International Journal of Mathematics and Soft Computing
Vol.2, No.1 (2012), 119 – 129.

ISSN 2249 – 3328

Characterization of complementary connected domination number of a graph

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Abstract

A set $S \subseteq V$ is a complementary connected dominating set if S is a dominating set of G and the induced subgraph $< V - S >$ is connected. The complementary connected domination number $\gamma_{cc}(G)$ is the minimum cardinality taken over all complementary connected dominating sets in G. The chromatic number is the minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour and is denoted by χ. In this paper we characterize all cubic graphs on 8, 10, 12 vertices for which $\gamma_{cc}=\chi=3$.

Keywords: Complementary connected domination number, Chromatic number.

AMS Subject Classification (2010): 05C.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. The degree of any vertex u in G is the number of edges incident with u and is denoted by $d(u)$. The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively. P_n denotes the path on n vertices. The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph. A colouring of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. An n-colouring of a graph G uses n colours. The chromatic number χ is defined to be the minimum n for which G has an n-colouring. If $\chi(G) = k$ but $\chi(H) < k$ for every proper subgraph H of G, then G is k-critical. A subset S of V is called a dominating set in G if every vertex in V-S is adjacent to atleast one vertex in S. The minimum cardinality taken over all dominating sets in G is called the domination number of G and is denoted by γ.

A Set $S \subseteq V$ is a complementary connected dominating set if S is a dominating set of G and the induced subgraph $< V- S >$ is connected. The complementary connected domination number $\gamma_{cc}(G)$ is the minimum cardinality taken over all complementary connected dominating sets in G. The chromatic number is the minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour and is denoted by χ.

In [16], Volkman studied graphs for which $\gamma = \beta_1$. He also investigated graphs for which $\gamma = \alpha_0$ [15]. In [12],J. Paulraj Joseph and S. Arumugam investigated graphs for which $\gamma = \gamma_c$. In [13],

J. Paulraj Joseph analyzed graphs for which the chromatic numbers are equal to domination parameters. In [5], G. Mahadevan A. Selvam Avadyappan and A. Mydeen bibi characterized all cubic graphs on 8, 10, 12 vertices for which $\gamma_2 = \chi = 3$. In [6, 7], G. Mahadevan and J. Paulraj Joseph characterized all cubic graphs on 8, 10, 12 vertices for which $\gamma = \chi = 3$. In this paper we characterize all cubic graphs on 8, 10, 12 vertices for which $\gamma_{cc} = \chi = 3$.

Theorem 1.1.[10] For any graph G, $\gamma_{cc}(G) \leq n - \delta$

Theorem 1.2.[10] For any graph G, $\gamma_{cc}(G) = n-1$ if and only if G is a star. If G is not a star, then $\gamma_{cc}(G) \leq n-2$, $(n \geq 3)$

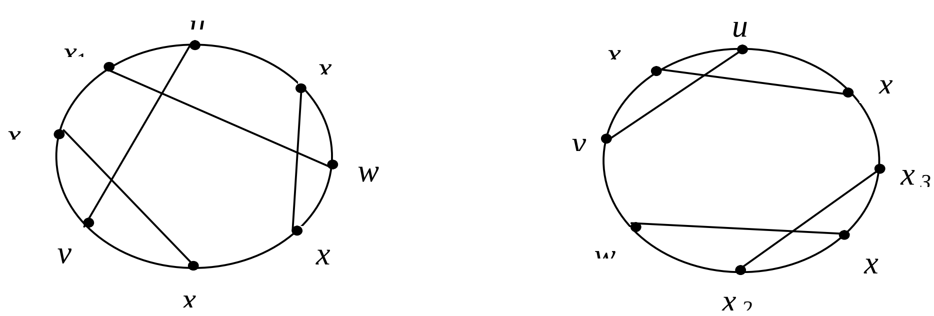
Theorem 1.3.[1] For any Graph G, $\chi(G) \leq \Delta(G) + 1$

Theorem 1.4.[2] If G is a graph of order p, with maximum degree Δ, then $\gamma \geq \lceil p / (\Delta + 1) \rceil$.

Let $G = (V, E)$ be a connected cubic graph of order p with $\gamma_{cc} = \chi$. By Theorem 1.3, $\chi \leq 3$. Clearly, $\chi \neq 1$. We consider cubic graphs for which $\gamma_{cc} = \chi = 3$. By Theorem 1.4, $\gamma_{cc} \geq \lceil p/4 \rceil$. Since $\gamma_{cc} = 3$, $6 < p \leq 15$ and $p \neq 14$. Since G is cubic we have p is even and hence the possible values of p are 8, 10 and 12.

2 Cubic graphs of order 8

Theorem 2.1. Let G be a connected cubic graph on 8 vertices. Then $\gamma_{cc} = \chi = 3$ if and only if G is isomorphic to any one of the graphs given in Figure 2.1.



G_1 G_2

Figure 2.1

Proof. Let $S = \{u, v, w\}$ be a minimum complementary connected dominating set of G and $V\text{-}S = \{x_1, x_2, x_3, x_4, x_5\}$. Clearly $< S > \neq K_3$. Hence we consider the following three cases.

Case 1. $< S > = \overline{K}_3$.

Without loss of generality, let u be adjacent to x_1, x_2 and x_3. Then v is adjacent to atleast one of the vertices of N (u) = $\{x_1, x_2, x_3\}$.

Subcase (a). Let v be adjacent to only one vertex of $N(u)$.

Without loss of generality, let v be adjacent to x_1. Then v is adjacent to x_4 and x_5. Now w is adjacent to x_1 or not adjacent to x_1. If w is adjacent to x_1, then w is adjacent to x_2 and x_3(or equivalently x_4 and x_5), or x_2 (or equivalently x_3) and x_4 (or equivalently x_5). If w is adjacent to x_2 and x_3, then x_2 is not adjacent to x_3. Hence x_2 must be adjacent to x_4(or equivalently x_5) and then x_5 is adjacent to x_3 and x_4, which is a contradiction. Hence no such graph exists. If w is adjacent to x_2 and x_4, then x_2 is not adjacent to x_4. Hence x_2 is adjacent to x_3 or x_5. Then in both the cases no graph exists. If w is not adjacent to x_1. Without loss of generality, let w be adjacent to x_2, x_3 and x_4. Then x_2 is adjacent to x_3 or x_1 or x_4 or x_5. If x_2 is adjacent to x_3 or x_1 or x_4, then no graph exists. If x_2 is adjacent to x_5, then x_4 is adjacent to x_5 or x_3 or x_1. Then also no graph exists.

Subcase (b). Let v be adjacent to two vertices of $N(u)$ say x_1 and x_2.

Without loss of generality, let v be adjacent to x_4. Now w is adjacent to x_1 (or equivalently x_2) or not adjacent to x_1 (or equivalently x_2). If w is adjacent to x_1, then x_2 is adjacent to x_3 (or equivalently x_4) or w or x_5. Then in all the cases, $\{u,v,w\}$ is not a complementary connected dominating set, which is a contradiction. If w is not adjacent to x_1, then without loss of generality, let w be adjacent to x_3, x_4 and x_5. Now x_5 is adjacent to x_3 and x_4 (or) x_1 and x_2 (or) x_1 and x_4 (or equivalently x_2 and x_3). In all the cases, $\{u,v,w\}$ is not a complementary connected dominating set, which is a contradiction.

Subcase (c)**.** Let v be adjacent to all the vertices of $N(u)$.

Now x_2 is adjacent to x_1(or equivalently x_3), or w (or equivalently x_4 or x_5). Since G is cubic, x_2 cannot be adjacent to x_1(or equivalently x_3). Hence x_2 must be adjacent to w. If x_2 is adjacent to w then w is adjacent to x_1 and x_3 (or) x_4 and x_5 (or) x_1 and x_5. Since G is cubic, w cannot be adjacent to x_1 and x_3. Also w cannot be adjacent to x_1 and x_5. Hence w must be adjacent to x_4 and x_5. Then x_1 must be adjacent to x_4 and x_5 is adjacent to x_3 and x_4. Consequently, $\{u,v,w\}$ is not a complementary connected dominating set, which is a contradiction.

Case 2. $<S> = K_2 \cup K_1$.

Let uv be an edge. Without loss of generality, let u be adjacent to x_1 and x_2. Now w is adjacent to x_1, x_2 and anyone of $\{x_3,x_4,x_5\}$(or) w is adjacent to x_1(or equivalently x_2) and any two of $\{x_3,x_4,x_5\}$. If w is adjacent to x_1, x_2 and x_3, then x_4 is adjacent to x_1(or equivalently x_2) or not adjacent to x_1(or equivalently x_2). If x_4 is not adjacent to x_1 (or equivalently x_2), then x_4 is adjacent to x_3, v and x_5. Also x_5 is adjacent to x_1 and x_2 is adjacent to x_3. Hence $G \cong G_1$. If x_4 is adjacent to x_1, then x_5 is adjacent to x_4 or not adjacent to x_4. If x_5 is adjacent to x_4, then x_4 is adjacent to v and x_2 is adjacent to x_3.Hence $G \cong G_1$. If x_4 is adjacent to x_1, then x_5 is adjacent to x_4, x_2 and x_3. Hence $G \cong G_1$. If x_4 is adjacent to x_1, then x_5 is not adjacent to x_4. Also x_5 is adjacent to x_2, x_3 and v, and x_3 is adjacent to x_4 and x_4 is adjacent to v. Hence $G \cong G_1$. If w is adjacent to x_1, x_2 and x_3, then x_5 is adjacent to x_1 or not adjacent to x_1. If x_5 is adjacent to x_1, x_2 and x_4, then x_3 is adjacent to x_2 and v. Also x_4 is adjacent to v. Hence $G \cong G_1$. If x_5 is not adjacent to x_1, then x_5 is adjacent to any three of $\{x_3, x_4, v, x_2\}$. Let x_5 be adjacent to x_3, x_4 and v. Also x_4 is adjacent to x_1 and x_2 is adjacent to $\{x_3, v\}$. Hence $G \cong G_1$.

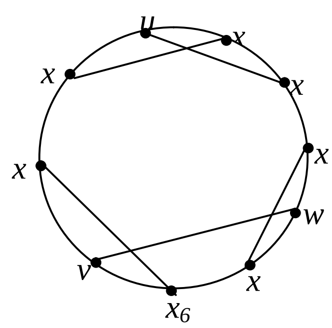
Case 3. $<S> = P_3$.

Let v be adjacent to u and w. Without loss of generality, let v be adjacent to x_1 and u be adjacent to $\{x_1, x_2\}$. Now x_2 is adjacent to u, w and anyone of $\{x_3, x_4, x_5\}$or w and any two of $\{x_3,x_4,x_5\}$. If x_2 is adjacent to u, w and x_3 then x_4 is adjacent to w (or equivalently u) or not adjacent to w (or equivalently u). If x_4 is adjacent to w (or equivalently u), then x_5 is adjacent to x_4 or not adjacent to x_4. If x_5 is adjacent to x_1, x_3 and x_4, then x_4 is adjacent to x_3 and x_3 is adjacent to x_2. Hence $G \cong G_1$. If x_5 is not adjacent to x_4, then x_5 is adjacent to x_1, x_2 and x_3, so that $\{u,v,w\}$ is not a complementary connected dominating set, which is a contradiction. If x_4 is not adjacent to w (or equivalently u), then x_4 is adjacent to x_5, x_3 and x_1. Also x_3 is adjacent to $\{x_2, w\}$ and x_5 is adjacent to $\{x_1, w\}$. Hence $G \cong G_1$. If x_2 is adjacent to w, x_3 and x_4, then x_5 is adjacent to w (or) not adjacent to w. If x_5 is adjacent to $\{w, x_1, x_4\}$, then x_3 is adjacent to $\{x_4, u\}$. Hence $G \cong G_1$. If x_5 is not adjacent to w, then x_5 is adjacent to any three of $\{x_1, u, x_3, x_4\}$. Let x_5 be adjacent to x_1, u, x_3. Also x_4 is adjacent to x_3 and x_5. Hence $G \cong G_3$. ■

3 Cubic graphs of order 10

Throughout this section, G is a connected cubic graph on 10 vertices with $V(G)=\{u, v, w, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Let $S = \{u, v, w\}$ be a minimum complementary connected dominating set. Let $S_1 = N(u) = \{x_1, x_2, x_3\}$. Clearly $<S> \neq K_3$ or P_3. Hence $<S> = K_2 \cup K_1$ or $\overline{K}_3$. If $<S> = K_2 \cup K_1$ then vw be the edge in $<S>$. Let x_6 and x_7 be the two remaining vertices adjacent to v and also x_4 and x_5 are the remaining two vertices adjacent to w. Let $S_2 = \{x_6, x_7\}$ and $S_3 = \{x_4, x_5\}$.

Lemma 3.1. Let G be a connected cubic graph on 10 vertices with $V(G)=\{u, v, w, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Let $S = \{u, v, w\}$ be a minimum complementary connected dominating set. Let $S_1 = N(u) = \{x_1, x_2, x_3\}$. If $\langle S\rangle = K_2\cup K_1$ and $\langle S_1\rangle = P_3$, then G is isomorphic to G_1 given in Figure 3.1.



G₁

Figure 3.1

Proof. Without loss of generality, Let $\langle S_1\rangle = P_3 = \{x_1, x_2, x_3\}$. We consider the following three cases.

Case 1. $\langle S_2\rangle = \langle S_3\rangle = K_2$.

Without loss of generality, let x_1 be adjacent to x_7. Since G is cubic, x_3 is adjacent to x_4 and x_5 is adjacent to x_6. Hence $G \cong G_1$.

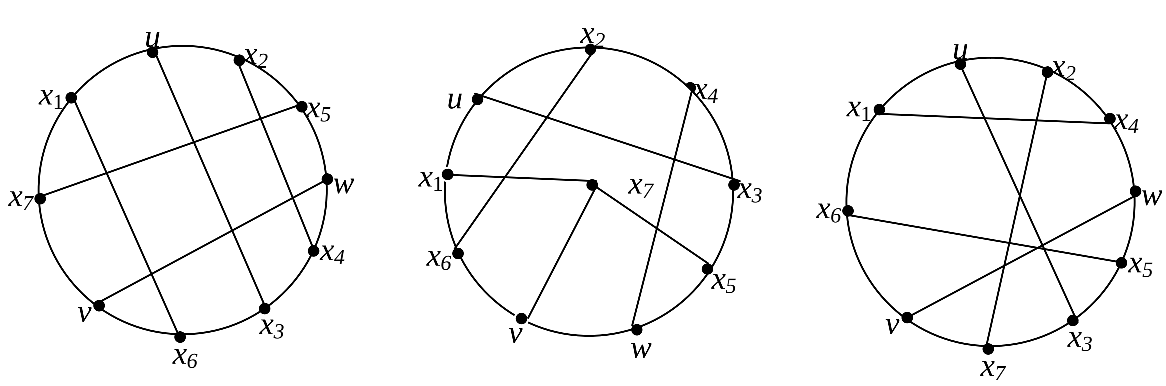
Case 2. $\langle S_2\rangle = K_2$ and $\langle S_3\rangle = \bar{K}_2$.

Let x_5 is adjacent to x_6 and x_7 or x_1 and x_3 or x_6 (or equivalently x_7) and x_1(or equivalently x_6). In all the above situation S fails to be a complementary connected dominating set, which is a contradiction. Hence no graph exists.

Case 3. $\langle S_2\rangle = \langle S_3\rangle = \bar{K}_2$.

Since G is cubic, without loss of generality let x_1 be adjacent to x_6. Also x_6 is adjacent to x_4 (or equivalently x_5) and x_5 is adjacent to x_3 and x_7 and also x_7 is adjacent to x_4. Hence $G \cong G_1$. ■

Lemma 3.2. If $\langle S\rangle = K_2\cup K_1$ and $\langle S_1\rangle = \bar{K}_3$, then G is isomorphic to any one of the graphs given in Figure 3.2.



G₂ **G₃** **G₄**

Figure 3.2

Proof. We consider the following three cases.

Case 1. $<S_2> = <S_3> = K_2$.

Since G is cubic, the three vertices in S_1 are to be incident with 6 edges. But the four vertices in S_2 and S_3 can be incident with four edges. This is a contradiction. Hence no graph exists in this case.

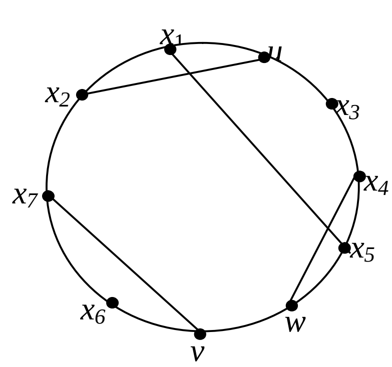
Case 2. $<S_2> = K_2$ *and* $<S_3> = \bar{K}_2$.

Now x_1 is adjacent to x_6 and x_7 or x_4 and x_5 or x_6 (or equivalently x_5). If x_1 is adjacent to x_6 and x_7 then $\{u, v, w\}$ is not a complementary connected dominating set, which is a contradiction. Hence no graph exists. If x_1 is adjacent to x_4 and x_5, then x_2 is adjacent to x_6 and x_7, or x_4 and x_5, x_6 (or equivalently x_7) and x_4 (or equivalently x_5). If x_2 is adjacent to x_6 and x_4, then x_3 is adjacent to x_5 and x_7. Hence $G \cong G_2$. If x_1 is adjacent to x_6 and x_4 then x_2 is adjacent to x_7 and x_5 or x_4 and x_5. If x_2 is adjacent to x_7 and x_5, then x_3 is adjacent to x_4 and x_5. Hence $G \cong G_2$. If x_2 is adjacent to x_4 and x_5, then x_3 is adjacent to x_5 and x_7. Hence $G \cong G_2$.

Case 3. $<S_2> = <S_3> = \bar{K}_2$.

x_1 is adjacent to x_6 and x_7 (or equivalently x_4 and x_5) or x_6 (or equivalently x_7) and x_4 (or equivalently x_7) and x_4 (or equivalently x_5). If x_1 is adjacent to x_6 and x_7, then x_2 is adjacent to x_4 and x_5 or x_6(or equivalently x_7) and x_4 (or equivalently x_5). If x_2 is adjacent to x_4 and x_5, then x_3 is adjacent to x_6 (or equivalently x_7) and x_4 (or equivalently x_5) and then x_5 is adjacent to x_7 and $G \cong G_2$. If x_2 is adjacent to x_6 and x_4, then x_5 is adjacent to x_7 and x_3 and also x_3 is adjacent to x_4. Hence $G \cong G_3$. If x_1 is adjacent to x_4 and x_6, then x_2 is adjacent to x_4 and x_6 or x_5 and x_7(or) x_4 and x_5 (or equivalently x_6 and x_7) or x_4 and x_7 (or equivalently x_5 and x_6). If x_2 is adjacent to x_4 and x_6, then x_3 is adjacent to x_5 and x_7 and then x_7 is adjacent to x_5 which is a contradiction. Hence no graph exists. If x_2 is adjacent to x_5 and x_7, then x_3 is adjacent to x_6 and x_4, or x_7 and x_5 or x_6 and x_5(or equivalently x_4 and x_5). If x_3 is adjacent to x_6 and x_4, then x_5 is adjacent to x_7 which is a contradiction and no graph exists. If x_3 is adjacent to x_7 and x_5, then x_4 is adjacent to x_6 which is a contradiction and no graph exists. If x_3 is adjacent to x_5 and x_6, then x_4 is adjacent to x_7 and hence $G \cong G_3$. If x_2 is adjacent to x_4 and x_5, then x_3 is adjacent to x_6 and x_7 and then x_7 is adjacent to x_5. Hence $G \cong G_2$. If x_2 is adjacent to x_4 and x_7, then x_3 is adjacent to x_6 and x_5 or x_5 and x_7. If x_3 is adjacent to x_5 and x_6, then x_5 is adjacent to x_7. Hence $G \cong G_4$. If x_3 is adjacent to x_7 and x_5, then x_5 is adjacent to x_6. Hence $G \cong G_4$. ■

Lemma 3.3. If $<S> = K_2 \cup K_1$ and $<S_1> = K_2 \cup K_{1,}$ then G is isomorphic to either G_5 given in Figure 3.1 or G_5 given in Figure 3.3.



G5

Figure 3.3

Proof. Let x_1x_2 be the edge in $<S_1>$. We consider the following three cases.

Case 1. $<S_2> = <S_3> = K_2$.

Now x_3 is adjacent to x_6 and x_7 (or equivalently x_4 and x_5) or x_4 and x_6 (or equivalently x_5 and x_7). If x_3 is adjacent to x_6 and x_7, then x_2 is adjacent to x_4 (or equivalently x_5). This implies that x_1 is adjacent to x_5, which is a contradiction. Hence no graph exists. If x_3 is adjacent to x_4 and x_6, then x_1 is adjacent to x_5(or equivalently x_7). And then x_2 is adjacent to x_7. Hence $G \cong G_5$.

Case 2. $<S_2> = K_2$ and $<S_3> = \overline{K_2}$

Since G is cubic x_3 is adjacent to x_4 and x_5 or x_6 (or equivalently x_7) and x_4(or equivalently x_5). If x_3 is adjacent to x_4 and x_5, then x_4 is adjacent to x_1 (or equivalently x_2) or x_6 (or equivalently x_7). If x_4 is adjacent to x_1, then x_5 is adjacent to x_6 or equivalently x_7) and x_7 is adjacent to x_2. Hence $G \cong G_1$. If x_4 is adjacent to x_6, then x_5 is adjacent to x_1(or equivalently x_2) and then x_2 is adjacent to x_7. Hence $G \cong G_1$. If x_3 is adjacent to x_6 and x_4, then x_5 is adjacent to x_1 and x_2 or x_1 (or equivalently x_2) and x_7. If x_5 is adjacent to x_1 and x_2, then x_4 is adjacent to x_7 which is contradiction. Hence no graph exists. If x_5 is adjacent to x_1 and x_7, then x_2 is adjacent to x_4. Hence $G \cong G_1$.

Case 3. $<S_2> = <S_3> = \overline{K_2}$.

Now x_3 is adjacent to x_4 and x_5 (or equivalently x_6 and x_7) or x_4 and x_6 (or equivalently x_5 and x_7). If x_3 is adjacent to x_4 and x_5 which is a contradiction. Hence no graph exists. If x_3 is adjacent to x_4 and x_6, then x_5 is adjacent to x_6 and x_7 or x_1 and x_6 or x_1 and x_7. If x_5 is adjacent to x_6 and x_7, then x_7 is adjacent to x_1 (or equivalently x_2) and then x_2 is adjacent to x_4. Hence $G \cong G_1$. If x_5 is adjacent to x_1 and x_6 then x_7 is adjacent to x_2 and x_4. Hence $G \cong G_1$. If x_5 is adjacent to x_1 and x_7 then x_2 is adjacent to x_6 or x_7. If x_2 is adjacent to x_6, then x_7 is adjacent to x_4. Hence $G \cong G_1$. If x_2 is adjacent to x_7, then x_6 is adjacent to x_4, which is a contradiction. Hence no graph exists. ■

Lemma 3.4. There is no connected cubic graph on 10 vertices with $<S> = \overline{K_3}$ and $<S_1> = P_3$.

Proof. If $<S> = \overline{K_3}$, then v can be adjacent to two of the three vertices not in $N(u)$. It is adjacent to two vertices say x_4 and x_5. Let $S_2 = \{x_4, x_5\}$, then w be adjacent to two other vertices say x_6, x_7. Let $S_3 = \{x_6, x_7\}$. Let $<S_1> = P_3 = x_1x_2x_3$. We consider the following three cases.

Case 1. $<S_2> = <S_3> = K_2$

Now x_1 is adjacent to anyone of $\{x_4, x_5, v\}$ (or equivalently any one of $\{x_6, x_7, w\}$). Since G is cubic, without loss of generality let x_1 be adjacent to v, and x_3 be adjacent to w. Then the remaining five vertices must be incident with one vertex, which is a contradiction. Hence no graph exists.

Case 2. $<S_2> = K_2$ and $<S_3> = \overline{K_2}$.

Let $x_4 x_5$ be the edge in S_2. If v is adjacent to x_6 (or equivalently x_7) then x_7 is adjacent to x_1 and x_3 (or) x_4 and x_5 (or) x_4 (or equivalently x_5) and x_1(or equivalently x_3).

Subcase (a). If x_7 is adjacent to x_1 and x_3 then x_6 is adjacent to x_4 (or equivalently x_5) and w is adjacent to x_5, which is a contradiction. Hence no graph exists.

Subcase (b). If x_7 is adjacent to x_1 and x_4, then x_5 is adjacent to x_6 or w. If x_5 is adjacent to x_6 then w is adjacent to x_3, which is a contradiction. Hence no graph exists. If x_5 is adjacent to w then x_3 is adjacent to x_6, which is a contradiction. Hence no graph exists.

Subcase (c). If x_7 is adjacent to x_5 and x_4 then x_6 is adjacent to x_1(or equivalently x_3). Then w is adjacent to x_3, which is a contradiction. Hence no graph exists.

Case 3. $<S_2> = <S_3> = \overline{K_2}$.

Now v is adjacent to x_1 (or equivalently x_3) or x_6 (or equivalently x_7) which is impossible. Hence no graph exists. ■

Lemma 3.5. If < S > = $\bar{K}_3$ and <S_1> = $K_2 \cup K_1$, then $G \cong G_5$ given in Figure 3.3 or $G \cong G_1$ given in Figure 3.1

Proof. Let x_1x_2 be the edge in <S_1>. We consider the following three cases.

Case 1. <S_2> = <S_3> = K_2.

x_1 is adjacent to any one of the vertices {x_4, x_5, v} (or equivalently any one of {x_6, x_7, w}) without loss of generality, let x_2 be adjacent to w. Then it can be verified that no cubic graph exists satisfying the hypothesis.

Case 2. < S_2> = K_2 *and* <S_3> = $\bar{K}_2$.

Now x_1 is adjacent to any one of {v, x_4, x_5} (or) w (or) x_6 (or equivalently x_7). If x_1 is adjacent to v, then w is adjacent to x_3 or x_4 (or equivalently x_5) or x_2. If w is adjacent to x_3 then x_2 is adjacent to x_7 and then x_6 is adjacent to x_3 and x_4 and also x_7 is adjacent to x_5. Hence by Lemma 3.3, $G \cong G_5$. If w is adjacent to x_4, then x_6 is adjacent to x_2 and x_3 and then x_3 is adjacent to x_7 and also x_7 is adjacent to x_5. Hence $G \cong G_5$, which falls under Lemma 3.3. If w is adjacent to x_2, then x_5 is adjacent to x_3 or x_6 (or equivalently x_7). If x_5 is adjacent to x_3, then x_3 is adjacent to x_6 and x_5 which is a contradiction. Hence no graph exists. If x_5 is adjacent to x_6, then x_7 is adjacent to x_3 and x_4 which is contradiction. Hence no graph exists. If x_1 is adjacent to w, then x_2 is adjacent to x_4 and then x_7 is adjacent to x_5 and x_3 and also x_6 is adjacent to x_3 and v. Hence $G \cong G_5$, which falls under Lemma 3.3.If x_1 is adjacent to x_6, then x_6 is adjacent to any one of {v, x_4, x_5} or x_3 or x_2. If x_6 is adjacent to v, then w is adjacent to x_3 or x_2 or x_4 (or equivalently x_5). If w is adjacent to x_3, then x_2 is adjacent to x_4 (or equivalently x_5) or x_7. If x_2 is adjacent to x_4, then x_7 is adjacent to x_3 and x_5. Hence $G \cong G_5$, which falls under Lemma 3.3. If x_2 is adjacent to x_7, then x_7 is adjacent to x_4 (or equivalently x_5) and then x_3 is adjacent to x_5. Hence $G \cong G_1$, as in Figure 3.1. which falls under Lemma 3.1. If w is adjacent to x_2, or x_4 we get a contradiction and hence no graph exists. If x_6 is adjacent to x_3, then x_3 is adjacent to x_4 and then x_3 is adjacent to x_4 and then x_2 is adjacent to w and x_7 is adjacent to v. Hence $G \cong G_1$ given in Figure 3.1. If x_6 is adjacent to x_2 then also we get a contradiction and hence no graph exists.

Case 3. <S_2> = <S_3> = $\bar{K}_2$

If x_1 be adjacent to v (or equivalently w) or x_4 (or equivalently x_5) { or equivalently x_6 (or equivalently x_7) }. In all the cases no new graph exists. ■

Lemma 3.6. There is no connected cubic graph on 10 vertices with < S > = $\bar{K}_3$ and <S_1>= $\bar{K}_3$.

Proof. We consider the following three cases.

Case 1. < S_2>= < S_3> = K_2.

Let x_4x_5 be the edge in <S_2> and x_6x_7 be the edge in <S_3>. Now x_1 is adjacent to any two of {v,x_4,x_5} (or equivalently any two of { w ,x_6, x_7}) or any of { v,x_4,x_5 } and anyone of {w, x_6 ,x_7}. In all the above cases,{u,v,w} is not a complementary connected dominating set ,which is a contradiction.

Case 2. < S_2> = K_2 and < S_3 >= $\bar{K}_2$.

Let w be adjacent to any one of {x_1,x_2,x_3}. Let w be adjacent to x_1. In this case also {u,v,w} is not a complementary connected dominating set, which is a contradiction.

Case 3. < S_2> = <S_3> = $\bar{K}_2$.

In this case, v is adjacent to x_1 (or equivalently x_3) or x_6 (or equivalently x_7). In the cases, {u, v, w} is not a complementary connected dominating set, which is a contradiction. ■

Theorem 3.7. Let G is connected cubic graph on 10 vertices. Then γ_{cc} = χ = 3 if and only if G is isomorphic to any one of graphs given in Figures 3.3, 3.2 and 3.3.

Proof. If G is any one of the graphs given in Figure 3.1, 3.2 and 3.3, then clearly γ_{cc} = χ = 3. Conversely if γ_{cc} = χ = 3, then the proof follows from the Lemmas 3.1 to 3.6. ■

4 Cubic graphs on 12 vertices

Let G be a connected cubic graph on 12 vertices with $V(G)=\{u, v, w, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$. Let $S = \{u, v, w\}$ be a complementary connected dominating set.

Let $<S_1> = N(u) = \{x_1, x_2, x_3\}$. Let $<S_2> = N(v) = \{x_4, x_5, x_6\}$ and $<S_3> = N(w) = \{x_7, x_8, x_9\}$.Then we consider the following cases:

Case 1: $< S_1> = < S_2> = < S_3> = P_3$.

Case 2: $< S_1> = < S_2> = P_3$ and $< S_3> = K_2 \cup K_1$.

Case 3: $< S_1> = < S_2> = P_3$, and $< S_3> = \overline{K_3}$

Case 4: $< S_1> = P_3$ and $< S_2> = <S_3> = K_2 \cup K_1$.

Case 5: $< S_1> = P_3$ and $< S_2> = \overline{K_3}$, and $< S_3> = K_2 \cup K_1$.

Case 6: $< S_1> = P_3$ and $< S_2> = < S_3> = \overline{K_3}$.

Case 7: $< S_1> = < S_2> = K_2 \cup K_1$, $< S_3> = \overline{K_3}$.

Case 8: $< S_1> = < S_2> = < S_3> = K_2 \cup K_1$.

Case 9: $< S_1> = < S_2> = < S_3> = \overline{K_3}$.

Case 10: $< S_1> = K_2 \cup K_1$, $< S_2> = < S_3> = \overline{K_3}$.

Lemma 4.1. If $< S_1> = < S_2> = < S_3> = P_3$, then G is isomorphic to G_1 given in Figure 4.1.

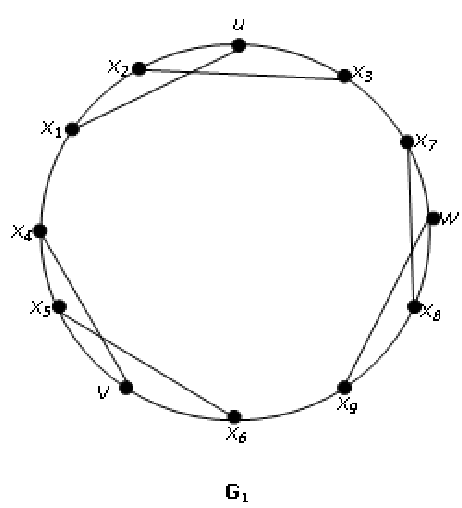
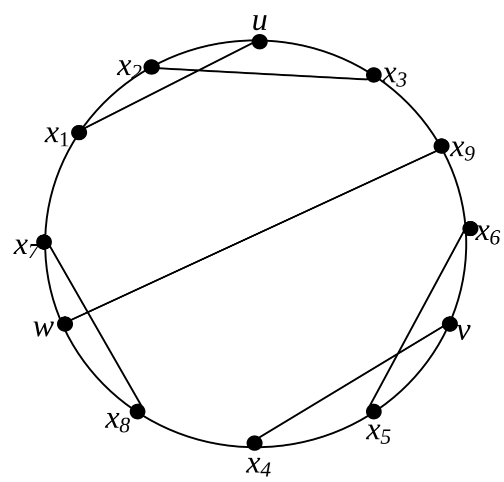


Figure 4.1

Proof. Let $< S_1> = P_3 = x_1 x_2 x_3$, $< S_2> = P_3 = x_4 x_5 x_6$ and $< S_3> = P_3 = x_7 x_8 x_9$.Without loss of generality, let x_1 be adjacent to x_4. Since G is cubic, x_3 must be adjacent to x_7 or x_9. Without loss of generality, let x_3 be adjacent to x_4. Then x_6 must be adjacent to x_9. Hence $G \cong G_1$. ■

Lemma 4.2. If $< S_1> = < S_2> = P_3$ and $< S_3> = K_2 \cup K_1$ then G is isomorphic to the graph G_2 given in Figure 4.2.

Proof. Let $< S_1> = P_3 = x_1 x_2 x_3$, $< S_2> = P_3 = x_4 x_5 x_6$. Let $x_7 x_8$ be the edge in $< S_3>$. Now x_7 is adjacent to anyone of $\{x_1, x_3, x_4, x_6\}$. Without loss of generality, let x_7 be adjacent to x_1. Then x_8 is adjacent to x_3 or anyone of $\{x_4, x_6\}$. If x_8 is adjacent to x_3 then x_9 must be adjacent to x_4 and x_6, which is a contradiction. Hence no graph exists. If x_8 is adjacent to x_4, then x_9 must be adjacent to x_3 and x_6. Hence $G \cong G_2$. ■



$\mathbf{G_2}$

Figure 4.2

Lemma 4.3. There exists no connected cubic graph on 12 vertices with $< S_1> = < S_2> = P_3$ and $< S_3> = \overline{K}_3$.

Proof. Let$< S_1> = P_3 = x_1, x_2, x_3,$ $< S_2> = P_3 = x_4, x_5, x_6$. Now x_7 is adjacent to one of $\{x_1, x_3\}$ and one of $\{x_4, x_6\}$ (or) x_7 is adjacent to x_1 and x_3 (or equivalently x_4 and x_6). In both the cases cubic graph does not exists. ■

Lemma 4.4. If $< S_1> = P_3$ and $< S_2> = < S_3> = K_2 \cup K_1$, then G isomorphic to the graph G_2 given in Figure 4.2.

Proof. Let $< S_1> = x_1, x_2, x_3$. Let x_4x_5 be the edge in $< S_2>$ and x_7x_8 be the edge in $< S_3>$. We consider the following two cases.

Case 1. Let x_1 be adjacent to any one of $\{x_4, x_5, x_7, x_8\}$. Without loss of generality, let x_1 be adjacent to x_4. Since G is cubic x_3 is adjacent to x_9 or x_7(or equivalently x_8). If x_3 is adjacent to x_9 is adjacent to x_5 or x_6. If x_9 is adjacent to x_5, then x_6 is adjacent to x_7 and x_8 which is a contradiction. Hence no graph exists. If x_9 is adjacent to x_6, then x_6 is adjacent to (or equivalently x_8) and x_5 is adjacent to x_8. Hence $G \cong G_2$ given in Figure 4.2. If x_3 is adjacent to x_7 (or equivalently x_8) then since G is cubic, x_8 must be adjacent to x_6 and x_9 is adjacent to x_5 and x_6. Hence $G \cong G_2$ given in Figure 4.2.

Case 2. Let x_1 be adjacent to any one of $\{x_6, x_9\}$. Without loss of generality, let x_1 be adjacent to x_6. Now x_4 is adjacent to x_3 or x_7(or equivalently x_8) or x_9. Since G is cubic, x_4 is not adjacent to x_3. If x_4 is adjacent to x_7, then x_5 is adjacent to x_8 or x_9. If x_5 is adjacent to x_8, then x_9 is adjacent to x_3 and x_6, which is a contradiction. Hence no graph exists. If x_5 is adjacent to x_9, then x_9 is adjacent to x_3 or x_6. If x_9 is adjacent to x_3 or x_6. If x_9 is adjacent to x_3, then x_6 is adjacent to x_8. Hence $G \cong G_2$ given in Figure 4.2. If x_9 is adjacent to x_6, then x_3 is adjacent to x_8. Hence $G \cong G_2$ given in Figure 4.2. If x_4 is adjacent to x_9, then x_5 is adjacent to x_8 and then x_3 is adjacent to x_7 and also x_6 is adjacent to x_9. Hence $G \cong G_2$ given in Figure 4.2. ■

Lemma 4.5. If $< S_1> = P_3$ and $< S_2> = \overline{K}_3$ and $< S_3> = K_2 \cup K_1$, then G is isomorphic to G_2 given in Figure 4.2.

Proof. If $< S_1> = x_1x_2x_3$. Let $x_7\, x_8$ be the edge in $< S_3>$. Since G is cubic, x_1 must be adjacent to any one of $\{x_4, x_5, x_6\}$. Without loss of generality let x_1 be adjacent to x_4. We consider the following three cases.

Case 1. x_4 is adjacent to x_3. In this case, x_5 is adjacent to x_7 (or equivalently x_8) and x_9 and then x_6 is adjacent to x_8 and x_9 which is a contradiction. Hence no graph exists.

Case 2. x_4 is adjacent to x_7 (or equivalently x_8). In this case, x_9 is adjacent to x_3 and x_5 (or equivalently x_6) (or) x_9 is adjacent to x_5 and x_6. Since G is cubic, x_9 cannot be adjacent to both x_3 and x_5. If x_9 is adjacent to x_5 and x_6, then x_5 is adjacent to x_3 or x_5. If x_5 is adjacent to x_3, then x_6 is adjacent to x_8. Hence $G \cong G_2$ given in Figure 4.2. If x_5 is adjacent to x_8, then x_3 is adjacent to x_6. Hence $G \cong G_2$ given in Figure 4.2.

Case 3. x_4 is adjacent to x_9. Since G is cubic, $x3$ must be adjacent to any one of $\{x_5, x_6\}$. Let x_3 be adjacent to x_5. Now x_6 is adjacent to both x_7 and x_8 (or) x_6 is adjacent to one of $\{x_7, x_8\}$ and x_9. If x_6 is adjacent to x_7 and x_8, then x_5 is adjacent to x_9, which is a contradiction. Hence no graph exists. If x_6 is adjacent to x_7 and x_9, then x_5 is adjacent to x_8. Hence, $G \cong G_2$ given in Figure 4.2. ■

Lemma 4.6. If $< S_1> = P_3$ and $< S_2> = < S_3> = \bar{K}_3$, then G is isomorphic to G_2 given in Figure 4.2.

Proof. Let $< S_1> = x_1\, x_2\, x_3$. Now x_1 is adjacent to anyone of $\{x_4, x_5, x_6, x_7, x_8, x_9\}$, without loss of generality, let x_1 be adjacent to x_4. Since G is cubic, x_4 is adjacent to anyone of $\{x_7, x_8, x_9\}$. Now without loss of generality, let x_4 be adjacent to x_7. Then x_7 is adjacent to x_3 or anyone of $\{x_5, x_6\}$. If x_7 is adjacent to x_3, then x_5 is adjacent to x_8 and x_9. Then x_6 is adjacent to x_8 and x_9, which is a contradiction. Hence no graph exists. If x_7 is adjacent to x_5 (or equivalently x_6), then x_5 is adjacent to x_8(or equivalently x_9). If x_5 is adjacent to x_8, x_3 is adjacent to x_9 and x_6 is adjacent to x_8 and x_9. Hence $G \cong G_2$ given in Figure 4.2. ■

Lemma 4.7. If $< S_1> = < S_2> = K_2 \cup K_1$, and $< S_3> = \bar{K}_3$, then G is isomorphic to G_2 given in Figure 4.2.

Proof. Let x_1x_2 be the edge in $<S_1>$ and x_4x_5 be the edge in $<S_2>$. Then x_1 is adjacent to any one of $\{x_4, x_5\}$ (or) x_6 (or) any one of $\{x_7, x_8, x_9\}$. In all the cases, it can be verified that G is isomorphic to G_2 given in Figure 4.2. ■

Lemma 4.8. If $< S_1> = < S_2> = < S_3> = K_2 \cup K_1$, then G is isomorphic to G_2 given in Figure 4.2.

Proof Let x_1x_2 be the edge in $<S_1>$, x_4x_5 be the edge in $< S_2 >$and x_7x_8 be the edge in $<S_3>$.Then x_1 is adjacent to any one of $\{x_4, x_5, x_7, x_8\}$ (or) any one of $\{x_6, x_9\}$. In all the cases, it can be verified that G is isomorphic to G_2 given in Figure 4.2. ■

Lemma 4.9. If $< S_1> = < S_2> = < S_3> = \bar{K}_3$, then G is isomorphic to G_2 given in Figure 4.2.

Proof. In this case, we consider the following two cases, x_1 is adjacent to any two of $\{x_4, x_5, x_6\}$ (or equivalently any two of $\{x_7, x_8, x_9\}$), (or) x_1 is adjacent to any one of $\{x_4, x_5, x_6\}$and any one of $\{x_7, x_8, x_9\}$.In both cases, it can be verified that G is isomorphic to G_2 given in Figure 4.2. ■

Lemma 4.10. If $< S_1> = K_2 \cup K_1$, $< S_2> = < S_3> = \bar{K}_3$, then G is isomorphic to the graph G_2 given in Figure 4.2.

Proof. Since G is cubic, x_1 is adjacent to any one of $\{x_4, x_5, x_6, x_7, x_8, x_9\}$. Without loss of generality, let x_1 be adjacent to x_4. Then x_2 is adjacent to x_4 (or) x_2 is adjacent to one of $\{x_5, x_6\}$ (or) x_2 is adjacent to anyone of $\{x_7, x_8, x_9\}$.

In all the cases, it can be verified that G is isomorphic to G_2 given in Figure 4.2. ■

Theorem 4.11. Let $G= (V, E)$ be a connected cubic graph on 12 vertices. Then G is isomorphic to any one of the graphs given in Figures 4.1 and 4.2 for which $\gamma_{cc} = \chi = 3$.

Proof. If G is any one of the graphs given in Figure 4.1 and 4.2, then clearly $\gamma_{cc} = \chi = 3$. Conversely, if $\gamma_{cc} = \chi = 3$, then the proof follows from the Lemmas 4.1 to 4.10. ■

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