in G_1' + $|N(N(x_i^{(2)}))|$ in G_1' = $12+4+4 = 20 = 5(5-1)$, for $0 \leq i \leq n-1$.
d_2 of each vertex in $C^{(7)}$, where $C^{(7)}$ is the cycle induced by the vertices $\{x_i^{(7)}/0 \leq i \leq n-1\}$ in G_2.

$d_2(x_i^{(7)})$ in G_2 = $d_2(x_i^{(7)})$ in G_1' + $|N(x_i^{(2)})|$ in G_1 + $|N(N(x_i^{(7)}))|$ in G_1. = $12+4+4 = 20 = 5(5-1)$, for $0 \leq i \leq n-1$.
Next consider the edge $x_i^{(3)}x_{i+1}^{(6)}$, for $0 \leq i \leq n-1$.

$N(x_i^{(3)}) = \{x_{i+2}^{(3)}, x_{i+n-2}^{(3)}, x_i^{(4)}, x_i^{(2)}\}$ in G_1 and $|N(x_i^{(3)})| = 4$ in G_1.
$N(N(x_i^{(3)})) = \{x_{i+n-2}^{(6)}, x_{i+n-1}^{(6)}, x_i^{(5)}, x_i^{(7)}\}$ in G_1' and $|N(N(x_i^{(3)}))| = 4$ in G_1'.
$N(x_{i+1}^{(6)}) = \{x_{i+2}^{(6)}, x_i^{(6)}, x_{i+1}^{(5)}, x_{i+1}^{(7)}\}$ in G_1' and $|N(x_{i+1}^{(6)})| = 4$ in G_1'.
$N(N(x_{i+1}^{(6)})) = \{x_{i+n-1}^{(3)}, x_{i+1}^{(3)}, x_{i+1}^{(4)}, x_{i+1}^{(2)}\}$ in G_1 and $|N(N(x_{i+1}^{(6)}))| = 4$ in G_1.
d_2 of each vertex in $C^{(3)}$, where $C^{(3)}$ is the cycle induced by the vertices $\{x_i^{(3)}/0 \leq i \leq n-1\}$ in G_2.

$d_2(x_i^{(3)})$ in G_2=$d_2(x_i^{(3)})$ in G_1+$|N(x_{i+1}^{(6)})|$ in G_1' + $|N(N(x_i^{(3)}))|$ in G_1'. = $12+4+4 = 20 = 5(5-1)$, for $0 \leq i \leq n-1$.
d_2 of each vertex in $C^{(6)}$, where $C^{(6)}$ is the cycle induced by the vertices $\{x_i^{(6)}/0 \leq i \leq n-1\}$ in G_2.

$d_2(x_{i+1}^{(6)})$ in G_2 = $d_2(x_{i+1}^{(6)})$ in G_1' + $|N(x_i^{(3)})|$ in G_1 + $|N(N(x_{i+1}^{(6)}))|$ in G_1 = $12+4+4 = 20 = 5(5-1)$, for $0 \leq i \leq n-1$.
Next consider the edge $x_i^{(4)}x_i^{(5)}$, for $0 \leq i \leq n-1$.

$N(x_i^{(4)}) = \{x_{i+1}^{(4)}, x_{i+n-1}^{(4)}, x_i^{(3)}, x_{i+n-1}^{(1)}\}$ in G_1 and $|N(x_i^{(4)})| = 4$ in G_1.
$N(N(x_i^{(4)})) = \{x_{i+1}^{(5)}, x_{i+n-1}^{(5)}, x_{i+1}^{(6)}, x_i^{(8)}\}$ in G_1' and $|N(N(x_i^{(4)}))| = 4$ in G_1'.
$N(x_i^{(5)}) = \{x_{i+2}^{(5)}, x_{i+n-2}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}\}$ in G_1' and $|N(x_i^{(5)})| = 4$ in G_1'.

$N(N(x_i^{(5)})) = \{x_{i+2}^{(4)}, x_{i+n-2}^{(4)}, x_{i+n-1}^{(3)}, x_i^{(1)}\}$ in G_1 and $|N(N(x_i^{(5)}))| = 4$ in G_1.
d_2 of each vertex in $C^{(4)}$, where $C^{(4)}$ is the cycle induced by the vertices $\{x_i^{(4)}/0 \leq i \leq n-1\}$ in G_2.

$d_2(x_i^{(4)})$ in G_2 = $d_2(x_i^{(4)})$ in G_1 + $|N(x_i^{(5)})|$ in G_1' + $|N(N(x_i^{(4)}))|$ in $G_1' = 12+4+4 = 20 = 5(5-1)$, for $0 \leq i \leq n-1$.
d_2 of each vertex in $C^{(5)}$, where $C^{(5)}$ is the cycle induced by the vertices $\{x_i^{(5)}/0 \leq i \leq n-1\}$ in G_2.

$d_2(x_i^{(5)})$ in G_2 = $d_2(x_i^{(5)})$ in G_1' +$|N(x_i^{(4)})|$ in G_1 +$|N(N(x_i^{(5)}))|$ in $G_1 = 12+4+4 = 20 = 5(5-1)$, for $(0 \leq i \leq n-1)$.
In G_2, for $1 \leq t \leq 8$, $d_2(x_i^{(t)}) = 5(5-1)$, for $0 \leq i \leq n-1$.
Therefore, G_2 is a $d(5, 2, 5(5-1))$-regular graph on $n \times 2^{5-2}$ vertices with the vertex set $V(G_2) = \{x_i^{(t)}/1 \leq t \leq 2^{5-2}, 0 \leq i \leq n-1\}$ and the edge set $E(G_2) = E(G_1) \cup E(G_1') \cup \{x_i^{(1)}x_{i+1}^{(8)}, x_i^{(2)}x_i^{(7)}, x_i^{(3)}x_{i+1}^{(6)}, x_i^{(4)}x_i^{(5)}/0 \leq i \leq n-1\}$ is of girth five. That is, the result is true for $r = 5$.

Assume that the result is true for $r = m+3$. There exists a $(m+3, 2, (m+3)(m+2))$-regular graph on $n \times 2^{m+1}$ vertices with the vertex set $V(G_m) = \{x_i^{(t)}/1 \leq t \leq 2^{m+1}, 0 \leq i \leq n-1\}$ and the edge set $E(G_m) = E(G_{m-1}) \cup E(G_{m-1}') \bigcup_{t=1}^{2^m} \{x_i^{(t)} x_{i+t(mod2)}^{2^{m+1}-t+1}/0 \leq i \leq n-1\}$ of girth 5.

That is, $d_2(x_i^{(t)}) = (m+3)(m+2)$ for $1 \leq t \leq 2^{m+1}$ and $d(x_i^{(t)}) = m+3$ for $0 \leq i \leq n-1$.

Take another copy of G_m as G_m' with the vertex set $V(G_m') = \{x_i^{(t)}/2^{m+1}+1 \leq t \leq 2^{m+2}, 0 \leq i \leq n-1\}$ and each $x_i^{(t)}, 2^{m+1}+1 \leq t \leq 2^{m+2}$ corresponds to $x_i^{(t)}, 1 \leq t \leq 2^{m+1}$ for $0 \leq i \leq n-1$.

The desired graph G_{m+1} has the vertex set $V(G_{m+1}) = V(G_m) \cup V(G_m')$ and the edge set $E(G_{m+1}) = E(G_m) \cup E(G_m') \bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)} x_{i+t(mod2)}^{2^{m+2}-t+1}/0 \leq i \leq n-1\}$. Now the resulting graph G_{m+1} is a $(m+4)$-regular graph having $n \times 2^{m+2}$ vertices with girth 5.

Consider the edges $\bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)} x_{i+t(mod2)}^{2^{m+2}-t+1}/0 \leq i \leq n-1\}$. d_2 of each vertex in $C^{(t)}$, $1 \leq t \leq 2^{m+1}$ where $C^{(t)}$ is the cycle induced by the vertices $\{x_i^{(t)}/0 \leq i \leq n-1\}$ in G_{m+1}.

$d_2(x_i^{(t)})$ in $G_{m+1} = d_2(x_i^{(t)})$ in G_m + $|N(x_{i+t(mod2)}^{2^{m+2}-t+1})|$ in G_m' + $|N(N(x_i^{(t)}))|$ in $G_m' = (m+3)(m+2) + (m+3) + (m+3) = (m+3)(m+2+1+1) = (m+3)(m+4)$, for $0 \leq i \leq n-1$.

d_2 of each vertex in $C^{(2^{m+2}-t+1)}$, $2^{m+1}+1 \leq t \leq 2^{m+2}$, where $C^{(2^{m+2}-t+1)}$ is the cycle induced by the vertices $\{x_i^{(2^{m+2}-t+1)}/0 \leq i \leq n-1\}$ in G_{m+1}.

$d_2(x_{i+t(mod2)}^{2^{m+2}-t+1})$ in $G_{m+1} = d_2(x_{i+t(mod2)}^{2^{m+2}-t+1})$ in G_m' + $|N(x_i^{(t)})|$ in G_m+ $|N(N(x_{i+t(mod2)}^{2^{m+2}-t+1}))|$ in G_m $= (m+3)(m+2) + (m+3) + (m+3) = (m+3)(m+4)$, for $1 \leq i \leq n-1$.

In G_{m+1}, for $1 \leq t \leq 2^{m+2}$, $d_2(x_i^{(t)}) = (m+3)(m+4), 0 \leq i \leq n-1$.

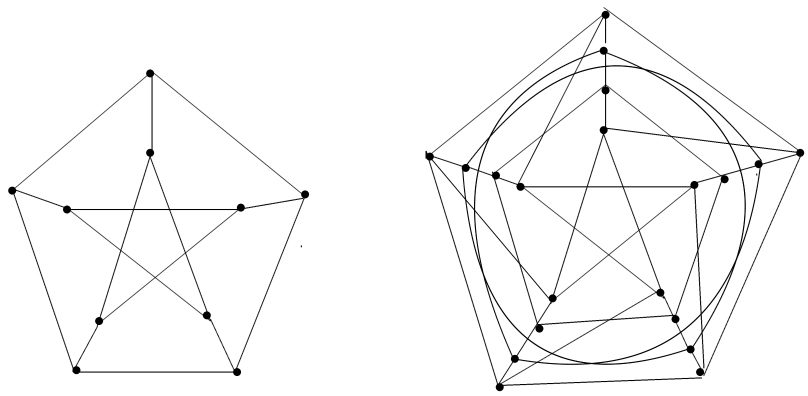
Thus, there exists a $(m+4, 2, (m+3)(m+4))$-regular graph on $n \times 2^{m+2}$ vertices with the vertex set $V(G_m) = \{x_i^{(t)}/1 \leq t \leq 2^{m+2}, 0 \leq i \leq n-1\}$ and the edge set $E(G_{m+1}) = E(G_m) \cup E(G_m') \bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)} x_{i+t(mod2)}^{2^{m+2}-t+1}/0 \leq i \leq n-1\}$ of girth 5.

Thus, for $1 \leq t \leq 2^{m+2}$, $d_2(x_i^{(t)}) = (m+3)(m+4)$ for $0 \leq i \leq n-1$ and $deg(x_i^{(t)}) = m+4$. If the result is true for $r = m+3$, then it is true for $r = m+4$. Therefore, the result is true for all $r > 2$.

That is, for any $n \geq 5$ $(n \neq 6, 8)$ and for any $r > 1$, there is a $(r, 2, r(r-1))$-regular graph on $n \times 2^{r-2}$ vertices with girth 5. ∎

Corollary 4.2. *For any $r > 1$, there is a $(r, 2, r(r-1))$-regular graph on $5 \times 2^{r-2}$ vertices.*

Example 4.3. *Figure 4. illustrates Corollary 4.2 for $n = 5$ and $r = 3, 4$.*



(a) n= 5, r= 3 r(r-1) = 6 *(b) n= 5, r= 4 r(r-1) = 12*

Figure 4.

Corollary 4.4. *For any $r > 1$, there is a $(r, 2, r(r-1))$-regular graph on $7 \times 2^{r-2}$ vertices with girth 5.*

Corollary 4.5. *For any $r > 1$, there is a $(r, 2, r(r-1))$-regular graph on $9 \times 2^{r-2}$ vertices with girth 5.*

Corollary 4.6. *For any $r > 1$, there exists a $(r, 2, r(r-1))$-regular graph on $10 \times 2^{r-2}$ vertices with girth 5.*

We observe that if the value of n increases then the order of the graph also increases.

5 Construction of $(r, 2, r(r-1))$-regular graphs of girth six

The Peterson graph $P(n, 3)$ has vertex set $\{x_i^{(1)}, x_i^{(2)} / 0 \leq i \leq n-1\}$ and the edge set $\{x_i^{(2)} x_{i+1}^{(2)} / i = 0, 1, 2, 3, \cdots n-1\} \cup \{x_i^{(1)} x_i^{(2)} / i = 0, 1, 2, 3, \cdots n-1\} \cup \{x_i^{(1)} x_{i+3}^{(1)} / i = 0, 1, 2, 3, \cdots n-1\}$ where the subscripts are taken modulo n.

Theorem 5.1. *For any $n \geq 7$, $n \neq 9, 12, 15$ and $r > 1$, there is a $(r, 2, r(r-1))$-regular graph on $n \times 2^{r-2}$ vertices with girth six.*

Proof. Replace $P(n, 2)$ with $P(n, 3)$ in Theorem 4.1. The result follows. ∎

Remark 5.2. *When $n = 15$, we get a $(r, 2, r(r-1))$-regular graph of order $n \times 2^{r-2}$ with girth 5. If*

$n = 9, 12$, *then we have graphs with girth* $3, 4$ *respectively as shown in Figure 5.*

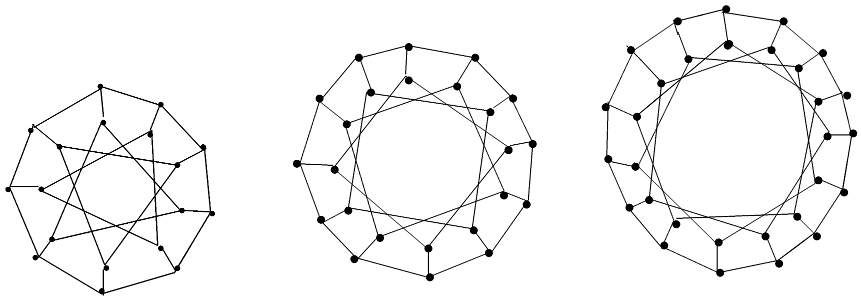


Figure 5.

6 Construction of $(r, 2, r(r-1))$-regular graphs of girth seven

Consider the Peterson graph $P(n, 4)$, which has the vertex set $\{x_i^{(1)}, x_i^{(2)}/0 \leq i \leq n-1\}$ and the edge set $\{x_i^{(2)}x_{i+1}^{(2)}/i = 0, 1, 2, 3, \cdots n-1\} \cup \{x_i^{(1)}x_i^{(2)}/i = 0, 1, 2, 3, \cdots n-1\} \cup \{x_i^{(1)}x_{i+4}^{(1)}/i = 0, 1, 2, 3, \cdots n-1\}$ (where the subscripts are taken modulo n).

Theorem 6.1. *For any* $n \geq 9$, $n \neq 12, 16, 20, 24$ *and any* $r > 1$, *there is a* $(r, 2, r(r-1))$*-regular graph on* $n \times 2^{r-2}$ *vertices with girth seven.*

Proof. Take $P(n, 4)$ instead of taking $P(n, 2)$ in Theorem 4.1. The result follows. ∎

Remark 6.2. *When* $n = 20, 24$, *we get a* $(r, 2, r(r-1))$*-regular graph of order* $n \times 2^{r-2}$ *with girth 5, 6 respectively. If* $n = 12, 16$, *then we have graphs with girth 3, 4 respectively as shown in Figure 6.*

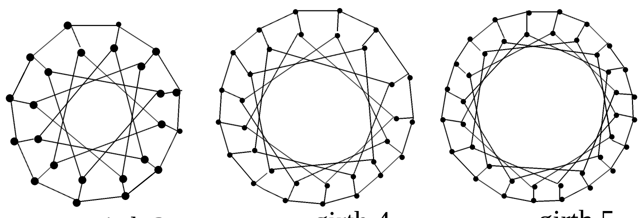


Figure 6.

Consider the generalized Peterson graph $P(n, k)$, for $n \geq 5$ and $(2 \leq k < n/2)$, having vertex set $V(G) = \{x_i^{(1)}, x_i^{(2)}/0 \leq i \leq n-1\}$ and the edge set $\{x_i^{(2)}x_{i+1}^{(2)}/0 \leq i \leq n-1\} \cup \{x_i^{(1)}x_i^{(2)}/0 \leq i \leq n-1\} \cup \{x_i^{(1)}x_{i+k}^{(1)}/0 \leq i \leq n-1\}$, where the subscripts are taken modulo n.

Theorem 6.3. *For any* $n \geq 5$ *and* $2 \leq k < n/2$ *(* $n \neq (2+t)k$ *where* $1 \leq t \leq 2$*) and for any* $r \geq 2$, *there is a* $(r, 2, r(r-1))$*-regular graph on* $(n \times 2^{r-2})$ *vertices with girth* $k+3$. *In particular, when* $n = (2+t)k$, $2 < t \leq k$ *there exist a* $(r, 2, r(r-1))$*-regular graph of order* $n \times 2^{r-2}$ *with girth* $(2+t)$.

Proof. We take the generalized Peterson graph $P(n, k)$, when $n \geq 5$ and $1 \leq k < n/2$ instead of taking $P(n, 2)$ in Theorem 4.1. ∎

Remark 6.4. *If* $n = (2+t)k$, $t = 1, 2$, *then we have graphs with girth 3 and 4 respectively. Therefore, we cannot have a* $(r, 2, r(r-1))$*-regular graph for* $n = (2+t)k$, *for* $t = 1, 2$ *by this construction.*

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