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A new property of quotients modulo pseudo-reflection groups

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Abstract

In this paper we prove that the induced homomorphism between local fundamental groups of germs of normal complex analytic spaces $\pi_1^0(W) \to \pi_1^0(W/G)$ is a surjection, where G is a finite group of automorphisms generated by pseudo-reflections.

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1 Introduction

Many interesting properties of rings of invariants of a normal local domain R with an action of a finite pseudo-reflection group in terms of those of R have been proved by several authors.

We state some of these properties for the sake of completeness. For simplicity, we state the results when the underlying field is $\mathbf{C}$. Unless otherwise stated, in these statements G always denotes a finite pseudo-reflection group.

1. In [11] and [3] it is proved that if a finite subgroup G of $GL(n, \mathbf{C})$ acts on the polynomial ring $R := \mathbf{C}[X_1, X_2, \ldots, X_n]$ then the ring of invariants R^G is again a polynomial ring if and only if G is generated by pseudo-reflections.

   In [8] a proof of this using homological algebra is given.

2. In [6] the above result is generalized using the proof in [3] to actions on affine graded domains R. In particular, it was shown that R is a free R^G-module. This proof also works when G acts on a local domain R.

   This implies by a standard argument that if R is Cohen-Macaulay then so is R^G.

3. In [9] it is proved that if G acts on a local UFD R then R^G is again a UFD.

4. In [10] it is proved that if G acts on a local domain (R, M) then for all $n \geq 0$, $dim\ M_S^n/M_S^{n+1} \leq dim\ M_R^n/M_R^{n+1}$, in case G is abelian or R is a hypersurface. Here $S := R^G$ and M_S is the maximal ideal of S.

5. In [4], [12] and [5] it is proved that if a pseudo-reflection group G acts on a local Gorenstein domain R then R^G is again Gorenstein.

6. In [2], with the notation in (4), it is proved that $dim\ S/M_S^n \leq dim\ R/M^n$ for all $n \geq 0$. This paper also contains several other interesting results about the relation between R and R^G.

The aim of this paper is to prove a special property of the action of such a G.

Theorem 1.1. *Let (W, q) be a germ of a normal complex space, G a finite pseudo-reflection group acting on (W, q) by complex analytic automorphisms and let $(V, p) := (W, q)/G$. Then the induced homomorphism between local fundamental groups, $\pi_1^q(W) \to \pi_1^p(V)$ is a surjection.*

As mentioned above, the analytic local ring $\mathcal{O}_{W,q}$ is a free module over the analytic local ring $\mathcal{O}_{V,p}$. We in fact prove the following stronger result which implies Theorem 1.1.

Theorem 1.2. *Let $f : (W, q) \to (V, p)$ be a finite complex analytic map of normal analytic germs such that the analytic local ring $\mathcal{O}_{W,q}$ is a free module over the analytic local ring $\mathcal{O}_{V,p}$. Then the induced homomorphism between local fundamental groups, $\pi_1^q(W) \to \pi_1^p(V)$ is a surjection.*

For the definitions of terms involved in the above results refer Section 2. For the converse, we have the following result which is analogous to Chevalley's result in [3].

Theorem 1.3. *Let $f : (W, q) \to (V, p)$ be a finite complex analytic map of normal analytic germs, where $(V, p) = (W, q)/G$ for a finite group of complex analytic automorphisms of (W, q). Assume that the ring $\mathcal{O}_{W,q}$ is a UFD. If the induced homomorphism $\pi_1^q(W) \to \pi_1^p(V)$ is a surjection then G is generated by pseudo-reflections.*

We give examples of finite morphisms between normal affine varieties where the analogue of the Theorem 1.2 is false.

2 Preliminaries

Let (Z, p) be a germ of a normal complex space. Then there exists a fundamental system of neighborhoods N_i of p in Z such that the group $\pi_1(N_i - Sing\ Z)$ is independent of i. This group is called the *local fundamental group of* Z, and denoted by $\pi_1^p(Z)$.

Suppose that $f : (T, q) \to (Z, p)$ is a finite analytic map of normal analytic germs. Then the codimension of $f^{-1}(Sing\ Z)$ in T is ≥ 2. It is well-known that if S is a closed complex analytic subspace of codimension ≥ 2 of a complex manifold M then the natural homomorphism $\pi_1(M - S) \to \pi_1(M)$ is an isomorphism. Since $T - Sing\ T$ is a complex manifold we see that the homomorphism $\pi_1(T - Sing\ T \cup f^{-1}(Sing\ Z)) \to \pi_1(T - Sing\ T)$ is an isomorphism. By considering the restricted map $T - Sing\ T \cup f^{-1}(Sing\ Z) \to Z - Sing\ Z$, from these observations we deduce that there is a natural homomorphism $\pi_1^q(T) \to \pi_1^p(Z)$.

An automorphism σ of finite order of an analytic domain $R := \mathbf{C}\{X_1, X_2, ..., X_n\}/P$ is called a *pseudo-reflection* if for suitable uniformizing coordinates for R we have

$\sigma(x_1, x_2, .., x_n) = (x_1, x_2, ..., x_{n-1}, \omega x_n)$, where x_i are the residues of X_i mod P and ω is an m^{th} root of unity for some $m \geq 1$.

3 Proofs of Theorems 1.1, 1.2 and 1.3

3.1 Proof of Theorem 1.2

It is a standard result that the image of the homomorphism $\pi_1^q(W) \to \pi_1^p(V)$ is a subgroup of finite index, say H, in $\pi_1^p(V)$ [7]. Suppose that H is a proper subgroup of $\pi_1^p(V)$. Then we can find a germ of a normal complex space $(\widetilde{V}, \widetilde{p})$ with a finite analytic map $(\widetilde{V}, \widetilde{p}) \to (V, p)$ which is unramified outside $Sing\ V$ and such that $\pi_1^{\widetilde{p}}(\widetilde{V}) = H$. By covering space theory, f factors as $(W, q) \to (\widetilde{V}, \widetilde{p}) \to (V, p)$. Using the trace map (or Reynold's operator), we know that $\mathcal{O}_{\widetilde{V},\widetilde{p}}$ is a direct summand of $\mathcal{O}_{W,q}$ as a $\mathcal{O}_{\widetilde{V},\widetilde{p}}$-module. By assumption, $\mathcal{O}_{W,q}$ is a free $\mathcal{O}_{V,p}$-module. Hence $\mathcal{O}_{\widetilde{V},\widetilde{p}}$ is a free $\mathcal{O}_{V,p}$-module. In this situation, purity of branch locus is valid and hence the branch locus of the map $(\widetilde{V}, \widetilde{p}) \to (V, p)$ has pure codimension one in (V, p), since the point p is certainly ramified as $\widetilde{p}$ is the only point of $\widetilde{V}$ lying over p [1]. This contradiction shows that $H = \pi_1^p(V)$.

3.2 Proof of Theorem 1.1

If G is generated by pseudo-reflections then $\mathcal{O}_{W,q}$ is a free module over $\mathcal{O}_{V,p}$ by Chevalley's argument, as used in Proposition 16 [6]. Hence Theorem 1.1 follows immediately from Theorem 1.2.

3.3 Proof of Theorem 1.3

Let H be the (normal) subgroup of G generated by pseudo-reflections. Let $(\widetilde{V}, \widetilde{p}) := (W, q)/H$. In Proposition 2 [9], it is proved that the map $(\widetilde{V}, \widetilde{p}) \to (V, p)$ is divisorially unramified. Then the homomorphism $\pi_1^{\widetilde{p}}(\widetilde{V}) \to \pi_1^p(V)$ is injective. By assumption, the map $\pi_1^q(W) \to \pi_1^p(V)$ is a surjection. It follows that $H = G$.

This completes the proof of Theorem 1.3.

Remark 3.1. *Let $f : W \to V$ be a finite unramified covering of smooth affine varieties which is galois with galois group G of order > 1. The coordinate ring $\Gamma(W, \mathcal{O})$ is a projective module over the coordinate ring $\Gamma(V, \mathcal{O})$. Hence there exists a non-empty Zariski-open affine subset $V_0 \subset V$ such that the coordinate ring of $f^{-1}(V_0)$ is a free module over the coordinate ring of V_0. Since the fundamental group of $f^{-1}(V_0)$ is a subgroup of index $|G|$ of the fundamental group of V_0, the analogue of Theorem 1.2 is false for the map $f^{-1}(V_0) \to V_0$.*

Remark 3.2. *If $\mathcal{O}_{W,q}$ is not a UFD then there are easy examples to show that in Theorem 1.3 the map $\pi_1^q(W) \to \pi_1^p(V)$ is surjective but G is not generated by pseudo-reflections.*
Let $\sigma,\ \tau$ act on the germ $(\mathbf{C}^2, O)$ as follows. $\sigma(X, Y) = (\omega\ X, Y),\ \tau(X, Y) = (X, \omega\ Y)$, where ω is a primitive cube root of unity. Then $\sigma \circ \tau(X, Y) = (\omega X, \omega Y)$. Clearly $\sigma \circ \tau$ is not a pseudo-reflection. The quotient $(W, q) := (\mathbf{C}^2, O)/(\sigma \circ \tau)$ has analytic coordinate ring $\mathbf{C}\{X^3, X^2Y, XY^2, Y^3\}$. The action of σ on this ring is $\sigma(X^3, X^2Y, XY^2, Y^3) = (X^3, \omega^2 XY, \omega^2 XY^2, Y^3)$. The ring of invariants

of this latter action is $\mathbf{C}\{X^3, Y^3\}$*. It follows that the induced homomorphism of local fundamental groups* $\pi_1^q(W) \to \pi_1^0(\mathbf{C}^2)$ *is a surjection. But the action of* σ *on* $\mathbf{C}\{X^3, X^2Y, XY^2, Y^3\}$ *is not a pseudo-reflection. It is easy to see that* $\mathbf{C}\{X^3, X^2Y, XY^2, Y^3\}$ *is not a UFD.*

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