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M / M $^{(k)}$ /1 Queuing model with varying bulk service

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Abstract

In this paper, we study single server bulk service queuing process in which customers are served in varying batch size and obtain various characteristics of the systems.

Keywords: Queuing system, service process, arrival process, bulk service, joint probability, marginal probabilities.

1 Introduction

The single server models with the interdependent structure had been studied by many authors with single service mechanism. In this paper we consider a generalized queuing model in which customers are served as a batch of size *k* at a time, except when there are less than *k* customers in the system at the time of service.For developing these interdependent models with bulk service rule we make use of the dependence structure given by Rao K .S (1986).

2 M / M$^{(k)}$/ 1 Interdependent queuing model with varying batch size

In this section, we consider the single server queuing system having the interdependent arrival and processes with bulk service.

In this system the interdependence can be induced by considering the dependence structures having a bivariate Poisson distribution of the form

$$P[X_1 = x_1, X_2 = x_2 / t] = e^{-(\lambda+\mu-\in)t} \sum_{j=0}^{\min.(x_1,x_2)} \frac{(\in t)^j \left[(\lambda-\in)t\right]^{x_1-j} \left[(\mu-\in)t\right]^{x_2-j}}{j!\,(x_1-j)!\,(x_2-j)!} \qquad (1)$$

where $x_1, x_2 = 0, 1, 2, ...$ and $0 < \lambda, \mu, \in < \min(\lambda, \mu)$. Here $P[X_1=x_1, X_2=x_2 / t]$ is the joint probability of x_1 arrivals and x_2 services during time t.

The marginal distribution of arrival and service are Poisson with parameters λ and μ respectively. Thus inter arrival time and service time follow negative exponentials distributions of the form $\lambda e^{-\lambda t}$ and $\mu e^{-\mu t}$ respectively where λ is the mean arrival rate and μ is the mean service rate (Feller 1969). Let □ be the covariance between the number of arrivals and services at time *t*. This dependence structure turns out to be an independent structure if □ = 0 (Teicher1954).

3 Postulates of the Model

The postulates of the model with this dependence structure are

1. The occurrence of the events in non-overlapping time intervals is statistically independent.
2. The probability that no arrivals and no service completions occur in an infinitesimal interval of time Δt is $1 - [(\lambda + \mu - \in)\, t] + O(\Delta t)$
3. The probability that no arrival and one service completion occurs in Δt is $(\mu - \in)\, t + O(\Delta t)$
4. The probability that one arrival and no service completion occurs in Δt is $(\lambda - \in)\, t + O(\Delta t)$
5. The probability that one arrival and one service completion occurs in Δt is $\in t + O(\Delta t)$.

This postulate is due to the dependence structure between the arrivals and service completions.

6. The probability that the occurrence of an event other than the above events during Δt is $O(\Delta t)$

For the given values of λ and μ the covariance $\in = r\sqrt{\lambda\,\mu}$, where r is the correlation coefficient between the arrival and the service.

Let $P_n(t)$ be the probability that there are n customers in the system at time t. The difference equations of the model are

$$P_n^{/}(t) = -(\lambda + \mu - 2\in)\, P_n(t) + (\lambda - \in)\, P_{n-1}(t) + (\mu - \in)\, P_{n+k}(t)$$

$$P_0^{/}(t) = -(\lambda - \in)\, P_0(t) + (\mu - \in) \sum_{i=1}^{k} P_i(t) \qquad (2)$$

In the steady-state the transition equations of the model are

$$-(\lambda + \mu - 2\in)\, P_n + (\lambda - \in)\, P_{n-1} + (\mu - \in)\, P_{n+k} = 0$$

$$-(\lambda - \in)\, P_0 + (\mu - \in) \sum_{i=1}^{k} P_i = 0 \qquad (3)$$

Using the heuristic arguments of Gross and Harries (1974), we can obtain the solution of these equations as

$$P_n = C\, r^n, \qquad n \geq 0 \text{ and } 0 < r < 1 \qquad (4)$$

where $r \in (0,1)$ is the root of the characteristic equation

$$\{(\mu - \in) D^{k+1} - (\lambda + \mu - 2\in) D + (\lambda - \in)\} P_n = 0 \qquad (5)$$

and D is the operator $\frac{d}{dt}$.

4 Measures and effectiveness

The probability that the system is empty is given by

$$P_0 = 1 - r \qquad (6)$$

For various values of □ and k for given value of λ and μ we computed P_0 values and are given in Table 2 the values of P_0 for fixed k, □ for various λ and μ are given in Table 4. The values of r is given for various values of □ and k for fixed values of $\lambda = 2$ and $\mu = 4$ in Table 1. The values of r for fixed λ, μ and for varying □ and k are given in Table 3.

Table 1: The values of r where $\lambda = 2$ and $\mu = 4$.

□ / k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.5	0.4872	0.4737	0.4595	0.4444	0.4286	0.4118	0.3939	0.375	0.3548
2	0.366	0.3586	0.3507	0.3423	0.3333	0.3238	0.3135	0.3025	0.2906	0.2777
3	0.3425	0.3362	0.3294	0.3222	0.3145	0.3061	0.2972	0.2875	0.277	0.2656
4	0.3362	0.3302	0.3238	0.317	0.3097	0.3018	0.2932	0.2839	0.2738	0.2628
5	0.3343	0.3284	0.3222	0.3155	0.3083	0.3005	0.2921	0.283	0.273	0.2621
6	0.3336	0.3279	0.3217	0.315	0.3079	0.3002	0.2918	0.2827	0.2728	0.262

Table 2: The values of P_0 where $\lambda = 2$ and $\mu = 4$

□ / k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.5	0.5128	0.5263	0.5405	0.5556	0.5714	0.5882	0.6061	0.625	0.6452
2	0.634	0.6414	0.6493	0.6577	0.6667	0.6762	0.6865	0.6975	0.7094	0.7223
3	0.6575	0.6638	0.6706	0.6778	0.6855	0.6939	0.7028	0.7125	0.723	0.7344
4	0.6638	0.6698	0.6762	0.683	0.6903	0.6982	0.7068	0.7161	0.7262	0.7372
5	0.6657	0.6716	0.6778	0.6845	0.6917	0.6995	0.7079	0.717	0.727	0.7379
6	0.6664	0.6721	0.6783	0.685	0.6921	0.6998	0.7082	0.7173	0.7272	0.738

Table 3: The values of r where $k = 2$ and □ = 0.2.

λ / μ	1	2	3	4	5	6	7	8	9
10	0.0759	0.1585	0.2319	0.2986	0.3601	0.4175	0.4715	0.5227	0.5714
11	0.0693	0.1455	0.2136	0.2758	0.3333	0.3872	0.4379	0.486	0.5319
12	0.0637	0.1345	0.1981	0.2563	0.3104	0.3611	0.409	0.4545	0.4979
13	0.059	0.125	0.1847	0.2395	0.2906	0.3385	0.3839	0.427	0.4682
14	0.055	0.1168	0.173	0.2248	0.2732	0.3187	0.3618	0.4029	0.4422
15	0.0514	0.1096	0.1627	0.2119	0.2578	0.3012	0.3423	0.3815	0.419

Table 4: The values of P_0 where k = 2 and □ = 0.2.

λ / μ	1	2	3	4	5	6	7	8	9
10	0.9241	0.8415	0.7681	0.7014	0.6399	0.5825	0.5285	0.4773	0.4286
11	0.9307	0.8545	0.7864	0.7242	0.6667	0.6128	0.5621	0.514	0.4681
12	0.9363	0.8655	0.8019	0.7437	0.6896	0.6389	0.591	0.5455	0.5021
13	0.941	0.875	0.8153	0.7605	0.7094	0.6615	0.6161	0.573	0.5318
14	0.945	0.8832	0.827	0.7752	0.7268	0.6813	0.6382	0.5971	0.5578
15	0.9486	0.8904	0.8373	0.7881	0.7422	0.6988	0.6577	0.6185	0.581

From Tables 2, 4 and equation (6) we observe that for fixed values of λ, μ and □ the value of P_0 increases as k increases. As the dependence parameter □ increases the value of P_0 increases for fixed values of λ, μ and k.

The value of P_0 decreases for fixed values of μ, *k* and □ as λ increases. As μ increases, the value of P_0 increases for fixed values of μ, *k* and □. If the mean dependence rate is zero then the value of P_0 is same as in the (M / M$^{(k)}$ / 1) model.

The average number of customers in the system is

$$L_s = \frac{r}{1-r} \tag{7}$$

The average number of customers in the queue is

$$L_q = \frac{r^2}{1-r} \tag{8}$$

where *r* is given in equation (4).

The values of L_s and L_q are computed and are given in Tables 5 and 7 for given values of λ, μ and for various values of □ and *k* respectively. The values of L_s and L_q for fixed values of □, *k* and for varying values of μ and λ are also given in Table 6 and 8 respectively.

Table 5: The values of L_s where λ = 2 and μ = 4.

k \ □	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	1	0.950078	0.900057	0.850139	0.799856	0.750088	0.700102	0.649893	0.6	0.549907
2	0.577287	0.559089	0.54012	0.52045	0.499925	0.478852	0.456664	0.433692	0.409642	0.384466
3	0.520913	0.506478	0.491202	0.475361	0.458789	0.44113	0.42288	0.403509	0.383126	0.361656
4	0.506478	0.492983	0.478852	0.464129	0.448646	0.432254	0.414827	0.396453	0.377031	0.356484
5	0.502178	0.488982	0.475361	0.46092	0.445713	0.429593	0.412629	0.3947	0.375516	0.355197
6	0.5006	0.487874	0.474274	0.459854	0.444878	0.42898	0.41203	0.394117	0.375138	0.355014

Table 6: The values of L_s where *k* = 2 and □ = 0.2.

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.082134	0.188354	0.301914	0.42572	0.562744	0.716738	0.892148	1.095118	1.333178
11	0.07446	0.170275	0.271617	0.380834	0.499925	0.631854	0.779043	0.945525	1.136296
12	0.068034	0.155402	0.247038	0.344628	0.450116	0.56519	0.692047	0.833181	0.991635
13	0.062699	0.142857	0.226542	0.314924	0.409642	0.511716	0.623113	0.745201	0.880406
14	0.058201	0.132246	0.20919	0.28999	0.375894	0.467782	0.566907	0.674761	0.792757
15	0.054185	0.123091	0.194315	0.268875	0.347346	0.431025	0.52045	0.616815	0.72117

Table 7: The values of L_q where $\lambda = 2$ and $\mu = 4$.

k \ □	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.5	0.462878	0.426357	0.390639	0.355456	0.321488	0.288302	0.255993	.225	0.195107
2	0.211287	0.200489	0.18942	0.17815	0.166625	0.155052	0.143164	0.131192	0.119042	0.106766
3	0.178413	0.170278	0.161802	0.153161	0.144289	0.13503	0.12568	0.116009	0.106126	0.096056
4	0.170278	0.162783	0.155052	0.147129	0.138946	0.130454	0.121627	0.112553	0.103231	0.093684
5	0.167878	0.160582	0.153161	0.14542	0.137413	0.129093	0.120529	0.1117	0.102516	0.093097
6	0.167	0.159974	0.152574	0.144854	0.136978	0.12878	0.12023	0.111417	0.102338	0.093014

Table 8: The values of L_q where $k = 2$ and □ = 0.2.

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.006234	0.029854	0.070014	0.12712	0.202644	0.299238	0.420648	0.572418	0.761778
11	0.00516	0.024775	0.058017	0.105034	0.166625	0.244654	0.341143	0.459525	0.604396
12	0.004334	0.020902	0.048938	0.088328	0.139716	0.20409	0.283047	0.378681	0.493735
13	0.003699	0.017857	0.041842	0.075424	0.119042	0.173216	0.239213	0.318201	0.412206
14	0.003201	0.015446	0.03619	0.06519	0.102694	0.149082	0.205107	0.271861	0.350557
15	0.002785	0.013491	0.031615	0.056975	0.089546	0.129825	0.17815	0.235315	0.30217

From equations (7) and (8) and from the corresponding tables we observe that as □ increases the values of L_s and L_q are decreasing and also as *k* increases the value of L_s and L_q are decreasing for fixed values of other parameters.

As the arrival rate increases the values of L_s and L_q are increasing for fixed values of μ, k and □. As μ increases the values of L_s and L_q are decreasing for fixed values of λ, k and □. When the dependence parameter □ = 0 then the mean queue length is same as that of $M/M^{(k)}/1$ model. When k=1 this is same as M/M/1 interdependence model.
The variability of this model can be obtained as

$$V = \frac{r}{(1-r)^2} \tag{9}$$

And the coefficient of variation of the model is

$$C.V = \frac{\sqrt{V}}{L_s} \tag{10}$$

where L_s and V are as given in equations (7) and (9).

The value of the variability of the system and coefficient of variation for various values of k, □ for fixed values of λ, μ are computed which are given Tables 8 and 9. The values of the variability of the system and coefficient of variation for fixed values of *k*, □ and for various values of λ, μ are given in Tables 10 and 11.

Table 9: The values of *V* where λ = 2 and μ = 4.

□ \ *k*	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	2	1.852726	1.71016	1.572875	1.439626	1.312719	1.190245	1.072253	0.96	0.852305
2	0.910547	0.871671	0.83185	0.791318	0.74985	0.708152	0.665206	0.62178	0.577448	0.532281
3	0.792262	0.762998	0.732481	0.70133	0.669277	0.635725	0.601707	0.566328	0.529911	0.492451
4	0.762998	0.736015	0.708152	0.679544	0.649928	0.619098	0.586909	0.553628	0.519184	0.483565
5	0.754361	0.728084	0.70133	0.673368	0.644374	0.614142	0.582892	0.550488	0.516528	
6	0.751201	0.725895	0.69921	0.67132	0.642794	0.613003	0.5818	0.549445	0.515866	0.481048

Table 10: The values of V where k = 2 and □ = 0.2.

λ \ μ	1	2	3	4	5	6	7	8	9
10	0.08888	0.223831	0.393066	0.606957	0.879425	1.230452	1.688075	2.294403	3.110541
11	0.080004	0.199269	0.345394	0.525869	0.74985	1.031093	1.385951	1.839543	2.427463
12	0.072662	0.179551	0.308066	0.463397	0.65272	0.88463	1.170977	1.52737	1.974975
13	0.06663	0.163265	0.277864	0.414102	0.577448	0.773569	1.011383	1.300525	1.655521
14	0.061588	0.149735	0.25295	0.374084	0.517191	0.686602	0.88829	1.130064	1.421221
15	0.057121	0.138242	0.232073	0.341168	0.467995	0.616807	0.791318	0.997275	1.241257

Table 11: The values of *C.V* where λ = 2 and μ = 4.

□ \ *k*	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	1.414214	1.432671	1.452942	1.475222	1.500075	1.527474	1.558321	1.593335	1.632993	1.678836
2	1.652949	1.669917	1.688621	1.709214	1.732137	1.757363	1.785999	1.818182	1.855035	1.897632
3	1.708715	1.724651	1.742361	1.761721	1.783157	1.807459	1.834322	1.86501	1.900029	1.940376
4	1.724651	1.740249	1.757363	1.776112	1.796923	1.820289	1.846792	1.876797	1.911099	1.950686
5	1.729545	1.745012	1.761721	1.780329	1.800998	1.824222	1.850266	1.879779	1.913898	1.953289
6	1.731358	1.746342	1.76309	1.781742	1.802168	1.825134	1.851217	1.880776	1.914599	1.953662

Table 12: The values of C.V where k = 2 and □ = 0.2.

μ \ λ	1	2	3	4	5	6	7	8	9
10	3.62977	2.511802	2.076585	1.830017	1.666435	1.547646	1.456328	1.383164	1.322909
11	3.798686	2.621613	2.163712	1.904158	1.732137	1.607061	1.511167	1.434438	1.37115
12	3.962144	2.726709	2.246766	1.975266	1.794895	1.664126	1.563646	1.483314	1.417193
13	4.116935	2.828427	2.32684	2.043371	1.855035	1.718781	1.613953	1.530333	1.461451
14	4.264014	2.926029	2.404235	2.109123	1.913197	1.771369	1.662516	1.575438	1.503802
15	4.410811	3.02061	2.47917	2.172374	1.969512	1.822101	1.709214	1.619022	1.544874

From equations (9) and (10) and from the corresponding tables we observe that as μ increases the variability of the system size decreases and the coefficient of variation increases. For fixed values of μ, □ and *k*, as λ increases the variability of the system size increases and the coefficient of variation decreases. We also observe that as □ increases the variability of the system size decreases and the coefficient of variation increases for fixed values of λ, μ and *k*. As *k* increases, the variability of the system decreases the coefficient of variation increases.

For □=0 and *k*=1, this model reduces to M/M/1 classical model. When *k*=1, this model becomes M/M/1 interdependent model and for □ = 0, this model is same as M/M$^{(k)}$/1 model.

Average time a customer spends in the system $Ws = 1/(\mu - \lambda)$ (11)

Average time a customer spends in the queue $W_q = r * W_s$ (12)

Table 13: $W_s = 1/(\mu - \lambda)$ where λ = 2 and μ = 4 then $W_s = 0.5$.

Table 14: $W_s = 1/(\mu - \lambda)$ where k = 2 and □ = 0.2.

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.111111	0.125	0.142857	0.166667	0.2	0.25	0.333333	0.5	1
11	0.1	0.111111	0.125	0.142857	0.166667	0.2	0.25	0.333333	0.5
12	0.090909	0.1	0.111111	0.125	0.142857	0.166667	0.2	0.25	0.333333
13	0.083333	0.090909	0.1	0.111111	0.125	0.142857	0.166667	0.2	0.25
14	0.076923	0.083333	0.090909	0.1	0.111111	0.125	0.142857	0.166667	0.2
15	0.071429	0.076923	0.083333	0.090909	0.1	0.111111	0.125	0.142857	0.166667

Table 15: $W_q = (r * w_s)$ where $\lambda = 2$ and $\mu = 4$.

k \ □	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.25	0.2436	0.23685	0.22975	0.2222	0.2143	0.2059	0.19695	0.1875	0.1774
2	0.183	0.1793	0.17535	0.17115	0.16665	0.1619	0.15675	0.15125	0.1453	0.13885
3	0.17125	0.1681	0.1647	0.1611	0.15725	0.15305	0.1486	0.14375	0.1385	0.1328
4	0.1681	0.1651	0.1619	0.1585	0.15485	0.1509	0.1466	0.14195	0.1369	0.1314
5	0.16715	0.1642	0.1611	0.15775	0.15415	0.15025	0.14605	0.1415	0.1365	0.13105
6	0.1668	0.16395	0.16085	0.1575	0.15395	0.1501	0.1459	0.14135	0.1364	0.131

Table 16: $W_q = r * W_s$ where $k = 2$ and □ = 0.2.

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.008433	0.019813	0.033129	0.049767	0.07202	0.104375	0.157167	0.26135	0.5714
11	0.006364	0.013379	0.019674	0.025458	0.030861	0.036019	0.041053	0.046286	0.05319
12	0.005349	0.011295	0.016636	0.021524	0.026068	0.030327	0.034352	0.038178	0.04184
13	0.004568	0.009678	0.0143	0.018543	0.022499	0.026208	0.029723	0.03306	0.03625
14	0.00395	0.008389	0.012426	0.016146	0.019623	0.022891	0.025987	0.028939	0.031761
15	0.003443	0.007342	0.010899	0.014195	0.017269	0.020177	0.02293	0.025556	0.028068

Table 17: P [The system size $\geq k$] = r^{k-1} where $\lambda = 2$ and $\mu = 4$.

k \ □	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	0.5	0.4872	0.4737	0.4595	0.4444	0.4286	0.4118	0.3939	0.375	0.3548
3	0.133956	0.128594	0.12299	0.117169	0.111089	0.104846	0.098282	0.091506	0.0844	0.0771
4	0.040177	0.038001	0.035741	0.033448	0.031107	0.028681	0.026251	0.023764	0.0212	0.0187
5	0.012776	0.011888	0.010993	0.010098	0.0092	0.008296	0.00739	0.006496	0.0056	0.0047
6	0.004175	0.00382	0.003472	0.003126	0.002785	0.00245	0.002126	0.001815	0.0015	0.0012
7	0.001378	0.001243	0.001108	0.000977	0.000852	0.000732	0.000617	0.00051	0.0004	0.0003

5 Conclusion

In this paper we extend the single server interdependent queuing model to bulk service queuing model with varying bulk service. These models are having wider applicability in transportation, inventory control, machine interference problems, neurophysiological systems and like for efficient design and to predict the system performance measures. In this paper, the behavior of the system is analyzed using the system characteristics like average number of customers in the system, average number of customers in the queue, variability of the system size, probability of the system emptiness and coefficient of variation of the system and also the average time of a customer spends in the system and the queue. It is observed that the positive dependence between the arrival and service completions can reduce the mean queue length and variability of system size.

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