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Radio labeling for some cycle related graphs

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Abstract

Let $G = (V(G), E(G))$ be a connected graph and let $d(u, v)$ denote the distance between any two vertices in G. The maximum distance between any pair of vertices is called the diameter of G denoted by $diam(G)$. A radio labeling(or multilevel distance labeling) for G is an injective function $f : V(G) \longrightarrow N \cup \{0\}$ such that for any vertices u and v, $|f(u) - f(v)| \geq diam(G) - d(u, v) + 1$. The span of f is the largest number in $f(V)$. The radio number of G, denoted by $r_n(G)$ is the minimum span of a radio labeling of G. In this paper we determine upper bounds of radio numbers for cycle with chords and $n/2$-petal graph. Further the radio number is completely determined for the split graph and middle graph of cycle C_n.

Keywords: Graph labeling, radio number, channel assignment.
AMS Subject Classification(2010): 05C15, 05C78.

1 Introduction

For standard terminology and notation we follow West [13]. We give brief summary of definitions and information which serves as prerequisites for our present work. Unless or otherwise mentioned, $G = (V(G), E(G))$ is a simple, finite, connected and undirected graph.

Definition 1.1. *A Chord of a cycle C is an edge not in C whose end vertices lie in C.*

Definition 1.2. *The* middle graph M(G) *of a graph G is the graph whose vertex set is $V(G) \bigcup E(G)$ in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.*

Definition 1.3. *For a graph G the* split graph *is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The new graph is denoted as* spl(G).

Definition 1.4. *A petal graph is a connected graph G with $\triangle(G) = 3$ and $\delta(G) = 2$ in which the set of vertices of degree three induces a 2-regular graph and the set of vertices of degree two induces an empty graph. In a petal graph G if w is a vertex of G with degree two, having neighbours v_1, v_2 then the path $P_w = v_1wv_2$ is called* petal *of G.*

We name w the *center of the petal* and v_1, v_2 the *basepoints*. If $d(v_1, v_2) = k$, we say that the size of the petal is k. If the size of each petal is k then it is called a *k-petal graph.*

In the next section we provide a brief introduction of radio labeling and some new results related to it.

2 Radio labeling and some new results

Radio labeling can be regarded as an extension of distance two labeling which is motivated by the channel assignment problem introduced by Hale [7]. For a set of given cities(or radio stations) the problem is to assign channel to each city, which is a non-negative integer so that interference is avoided. The task is to find a valid channel assignment with the minimum span of the channels used. The level of interference is closely related to the geographical location of the stations - the closer are the stations the stronger the interference between them may occur. In order to avoid interference, the separation between the channels assigned to a pair of nearby stations must be large enough. To model this problem, we construct a graph G in which each station is represented by a vertex and connect two vertices by an edge if the geographical locations of the corresponding stations are very close. Two close stations are represented in the corresponding graph G by a pair of vertices that are distance - 2 apart. In this context Griggs and Yeh [2] introduced $L(2, 1)$ labeling as follows.

Definition 2.1. *[2] For a graph G, $L(2, 1)$ labeling (or distance two labeling) with span k is a function $f : V(G) \longrightarrow \{0, 1, \dots, k\}$ such that the following conditions are satisfied:*
$(1)|f(x) - f(y)| \geq 2$ *if* $d(x, y) = 1$
$(2)|f(x) - f(y)| \geq 1$ *if* $d(x, y) = 2$

In other words the $L(2, 1)$-labeling of a graph is an abstraction of assigning integer frequencies to radio transmitters such that
(1) Transmitters that are one unit of distance apart receive frequencies that differ by at least two and
(2) Transmitters that are two units of distance apart receive frequencies that differ by at least one.

$L(2, 1)$-labeling has been extensively studied in recent past by Georges and Mauro [3], Georges et al. [4–6], Liu and Yeh [8], Vaidya et al. [10], Vaidya and Bantva [11, 12]. Practically it has been observed that the interference among channels might go beyond two levels. Radio labeling extends the number of interference level considered in $L(2, 1)$-labeling from two to the largest possible - the diameter of G. The diameter of G is denoted by *diam*(G) which is the maximum distance among all pairs of vertices in G. Motivated by the problem of channel assignments to FM radio stations, Chartrand et al. [1] introduced the concept of radio labeling of graph as follows.

Definition 2.2. *A* radio labeling *of a graph G is an injective function* $f : V(G) \longrightarrow N \cup \{0\}$ *such that for every* $u, v \in V$, $|f(u) - f(v)| \geq diam(G) - d(u,v) + 1$.
The span of f *is the difference of the largest and the smallest channels used, that is* $\max\limits_{u,v \in V} \{f(u) - f(v)\}$, *for every* $u, v \in V$.

The *radio number* of G is defined as the minimum span of a radio labeling of G and is denoted as $r_n(G)$. The radio number for paths and cycles are discussed in [1] by Chartrand et al and in [9] by Liu and Zhu.

The common flavor of most of the research papers available in this area is either to determine upper bounds of the radio number or the exact value of the radio number for the graph under consideration. The natural lower bound is 0 for all the graphs considered here as we initiate the labeling from 0. It is obvious that any change in lower bound will affect the optimality(radio number) of the labeling.

We determine upper bounds of radio number for cycle with chords and $n/2$-petal graph, while the radio number is completely determined for $spl(C_n)$ and for $M(C_n)$. Following the general trend we contribute four results. In the last two results we slightly deviate and discuss radio labeling in the context of some graph operations.

Theorem 2.3. *Let G be a cycle with chords. Then,*

$$r_n(G) = \begin{cases} (k+2)(2k-1) + 1 + \sum\limits_i (d_i - d_i'), & n \equiv 0(\bmod 4) \\ 2k(k+2) + 1 + \sum\limits_i (d_i - d_i'), & n \equiv 2(\bmod 4) \\ 2k(k+1) + \sum\limits_i (d_i - d_i'), & n \equiv 1(\bmod 4) \\ (k+2)(2k+1) + \sum\limits_i (d_i - d_i'), & n \equiv 3(\bmod 4) \end{cases}$$

Proof. Let C_n denote the cycle on n vertices and $V(C_n) = \{v_0, v_1 \ldots v_{n-1}\}$ be such that v_i is adjacent to v_{i+1} and v_{n-1} is adjacent to v_0. We denote $d = diam(C_n)$.
The labels are assigned with the help of the following two sequences.

- the distance gap sequence $D = (d_0, d_1 \ldots d_{n-2})$
- the color gap sequence $F = (f_0, f_1 \ldots f_{n-2})$

The distance gap sequence in which each $d_i \leq d$ is a positive integer is used to generate an ordering of the vertices of C_n. Let $\tau : \{0, 1, \ldots, n-1\} \longrightarrow \{0, 1, \ldots, n-1\}$ be defined as $\tau(0) = 0$ and $\tau(i+1) =$

$\tau(i)+d_i\ (mod\, n)$. Here τ is a corresponding permutation. Let $x_i = v_{\tau(i)}$ for $i = 0, 1, 2, \ldots n-1$. Then $\{x_0, x_1 \ldots x_{n-1}\}$ is an ordering of the vertices of C_n. Let us denote $d(x_i, x_{i+1}) = d_i$.
The color gap sequence is used to assign labels to the vertices of C_n. Let f be the labeling defined by $f(x_0) = 0$ and for $i \geq 1, f(x_{i+1}) = f(x_i) + f_i$. By the definition of radio labeling, $f_i \geq d - d_i + 1$ for all i. We adopt the scheme for distance gap sequence and color gap sequence reported in [9] and proceed as follows.

Case 1: $n = 4k$. In this case, $diam(G) = 2k$.
Using the sequences given below we can generate the radio labeling of cycle C_n for $n \equiv 0(\text{mod } 4)$ with minimum span.
The distance gap sequence is given by

$$\begin{aligned} d_i &= 2k && \text{if } i \text{ is even} \\ &= k && \text{if } i \equiv 1(mod\, 4) \\ &= k+1 && \text{if } i \equiv 3(mod\, 4) \end{aligned}$$

and the color gap sequence is given by

$$\begin{aligned} f_i &= 1 && \text{if } i \text{ is even} \\ &= k+1 && \text{if } i \text{ is odd} \end{aligned}$$

Then, for $i = 0, 1, 2, \ldots k-1$ we have the following permutation,

$$\begin{aligned} \tau(4i) &= 2ik + i(mod\, n) \\ \tau(4i+1) &= (2i+2)k + i(mod\, n) \\ \tau(4i+2) &= (2i+3)k + i(mod\, n) \\ \tau(4i+3) &= (2i+1)k + i(mod\, n) \end{aligned}$$

Now we add chords in cycle C_n such that diameter of the cycle remains unchanged. Label the vertices of this newly obtained graph using the above permutation. Suppose the new distance between x_i and x_{i+1} is $d_i^{'}(x_i, x_{i+1})$, then due to chords in the cycle it is obvious that $d_i \geq d_i^{'}$.
We define the color gap sequence as
$f_i^{'} = f_i + (d_i - d_i^{'}), 0 \leq i \leq n-2$
So, that span $f^{'}$ for cycle with chords is

$$\begin{aligned} f_0^{'} + f_1^{'} \ldots + f_{n-2}^{'} &= f_0 + f_1 + f_2 \ldots + f_{n-2} + \sum(d_i - d_i^{'}) \\ &= r_n(C_n) + \sum(d_i - d_i^{'}) \\ &= (k+2)(2k-1) + 1 + \sum(d_i - d_i^{'}).\ 0 \leq i \leq n-2 \end{aligned}$$

which is an upper bound for the radio number for the cycle with arbitrary number of chords when $n = 4k$.

Case 2: $n = 4k+2$. In this case $diam(G) = 2k+1$.
Using the sequences given below we can generate radio labeling of the cycle C_n for $n \equiv 2(\text{mod } 4)$ with minimum span.
The distance gap sequence is given by

d_i = $2k+1$ if i is even
= $k+1$ if i is odd

and the color gap sequence is given by

f_i = 1 if i is even
= $k+1$ if i is odd

Hence for $i = 0, 1, \ldots 2k$, we have the following permutation,

$\tau(2i)$ = $i(3k+2)(mod\, n)$
$\tau(2i+1)$ = $i(3k+2) + 2k + 1 \quad (mod\, n)$

Now we add chords in the cycle C_n such that the diameter of the cycle remains unchanged. Label the vertices of this newly obtained graph by using the above permutation. So, that span $f^{'}$ for cycle with chords is

$$\begin{aligned} f_0^{'} + f_1^{'} \ldots + f_{n-2}^{'} &= f_0 + f_1 + f_2 \ldots + f_{n-2} + \sum(d_i - d_i^{'}) \\ &= r_n(C_n) + \sum(d_i - d_i^{'}) \\ &= 2k(k+2) + 1 + \sum(d_i - d_i^{'}).\ 0 \leq i \leq n-2 \end{aligned}$$

which is an upper bound for the radio number for cycle with arbitrary number of chords. when $n = 4k+2$.

Case 3: $n = 4k+1$. In this case $diam(G) = 2k$.

Using the sequences given below we can generate radio labeling of cycle C_n for $n \equiv 1$(mod 4) with minimum span.

The distance gap sequence is given by

$d_{4i} = d_{4i+2} = 2k - i$
$d_{4i+1} = d_{4i+3} = k + 1 + i$

and the color gap sequence is given by

$f_i = 2k - d_i + 1$

Then we have.

$\tau(2i)$ =$i(3k+1)$ $(mod\, n)$, $0 \leq i \leq 2k$
$\tau(4i+1)$ = $2(i+1)k$ $(mod\, n)$, $0 \leq i \leq k-1$
$\tau(4i+3)$ = $(2i+1)k$ $(mod\, n)$, $0 \leq i \leq k-1$

Label the vertices of this newly obtained graph by using the above permutation. So, that span of $f^{'}$ for cycle with chords is

$$\begin{aligned} f_0^{'} + f_1^{'} \ldots + f_{n-2}^{'} &= f_0 + f_1 + f_2 \ldots + f_{n-2} + \sum(d_i - d_i^{'}) \\ &= r_n(C_n) + \sum(d_i - d_i^{'}) \\ &= 2k(k+1) + \sum(d_i - d_i^{'}).\ 0 \leq i \leq n-2 \end{aligned}$$

which is an upper bound for the radio number for cycle with arbitrary number of chords. when $n = 4k+1$.

Case 4: $n = 4k+3$. In this case $diam(G) = 2k+1$.

Using the sequences below we can give radio labeling of the cycle C_n for $n \equiv 3$(mod 4) with minimum span.

The distance gap sequence is given by
$d_{4i} = d_{4i+2} = 2k + 1 - i$
$d_{4i+1} = k + 1 + i$
$d_{4i+3} = k + 2 + i$
and the color gap sequence is given by

$$
\begin{aligned}
f_i &= 2k - d_i + 2 \quad \text{if } i \not\equiv 3(mod\,4) \\
&= 2k - d_i + 3 \quad \text{otherwise}
\end{aligned}
$$

Then we have the following permutation,

$$
\begin{aligned}
\tau(4i) &= 2i(k+1)\ (mod\,n)\ , && 0 \le i \le k \\
\tau(4i+1) &= (i+1)(2k+1)\ (mod\,n)\ , && 0 \le i \le k \\
\tau(4i+2) &= (2i-1)(k+1)\ (mod\,n)\ , && 0 \le i \le k \\
\tau(4i+3) &= i(2k+1) + k\ (mod\,n), && 0 \le i \le k-1, 1 \le i \le n-2
\end{aligned}
$$

Label the vertices of this newly obtained graph by using the above permutation. So, that span of f' for cycle with chords is

$$
\begin{aligned}
f'_0 + f'_1 \ldots + f'_{n-2} &= f_0 + f_1 + f_2 \ldots + f_{n-2} + \sum(d_i - d'_i) \\
&= r_n(C_n) + \sum(d_i - d'_i) \\
&= (k+2)(2k+1) + \sum(d_i - d'_i)
\end{aligned}
$$

which is an upper bound for the radio number for cycle with arbitrary number of chords for the cycle with chords.
Thus, in all the possibilities we have obtained the upper bounds of the radio numbers. ■

Illustration 2.4. *Consider the graph C_{12} with 5 chords. The radio labeling is shown in Figure 1.*

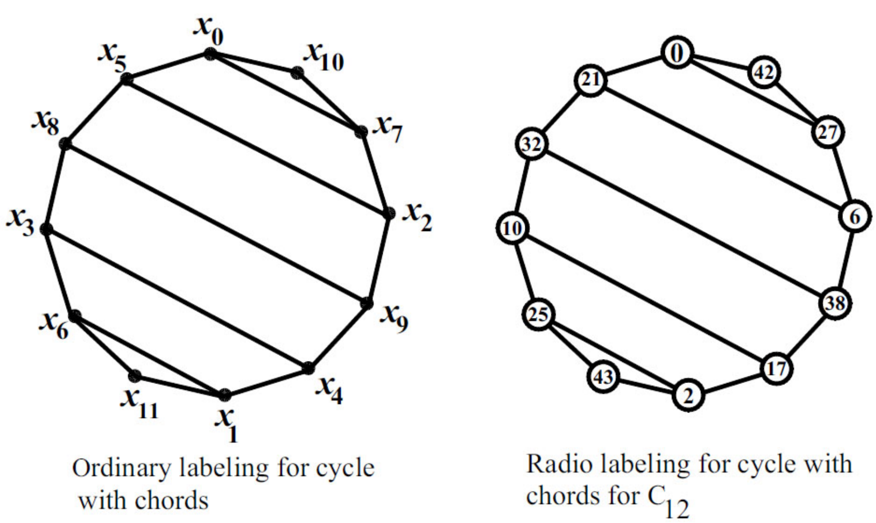

Figure 1: Ordinary and Radio labeling for cycle with chords for C_{12}.

Theorem 2.5. *Let G be an $n/2$-petal graph constructed from an even cycle C_n. Then*

$$r_n(G) \leq \begin{cases} \frac{3p}{2} + n\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor + 2n - 2 \\ \\ (p-1) + n\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor + 2n \end{cases}$$

Proof. Let G be an $n/2$-petal graph with vertices $v_0, v_1 \dots v_{n-1}$ of degree 3 and $v_1^{'}, v_2^{'} \dots v_p^{'}$ of degree 2. Here v_i is adjacent to v_{i+1} and v_{n-1} is adjacent to v_0.
Case 1: $n \equiv 0(mod4)$ and $diam(G) = \lfloor n/4 \rfloor + 2$.
First we label the vertices of degree 2. Let $v_1^{'}, v_2^{'} \dots v_p^{'}$ be the vertices on the petals satisfying the order defined by the following distance sequence.

$d_i^{'} \quad = \lfloor n/4 \rfloor + 2$ if i is even
$\quad\;\; = \lfloor n/4 \rfloor + 1$ if i is odd

The color gap sequence for vertices on the petals is defined as

$f_i^{'} \quad = 1$ if i is even
$\quad\;\; = 2$ if i is odd

Let v_1 be the vertex on the cycle C_n such that $d(v_p^{'}, v_1) = \lfloor n/8 \rfloor + 1 = d(v_{p-1}^{'}, v_1)$.
Label v_1 as $f(v_1) = f(v_p^{'}) + diam(G) - \lfloor n/8 \rfloor$.
For the remaining vertices of degree 3, we use the permutation defined for the cycle C_n in case 1 of Theorem 2.3.
The color gap sequence for the same vertices is defined as
$f_i = \lfloor n/4 \rfloor + 2$, $0 \leq$i$\leq n-2$.
Then span of $f = 3p/2 + n\lfloor n/4 \rfloor - \lfloor n/8 \rfloor + 2n - 2$
which is an upper bound for the radio number of the $n/2$-petal graph when $n \equiv 0(mod4)$.
Case 2: $n \equiv 2(mod4)$ and $diam(G) = \lfloor n/4 \rfloor + 2$.
First we label the vertices of degree 2. Let $v_1^{'}, v_2^{'} \dots v_p^{'}$ be the vertices on the petals satisfying the order defined by the following distance sequence.
$d(v_i^{'}, v_{i+1}^{'}) = \lfloor n/4 \rfloor + 2$.
The color gap sequence for the vertices on the petals is defined as
$f_i^{'} = 1, 1 \leq i \leq p$.
Let v_1 be the vertex on the cycle C_n such that $d(v_p^{'}, v_1) = \lfloor n/8 \rfloor + 1 = d(v_{p-1}^{'}, v_1)$.
Label v_1 as $f(v_1) = f(v_p^{'}) + diam(G) - \lfloor n/8 \rfloor$.

For the remaining vertices of degree 3, we use the permutation defined for the cycle C_n in case 2 of Theorem 2.3.
The color gap sequence for the same vertices is defined as
$f_i = \lfloor n/4 \rfloor + 2$, $0 \leq$i$\leq n-2$.
Then span of $f = p - 1 + n\lfloor n/4 \rfloor - \lfloor n/8 \rfloor + 2n$

which is an upper bound for the radio number of the $n/2$-petal graph when $n \equiv 2(mod4)$. ∎

Illustration 2.6. *Consider the n/2-petal graph of C_8. The radio labeling is shown in Figure 2.*

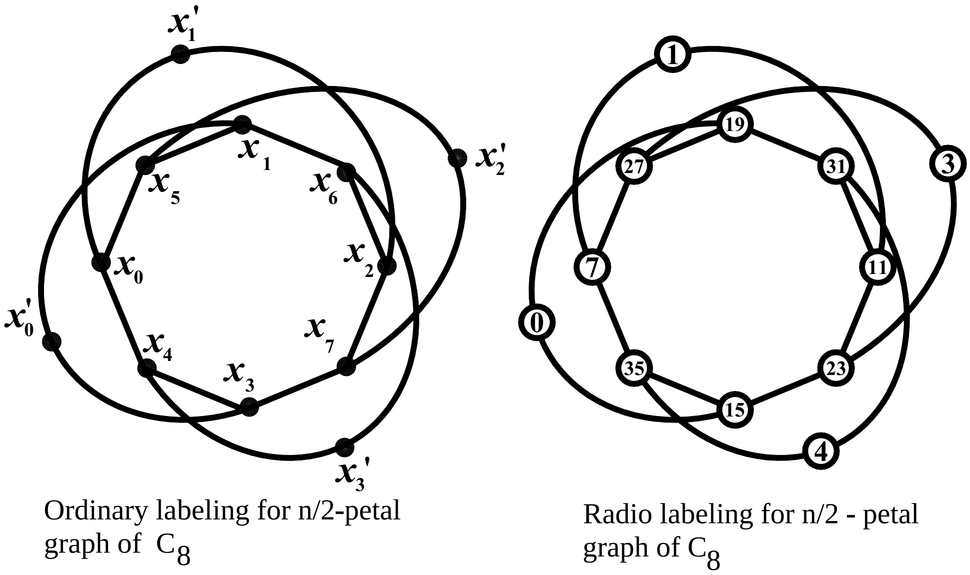

Figure 2: Ordinary and radio labeling for an n/2-petal graph of C_8.

Theorem 2.7. *For any cycle C_n,*

$$r_n(spl(C_n)) = \begin{cases} 2[(k+2)(2k-1)+1]+k+1 & n \equiv 0\,(\mathrm{mod} 4) \\ 2[2k(k+2)+1]+k+1 & n \equiv 2(\mathrm{mod}\ 4) \\ 2[2k(k+1)]\ +\ k & n \equiv 1(\mathrm{mod}\ 4) \\ 2[(k+2)(2k+1)]+k+1 & n \equiv 3\,(\mathrm{mod}\ 4) \end{cases}$$

Proof. Let G be the split graph of C_n. Let $v_1^{'}, v_2^{'}, v_3^{'} \ldots v_n^{'}$ be the duplicated vertices corresponding to $v_1, v_2, v_3 \ldots v_n$. We initiate the labeling by assigning the labels to vertices of cycle and then to their duplicated vertices because $d(v_i, v_j) = d(v_i, v_j^{'})$. In order to obtain the labeling with minimum span we employ twice the distance gap sequence, the color gap sequence and the permutation scheme used by Liu and Zhu in [9]. This labeling procedure will generate exact radio number as optimality of the permutation is established in [9].

Case 1: $n \equiv 4k(n > 4)$. Then, $diam(G) = 2k$.

We first label the vertices $v_1, v_2, v_3 \ldots v_n$ as in Case 1 of Theorem 2.3 and then we label the vertices $v_1^{'}, v_2^{'}, v_3^{'} \ldots v_n^{'}$ as follows:

Define $f(v_j^{'}) = f(v_k) + k + 1$ where v_k is the last labeled vertex in the cycle and $v_j^{'}$ is the vertex such that $d(v_k, v_j^{'}) = k$.

Now using the permutation used in Case 1 of Theorem 2.3 for cycle C_n with $n \equiv 0(mod\, 4)$, label the duplicated vertices starting from $v_j^{'}$.

Then $r_n(spl(C_n))$= $f_0 + f_1 + f_2 \ldots + f_{n-2} + 2k - k + 1 + f_0^{'} + f_1^{'} + \cdots + f_{n-2}^{'}$.

As $f_i = f_i^{'}$, for $i = 0, 1, 2 \ldots n-2$, we have

$$
\begin{aligned}
r_n(spl(C_n)) &= 2f_0 + 2f_1 + 2f_2 \ldots + 2f_{n-2} + k + 1 \\
&= 2[(k+2)(2k-1)+1] + k + 1.
\end{aligned}
$$

Case 2: $n \equiv 4k + 2 (n > 6)$. Then $diam(G) = 2k + 1$.

We first label the vertices $v_1, v_2, v_3 \ldots v_n$ as in Case 2 of Theorem 2.3 and then we label the vertices $v_1^{'}, v_2^{'}, v_3^{'} \ldots v_n^{'}$ as follows:

Define $f(v_j^{'}) = f(v_k) + k + 1$ where v_k is the last labeled vertex in the cycle and $v_j^{'}$ is the vertex such that $d(v_k, v_j^{'}) = k + 1$

Using the permutation for cycle C_n with $n \equiv 2(mod\, 4)$ in Case 2 of Theorem 2.3, label the duplicated vertices starting from $v_j^{'}$.

Then, $r_n(spl(C_n))$= $f_0 + f_1 + f_2 \ldots + f_{n-2} + 2k + 1 - k - 1 + 1 + f_0^{'} + f_1^{'} + \cdots + f_{n-2}^{'}$.

As $f_i = f_i^{'}$, for $i = 0, 1, 2 \ldots n-2$, we have

$$
\begin{aligned}
r_n(spl(C_n)) &= 2f_0 + 2f_1 + 2f_2 \ldots + 2f_{n-2} + k + 1 \\
&= 2[2k(k+2)+1] + k + 1.
\end{aligned}
$$

Case 3: $n \equiv 4k + 1 (n > 5)$. Then $diam(G) = 2k$.

We first label the vertices $v_1, v_2, v_3 \ldots v_n$ as in Case 3 of Theorem 2.3. Now we label the vertices $v_1^{'}, v_2^{'}, v_3^{'} \ldots v_n^{'}$ as follows:

Define $f(v_j^{'}) = f(v_k) + k$ where v_k is the last labeled vertex in the cycle and $v_j^{'}$ is the vertex such that $d(v_k, v_j^{'}) = k + 1$.

Using the permutation for cycle C_n with $n \equiv 1(mod\, 4)$ in Case 3 of Theorem 2.3, label the duplicated vertices starting from $v_j^{'}$.

Then $r_n(spl(C_n))$= $f_0 + f_1 + f_2 \ldots + f_{n-2} + 2k - k - 1 + 1 + f_0^{'} + f_1^{'} + \cdots + f_{n-2}^{'}$.

As $f_i = f_i^{'}$, for $i = 0, 1, 2 \ldots n-2$, we have

$$
\begin{aligned}
r_n(spl(C_n)) &= 2f_0 + 2f_1 + 2f_2 \ldots + 2f_{n-2} + k \\
&= 2[2k(k+1)] + k.
\end{aligned}
$$

Case 4: $n \equiv 4k + 3 (n > 3)$. Then $diam(G) = 2k + 1$.

We first label the vertices $v_1, v_2, v_3 \ldots v_n$ as in Case 4 of Theorem 2.3. Now we label the vertices $v_1^{'}, v_2^{'}, v_3^{'} \ldots v_n^{'}$ as follows:

Let $v_j^{'}$ be the vertex such that $d(v_k, v_j^{'}) = k + 1$.

Define $f(v_j^{'}) = f(v_{n-1}) + k + 1$ where v_k is the last labeled vertex in the cycle and $v_j^{'}$ is the vertex such that $d(v_k, v_j^{'}) = k + 1$.

Using the permutation for cycle C_n with $n \equiv 3(mod\, 4)$ in Case 4 of Theorem 2.3, label the duplicated

vertices starting from v_j'.
Then $r_n(spl(C_n))$= $f_0 + f_1 + f_2 \ldots + f_{n-2} + 2k + 1 - k - 1 + 1 + f_0' + f_1' + \cdots + f_{n-2}'$.
As $f_i = f_i'$, for $i = 0, 1, 2 \ldots n-2$, we have

$$\begin{aligned} r_n(spl(C_n)) &= 2f_0 + 2f_1 + 2f_2 \ldots + 2f_{n-2} + k + 1 \\ &= 2[(k+2)(2k+1)] + k + 1. \end{aligned}$$

Thus in all the four cases we have determined the radio number of G. ∎

Illustration 2.8. *Consider the graph $spl(C_{10})$. The radio labeling is shown in Figure 3.*

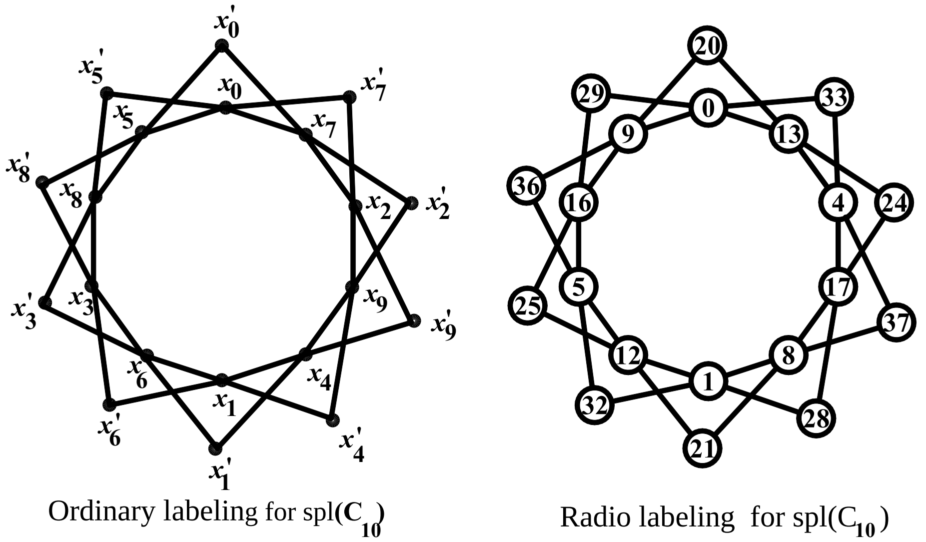

Figure 3: Ordinary and radio labeling for $spl(C_{10})$.

Application

Above result can be applied for the purpose of expansion of existing circular network of radio transmitters. By applying the concept of duplication of vertex the number of radio transmitters are doubled and separation of the channels assigned to the stations is such that interference can be avoided. Thus our result can play a vital role for the expansion of radio transmitter network without disturbing the existing one. In the expanded network the distance between any two transmitters is large enough to avoid the interference.

Theorem 2.9. *For any cycle C_n,*

$$r_n(M(C_n)) = \begin{cases} 2(k+2)(2k-1) + n + 3 & n \equiv 0 \pmod 4 \\ 4k(k+2) + k + n + 3 & n \equiv 2 \pmod 4 \\ 4k(k+1) + k + n & n \equiv 1 \pmod 4 \\ 2(k+2)(2k+1) + k + n + 1 & n \equiv 3 \pmod 4 \end{cases}$$

Proof. Let $u_1, u_2, \dots, u_n$ be the vertices of the cycle C_n and $u_1^{'}, u_2^{'}, \dots, u_n^{'}$ be the newly inserted vertices corresponding to the edges of C_n to obtain $M(C_n)$. In $M(C_n)$ the diameter is increased by 1. Here $d(u_i, u_j) \geq d(u_i, u_j^{'})$ for $n \equiv 0, 2(mod\,4)$ and $d(u_i, u_j) = d(u_i, u_j^{'})$ for $n \equiv 1, 3(mod\,4)$. Through out the discussion, first we label the vertices $u_1, u_2, \dots, u_n$ and then the newly inserted vertices $u_1^{'}, u_2^{'}, \dots, u_n^{'}$. For this purpose we will employ twice the permutation scheme for respective cycles as in Theorem 2.3.

Case 1: $n \equiv 4k$. In this case $diam(M(C_n)) = 2k + 1$.

Since $f_i + f_{i+1} \leq f_i^{'} + f_{i+1}^{'}$, for all i, the distance gap sequence to order the vertices of the original cycle C_n is defined as follows:

d_i = $2k + 1$ if i is even
= $k + 1$ if $i \equiv 1(mod\,4)$
= $k + 2$ if $i \equiv 3(mod\,4)$

The color gap sequence is defined as follows:

f_i = 1 if i is even
= $k + 1$ if i is odd

Let $u_1^{'}$ be the vertex on the inscribed cycle such that $d(u_k, u_1^{'}) = k + 1$ and $f = k + 1$, where u_k is the last labeled vertex in the cycle.

The distance gap sequence to order the vertices of the inscribed cycle C_n is defined as follows:

d_i = $2k$ if i is even
= k if $i \equiv 1(mod\,4)$
= $k + 1$ if $i \equiv 3(mod\,4)$

The color gap sequence is defined as follows:

$f_i^{'}$ = 2 if i is even
= $k + 2$ if $i \equiv 1(mod\,4)$
= $k + 1$ if $i \equiv 3(mod\,4)$

Then, $r_n(M(C_n)) = 2(k + 2)(2k - 1) + n + 3$.

Case 2: $n = 4k + 2$. In this case $diam(M(C_n)) = 2k + 2$.

Since $f_i + f_{i+1} \leq f_i^{'} + f_{i+1}^{'}$, for all i, the distance gap sequence to order the vertices of the original cycle C_n is defined as follows:

d_i = $2k + 2$ if i is even
= $k + 3$ if i is odd

and the color gap sequence is given by

f_i = 1 if i is even
= $k + 1$ if i is odd

Let $u_1^{'}$ be the vertex on the inscribed cycle such that $d(u_k, u_1^{'}) = k + 1$ and $f = k + 2$, where u_k is the last labeled vertex in the cycle.

The distance gap sequence to order the vertices of the inscribed cycle C_n is defined as follows:

d_i = $2k+1$ if i is even

= $k+1$ if i is odd

and the color gap sequence is given by

$f_i^{'}$ = 2 if i is even

= $k+2$ if i is odd

Thus $r_n(M(C_n)) = 4k(k+2)+k+n+3$.

Case 3: $n = 4k+1$. In this case, $diam(M(C_n)) = 2k+1$.

Since $f_i + f_{i+1} \leq f_i^{'} + f_{i+1}^{'}$ for all i, the distance gap sequence to order the vertices of the original cycle C_n is defined as follows:

d_{4i} = $d_{4i+2} = 2k+1-i$

d_{4i+1} = $d_{4i+3} = k+2+i$

and the color gap sequence is given by

$f_i = (2k+1) - d_i + 1$

Let $u_1^{'}$ be the vertex on the inscribed cycle such that $d(u_k, u_1^{'}) = k+1$ and $f = k+1$, where u_k is the last labeled vertex in the cycle.

The distance gap sequence to order the vertices of the inscribed cycle C_n is defined as follows:

d_{4i} = $d_{4i+2} = 2k-i$

d_{4i+1} = $d_{4i+3} = k+1+i$

and the color gap sequence is given by

$f_i^{'} = 2k - d_i + 2$

Thus, $r_n(M(C_n)) = 4k(k+1)+k+n$.

Case 4: $n = 4k+3$. In this case, $diam(M(C_n)) = 2k+2$.

Since $f_i + f_{i+1} \leq f_i^{'} + f_{i+1}^{'}$, for all i, the distance gap sequence to order the vertices of the original cycle C_n is defined as follows:

$d_{4i} = d_{4i+2} = 2k+2-i$

$d_{4i+1} = d_{4i+3} = k+2+i$

and the color gap sequence is given by

f_i= $2k - d_i + 3$.

Let $u_1^{'}$ be the vertex on the inscribed cycle such that $d(u_k, u_1^{'}) = k+2$ and $f = k+1$, where u_k is the last labeled vertex in the cycle.

The distance gap sequence to order the vertices of the inscribed cycle C_n is defined as follows:

$d_{4i} = d_{4i+2} = 2k+1-i$

$d_{4i+1} = k+1+i$

$d_{4i+3} = k+2+i$

and the color gap sequence is given by

$f_i^{'}$ = $2k - d_i + 3$.

Thus, $r_n(M(C_n)) = 2(k+2)(2k+1)+n$.
Thus the radio number is completely determined for the graph $M(C_n)$. ∎

Illustration 2.10. *Consider the graph $M(C_8)$. The radio labeling is shown in Figure 4.*

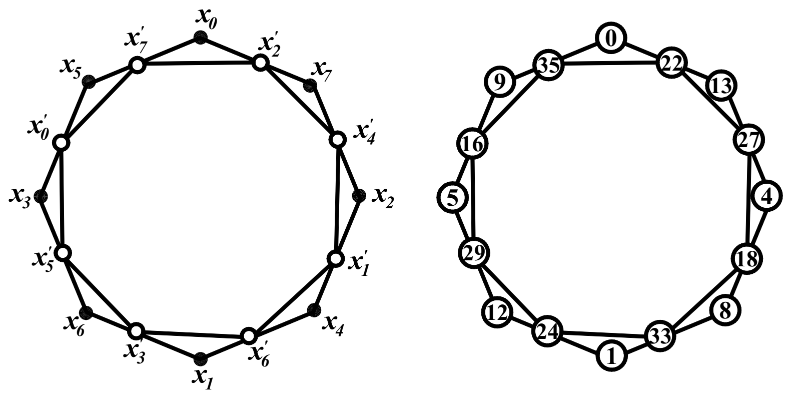

Figure 4: Ordinary and radio labeling of $M(C_8)$.

Application

Above result is useful for the expansion of an existing radio transmitters network. In the expanded network two newly installed nearby transmitters are connected and interference is also avoided between them. Thus the radio labeling described in Theorem 2.10 is rigourously applicable to accomplish the task of channel assignment for the feasible network.

The comparison between radio numbers of C_n, $spl(C_n)$ and $M(C_n)$ is tabulated in the following Table 1.

n	Radio number of Cn	Radio number of spl(C_n)	Radio number of M(C_n)
0(mod 4)	(k+2)(2k-1)+1	2[(k+2)(2k-1)+1]+k+1	2(k+2)(2k-1)+n+3
2(mod 4)	2k(k+2)+1	2[2k(k+2)+1]+k+1	4k(k+2)+k+n+3
1(m0d 4)	2k(k+1)	2[2k(k+1)]+k	4k(k+1)+k+n
3(mod 4)	(k+2)(2k+1)	2[(k+2)(2k+1)]+k+1	2(k+2)(2k+1)+n

Table 1: Comparison of radio numbers of C_n, $spl(C_n)$ and $M(C_n)$.

3 Concluding Remarks

We have studied four new results corresponding to radio labeling. The upper bounds for the radio numbers of cycle with chords and $n/2$-petal graph are obtained. The radio numbers are completely determined for the graphs $spl(C_n)$ and $M(C_n)$. Last two results is an effort to relate graph operations and radio labeling. We have completely determined the radio number for larger graphs resulted due to graph operations on standard graphs. This work is having potential to play a vital role for the solution of real life channel assignment problem.

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