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The All-Ones Problem for Binomial Trees, Butterfly and Benes Networks

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Abstract

The *all-ones problem* is an NP-complete problem introduced by Sutner [11], with wide applications in linear cellular automata. In this paper, we solve the *all-ones problem* for some of the widely studied architectures like binomial trees, butterfly, and benes networks.

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1 Introduction

The *All-Ones Problem* was introduced by Sutner [11] where he also discussed wide applications of the *all-ones problem* in linear cellular automata. The problem has been studied under various names by various authors in different papers. The *all-ones problem* can be stated as follows: Suppose each square of an $n \times n$ chessboard is equipped with an indicator light and a button. If the button of a square is pressed, the light of that square will change from off to on, and vice versa; the same happens to the lights of all the edge-adjacent squares. Initially all lights are off. Now, we consider the following questions: is it possible to press a sequence of buttons in such a way that in the end all lights are on? This is referred to as the all-ones problem. If there is such a solution, how can we find it? And finally, how can we find a solution that presses as few buttons as possible? This is referred to as the *minimum all-ones problem.*

The above questions can be posed for any arbitrary graph. In this chapter, we consider connected, simple, undirected graphs only. One can deal with disconnected graphs, component by component. For all notations and terminology used in this chapter, we refer to [2]. An equivalent version of the all-ones problem was proposed by Peled [8], where it was called the Lamp Lighting problem. The rule of the *all-ones problem* is called σ^+ rule on graphs, which means that a button lights not only its neighbours but also its own light. If a button lights only its neighbours but not its own light, this rule on graphs is called σ rule.

There have been many publications on the All-Ones Problem [1, 6, 12]. Using linear algebra, Sutner [13] proved that it is always possible to light every lamp in any graph by σ^+ rule. A graph-theoretic proof was given by Erikisson et al. [7]. In [9] Sutner proved that the *Minimum all-ones problem* is

NP-complete in general. Li et al. [3] have proved that the problem is *NP-complete* even when restricted to bipartite graphs. Li et al. [5] have given a linear time algorithm for finding optimal solutions for trees.

2 Notations

Let $G = (V, E)$ be a simple, connected graph. Let $|V| = n$ and $|E| = m$. The open neighbourhood of a vertex v is $\{u \in V : u$ is adjacent to $v\}$ and is denoted by $N(v)$. The closed neighbourhood of a vertex v is $\{v\} \cup N(v)$ and is denoted by $N[v]$. The degree of a vertex v in G is the number of neighbours of v in G. The *all-ones problem* is equivalent to the following dominating set problem [4] from a graph-theoretic point of view.

A set S of vertices is *independent* if no two vertices in S are adjacent in G. A *clique* is a graph where all the vertices are mutually adjacent. A maximal clique (independent set) of G is one in which we cannot add a vertex and still have a clique(independent set) in G. For any $X \subseteq V$ if $G[V \backslash X]$ has more than one connected component, we say that X is a *separator* of G. A set S of vertices in a graph G is a *dominating set* if every vertex, not in S, is adjacent to at least one vertex in S.

Definition 2.1. *Given a graph $G = (V, E)$, where V and E denote the vertex-set and the edge-set of G respectively, the all-ones problem is to find a subset $S \subseteq V$ with the property that every vertex in S has an even number of neighbours in S, while every vertex in $V \backslash S$ has an odd number of neighbours in S. Since this implies that every vertex is dominated by an odd number of vertices in S (including the vertex itself if it belongs to S), the solution set S is called an odd dominating set or a OD-set of G. If we are able to find such a set with minimum cardinality, then such an S is said to be a solution to the minimum all-ones problem and is called a minimum all-ones dominating set or typically a MOD-set of G.*

3 Binomial Trees

In this section we solve the minimum all-ones problem for the binomial trees.

A binomial tree of height 0 is a single vertex. For all $h > 0$, a binomial tree of height h is a tree formed by joining the roots of two binomial trees of height $h - 1$ with a new edge and designating one of these roots to be the root of the new tree. A binomial tree of height n has 2^n vertices.

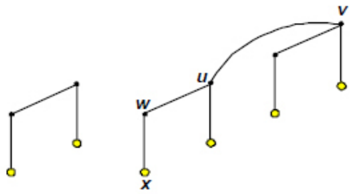

Figure 1: MOD-set of B_2 and B_3.

Lemma 3.1. *Let B_n $(n > 2)$ be a binomial tree which contains two binomial trees of height $n - 1$ with roots u and v respectively. If S is a solution to the minimum all-ones problem $(MOD$-set$)$ of B_n, then both u and v cannot lie in S.*

Proof. Suppose u lies in S. By definition of binomial tree, u lies in some B_2. Consider the vertex of degree 2 in that B_2, adjacent to u and let us call it w. Let the pendant vertex adjacent to w be x (refer

Figure 1). Since x is of degree 1, either w or x should lie in S. Since u lies in S, in either case, w will be evenly dominated by S which is a contradiction. By a similar argument we can prove that v also cannot lie in S. ∎

Lemma 3.2. *Let B_n be a binomial tree which contains two binomial trees of height $n-1$ say B'_{n-1} and B''_{n-1} with MOD-sets S_1 and S_2 respectively. Then the MOD-set for B_n is $S = S_1 \cup S_2$.*

Proof. Let W be any other odd-dominating set of B_n. Suppose $W_1 = W \cap V(B'_{n-1})$ and $W_2 = W \cap V(B''_{n-1})$. Then by Lemma 3.1, the roots of B'_{n-1} and B''_{n-1} cannot lie in W. Hence W_1 and W_2 will necessarily be odd-dominating sets of B'_{n-1} and B''_{n-1} respectively. Since S_1 and S_2 are MOD-sets of B'_{n-1} and B''_{n-1} respectively, $|S_1| \leq |W_1|$ and $|S_2| \leq |W_2|$. Hence $|S_1 \cup S_2| \leq |W_1 \cup W_2|$ proving that $|S| \leq |W|$. ∎

Theorem 3.3. *The solution to the minimum all-ones problem for a binomial tree B_n is the set of all its pendant vertices.*

Proof. The proof is by induction on n. Consider the case $n = 2$. B_2 is isomorphic to P_4. Hence the MOD-set of B_2 are its two terminal vertices. Using Lemma 3.2 repeatedly by induction, one can easily find the required MOD-set of B_n for all n. ∎

4 Butterfly networks

In this section, we solve the minimum all-ones problem for butterfly architectures.

The vertices of the n - dimensional butterfly network (BF_n) are the pairs (r, x), where r is a non-negative integer $0 \leq r \leq n$ called the rank, and $x = x_1x_2...x_n$ is a binary string of length n. A vertex (r, x), $0 \leq r \leq n-1$, is joined to the vertices $(r+1, x)$ and $(r+1, x_1x_2...x_r\overline{x_{r+1}}x_{r+2}...x_n)$. The edges of the form $((r, x), (r+1, x))$ are called straight edges, edges of the form $((r, x), (r+1, x_1x_2...x_r\overline{x_{r+1}}x_{r+2}...x_n))$ are called cross edges and the edges connecting vertices on ranks i and $i+1$ are called level i edges. The Butterfly network (BF_n) has $(n+1)2^n$ vertices.

Consider BF_{k+1}. It contains two k-dimensional butterflies say BF'_k and BF''_k. Let $B = \{v \in V(BF_{k+1})/v \notin V(BF'_k)$ and $v \notin V(BF''_k)\}$. The vertices in B can be partitioned into ordered pairs (x, y) lying on the same 4-cycle such that $d(x) = d(y) = 2$ and $d(x, y) = 2$. We call such an ordered pair of vertices as 2-distant, 2-degree vertices or a "$2DD$-*pair*" in short. Also if (x, y) is a $2DD$-pair, then $N(x) = N(y)$. See Figure 2.

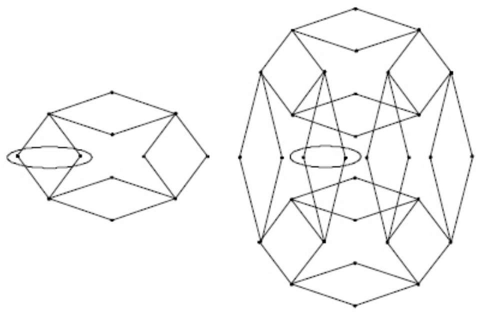

Figure 2: A 2DD-pair of BF_2 and BF_3.

Lemma 4.1. *If (x, y) is a $2DD$-pair of BF_{k+1}, then either x and $y \in S$ or x and $y \notin S$.*

Proof. Let $N(x) = N(y) = \{a, b\}$ such that $a \in V(BF'_k)$ and $b \in V(BF''_k)$. Suppose only one vertex of the $2DD$-pair (x, y) lies in S. Without loss of generality say $x \in S$ and $y \notin S$. Since $x \in S$, either both a and $b \in S$ or both a and $b \notin S$ as x has to be odd-dominated by S.

Case 1: $a \in S$ and $b \in S$.

Then y is evenly-dominated by S which contradicts our hypothesis that S is an odd-dominating set.

Case 2: a and $b \notin S$

In this case, y is not dominated by S at all, contradicting our hypothesis that S is a solution set. ■

Lemma 4.2. *Let (x, y) be a $2DD$-pair of BF_{k+1} and $N(x) = N(y) = \{a, b\}$ such that $a \in V(BF'_k)$ and $b \in V(BF''_k)$. If x and $y \notin S$, then either $a \in S$ or $b \in S$.*

Proof. (x, y) is a $2DD$-pair of BF_{k+1}. Suppose if both a and $b \in S$, then x and y will be dominated by an even number of vertices in S. If both a and $b \notin S$, then, both x and y are not dominated by S at all. ■

Theorem 4.3. *$V(BF_r)$ is the only solution to the all-ones problem of the r-dimensional butterfly BF_r.*

Proof. The proof is by induction on the dimension r of BF_r. For $r = 1$, the proof is trivial. Assume that the result is true for $r = k$. Consider BF_{k+1}. It contains two k-dimensional butterflies say BF'_k and BF''_k. Let S, S',and S'', be the solution sets to the minimum all–ones problem for BF_{k+1}, BF'_k and BF''_k respectively. By induction hypothesis, $S' = V(BF'_k)$ and $S'' = V(BF''_k)$. Now let (x_1, y_1), (x_2, y_2) ...(x_n, y_n) where $n = 2k$, be the $2DD$-pairs of BF_{k+1}. By Lemma 4.1, if (x_i, y_i) is a $2DD$-pair of BF_{k+1} then x_i and $y_i \in S$ or x_i and $y_i \notin S$. Suppose there exists a $2DD$-pair (x_i, y_i) such that x_i and $y_i \notin S$. Then by Lemma 4.2, either $a_i \in S$ or $b_i \in S$. Without loss of generality let $a_i \in S$ and $b_i \notin S$. Now suppose $b_i \in V(BF''_k)$. Then, x_i and y_i are not in S implies that b_i is being dominated by vertices of BF''_k only. This in turn implies that $S'' \subseteq V(BF''_k)\backslash b_i$ which is a contradiction to our induction hypothesis that $S'' = V(BF''_k)$. ■

5 Benes networks

The r -dimensional Benes network consists of back-to-back butterflies, denoted by $B(r)$. The $B(r)$ has $2r + 1$ levels, each with $2r$ vertices. The first and last levels in the $B(r)$ form two butterflies BF_r respectively, while the middle level is shared by these butterfly networks. The r-dimensional Benes network has $(n + 1)2^{n+1}$ vertices and $n2^{n+2}$ edges.

The level zero to level r vertices in the network form an r-dimensional butterfly. The middle level of the Benes network is shared by these butterflies An r-dimensional Benes is denoted by $B(r)$. Figure 3 shows a $B(2)$ network. A Benes network is bipartite. No two 4-cycles of $B(r)$ have a common edge. The edge set of $B(r)$ is disjoint union of 4-cycles.

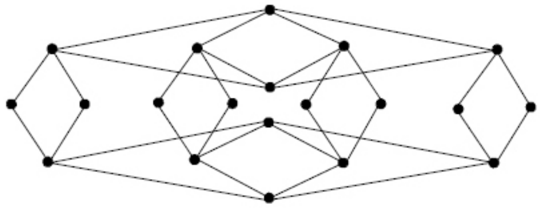

Figure 3:Diamond representation of $B(2)$.

Theorem 5.1. *$V(B(r))$ is the only solution to the all-ones problem of the r-dimensional Benes $B(r)$.*

Proof. The benes network consists of back-to-back butterflies. Hence similar to the butterfly networks, we identify all the $2DD$-pairs of the benes networks. Tracing the same steps as in Theorem 4.3, we obtain the desired result. ∎

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