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Minimum cycle covers of Butterfly and Benes networks

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Abstract

Butterfly network is the most popular bounded-degree derivative of the hypercube network. The *benes* network consists of back-to-back butterflies. In this paper, we obtain the minimum vertex-disjoint cycle cover number for the odd dimensional butterfly networks and prove that it is not possible to find the same for the even dimensional butterfly networks and benes networks. Further we obtain the minimum edge-disjoint cycle cover number for butterfly networks.

Keywords: Butterfly networks, Benes networks, cycle partition, vertex-disjoint cycle cover, edge-disjoint cycle cover.
AMS Subject Classification(2010): 05B40, 68R10.

1 Introduction

Butterfly graphs are defined as the underlying graphs of FFT networks which can perform the Fast Fourier Transforms very efficiently. The butterfly network consists of a series of switch stages and interconnection patterns, which allows n inputs to be connected to n outputs. The benes network consists of back-to-back butterflies. As butterfly is known for FFT, benes is known for permutation routing. The butterfly and benes networks are important multistage interconnection networks, which possess attractive topologies for communication networks [8]. They have been used in parallel computing systems such as IBM, SP1/SP2, MIT Transit Project, NEC Cenju-3 and used as well in the internal structures of optical couplers [7, 13]. The multistage networks have long been used as communication networks for parallel computing [5].

Cycles in interconnection networks are useful in many applications such as indexing, embedding linear arrays and rings, computing FFT and so on [4]. The cycle partition of star graphs and arrangement graphs are studied in [1, 10]. The pancycle problem of butterfly networks is investigated in [4].

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. Let H be a subgraph of G. Two subgraphs of G are said to be vertex-disjoint if they have no vertices in common and edge-disjoint if they have no edges in common. Given $G(V, E)$, a vertex-disjoint cycle cover is a partition $\{V_1, V_2, ..., V_k\}$ of the vertex set $V(G)$ into subsets of V such that each V_i, $1 \le i \le k$ induces a cycle [9]. The minimum vertex-disjoint cycle cover problem is to minimise k [2]. We denote this minimum number by $\eta(G)$.

An edge-disjoint cycle cover is a partition $\{E_1, E_2, \dots E_k\}$ of the edge set $E(G)$ into subsets of E such that each E_i, $1 \le i \le k$ induces a cycle [9]. The minimum edge-disjoint cycle cover problem is to minimise k [2]. We denote this minimum number by $\eta'(G)$. Minimum vertex-disjoint cycle cover and minimum edge-disjoint cycle cover problems are *NP*-complete [2].

In this paper, we solve the minimum vertex-disjoint cycle cover problem for odd dimensional butterfly networks and prove that vertex-disjoint cycle cover is not possible for benes networks. Also, edge-disjoint cycle cover number is determined for butterfly networks.

2 Vertex-disjoint Cycle Cover

Butterfly and benes networks are represented as undirected graphs whose nodes represent processors and edges represent interprocessor communication links.

The set V of nodes of an r-dimensional butterfly network correspond to pairs $[w, i]$, where i is the dimension or level of a node ($0 \leq i \leq r$) and w is an r-bit binary number that denotes the row of the node. Two nodes $[w, i]$ and $[w', i']$ are linked by an edge if and only if $i' = i + 1$ and either w and w' are identical or w and w' differ in precisely the i^{th} bit. The r-dimensional butterfly is denoted by $BF(r)$. It has $(r+1)2^r$ vertices and $r2^{r+1}$ edges [8, 11].

An r-dimensional benes network has $2r+1$ levels, each level with 2^r nodes. The level 0 to level r nodes in the network form an r-dimensional butterfly. The benes network consists of back-to-back butterflies. The middle level of the benes network is shared by these butterflies [6]. An r-dimensional benes network is denoted by $BB(r)$ [13].

Manuel et al. [8] have identified new topological representations for butterfly and benes networks as diamond representations and proved that the normal and diamond representations of butterfly and benes networks are isomorphic. Normal and Diamond representations of $BF(2)$ are given in Figure 1.

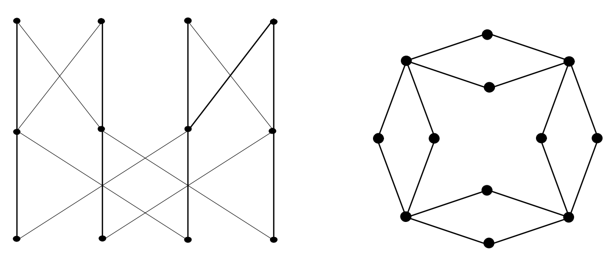

Figure 1: Normal and Diamond representations of $BF(2)$.

We denote by $N(u)$, the set of all vertices of G adjacent to u and call this set as the neighbourhood of u [3].

Two nodes $[w, i]$ and $[w', i']$ in $BF(r)$ are said to be mirror images of each other if w and w' differ precisely in the first bit. The removal of level 0 vertices $\{v_1, v_2, \ldots, v_{2^r}\}$ of $BF(r)$ gives two subgraphs H_1 and H_2 of $BF(r)$, each isomorphic to $BF(r\text{-}1)$. Since $\{v_1, v_2, \ldots, v_{2^r}\}$ is a vertex-cut of $BF(r)$, the vertices are called binding vertices of $BF(r)$. If a 4-cycle in $BF(r)$ has binding vertices then it is called a binding diamond. The edges of binding diamonds are called binding edges [8]. Such diamonds are also obtained when vertices of $BF(r)$ at level $(n + 1)$ are removed. To distinguish between the two, we call the binding diamonds defined by removing the vertices at level 0 as vertical binding diamonds and those defined by removing vertices at level $(n + 1)$ as horizontal binding diamonds. The two types of diamonds are given in Figure 2.

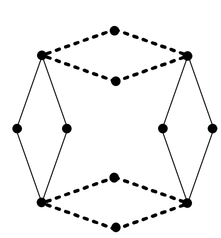

Figure 2: Horizontal and vertical binding diamonds of $BF(2)$

Lemma 2.1. [4] $BF(r)$ contains cycles of length $4d$ for all d, $1 \le d \le r$.

Lemma 2.2. [4] $BF(r)$ contains cycles of length $4d+2$ for all d, $3 \le d \le r$.

Theorem 2.3. Let $BF(2r+1)$ be an odd dimensional butterfly network. Then, $\eta(BF(2r+1)) = (2r+2)2^{2r-1}$, $r \ge 1$.

Proof. We prove the result by induction on r.
Consider $BF(3)$. There are 8 binding diamonds in $BF(3)$ and they cover all the 32 vertices of $BF(3)$. Hence, $\eta(BF(3)) = 8 = (2+2)2^{2-1}$.

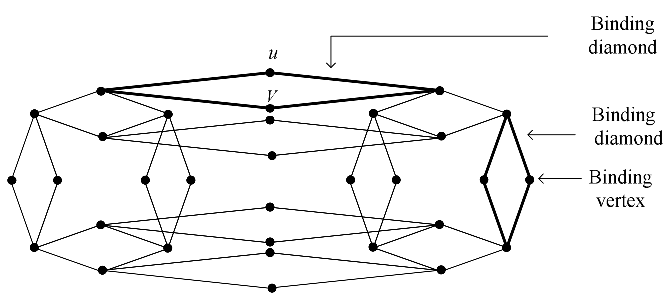

Figure 3: Binding diamonds of $BF(3)$

Assume the result to be true for $BF(k)$,where k is odd, $k \le 2r-1$. The hypothesis implies that the cycles in any vertex-disjoint cycle cover of $BF(k)$, k odd and $k \le 2r-1$ are all 4-cycles.

Consider $BF(2r+1)$. Let $\mathcal{B}$ be the set of all binding diamonds of $BF(2r+1)$. There are 2^{2r+1} number of binding diamonds in $BF(2r+1)$. We have exactly two binding vertices through which a binding diamond passes. Further, two binding vertices u and v belonging to a binding diamond cannot lie on two different vertex-disjoint cycles as $N(u)$ and $N(v)$ are equal and are of cardinality 2. Removal of all the binding vertices leaves 4 copies of $BF(2r-1)$. By induction hypothesis, the binding vertices of the 4 copies of $BF(2r-1)$ are all covered by the 4-cycles in $\mathcal{B}$. Removal of the binding vertices of all the 4 copies of $BF(2r-1)$ leave 16 copies of $BF(2r-3)$. Again by induction hypothesis, each $BF(2r-3)$ is covered by only 4-cycles and $\eta(BF(2r-3)) = (2r-2)2^{2r-5}$. Let τ be the collection of all these 4-cycles. Then $|\tau| = 16 \times (2r-2)2^{2r-5} = (2r-2)2^{2r-1}$. Now, $\mathcal{B} \cup \tau$ is a vertex-disjoint cycle cover of $BF(2r+1)$ such that every member of $\mathcal{B} \cup \tau$ is a 4-cycle. Further, $|\mathcal{B} \cup \tau| = \eta(BF(2r+1)) = 2^{2r+1} + (2r-2)2^{2r-1} = (2r+2)2^{2r-1}$. ■

Remark 2.4. Any vertex-disjoint cycle cover of $BF(r)$ where r is even, contains $r2^{r-2}$ number of 4-cycles and 2^r isolated vertices. Isolated vertices of $BF(2)$ are given in Figure 4.

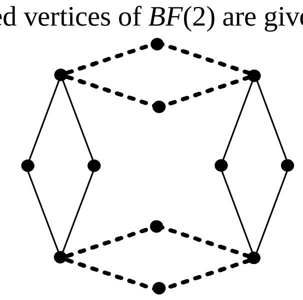

Figure 4: Isolated vertices of $BF(2)$.

Theorem 2.5. There does not exist a vertex-disjoint cycle cover for the r-dimensional benes network $BB(r)$, $r \geq 1$.

Proof. Case 1. r is odd.

Let $r = 2k+1$, $k \geq 1$. Since $BB(r)$ consists of back to back butterflies $BF(2k+1)$ obtained by merging level 0 vertices, by Theorem 2.3, one copy of $BF(2k+1)$ has a vertex-disjoint cycle cover, say $\mathcal{C}$. Deleting vertices of level 0 which are already covered by cycles in $\mathcal{C}$ from the second copy of $BF(2k+1)$ leave 2 copies of $BF(2k)$ for which there does not exist a vertex-disjoint cycle cover. Hence, there does not exist a vertex-disjoint cycle cover for $BB(r)$.

Case 2. r is even.

Let $r = 2k$. The vertex set of $BB(r)$ can be partitioned into three sets, one inducing $BF(2k)$ and the other two inducing two copies of $BF(2k–1)$. Even though there exists a vertex-disjoint cycle cover for $BF(2k–1)$, there does not exist a vertex-disjoint cycle cover for $BF(2k)$. ■

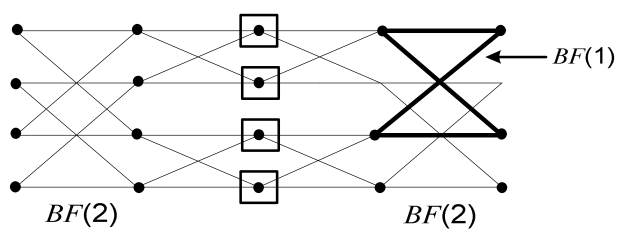

Figure 5: A 2-dimensional benes network with level zero vertices marked.

3 Edge-disjoint Cycle Cover

For any graph G, a lower bound for $\eta'(G)$ has been obtained in [12]. In this section, we prove that the bound is sharp for butterfly network.

Theorem 3.1. [12] Let G be a graph with an edge-disjoint cycle cover. Let Δ be the maximum degree in G. Then $\eta'(G) \geq (\Delta/2)$.

Theorem 3.2. Let $BF(r)$, $r \geq 1$ be a butterfly network. Then $\eta'(BF(r)) = 2$.

Proof. $BF(r)$ contains 2^{r-1} horizontal binding diamonds HB_i and 2^{r-1} vertical binding diamonds VB_i, $1 \leq i \leq 2^{r-1}$. Let u_i be a 2 degree vertex in HB_i. Then the neighbourhood of u_i has two vertices $u_i(1)$ and $u_i(2)$ of degree four.

Similarly VB_i has a 2 degree vertex v_i and its neighbouring vertices are $v_i(1)$ and $v_i(2)$ with degree four. In other words, $N(u_i) = \{ u_i(1) , u_i(2) \}$ and $N(v_i) = \{ v_i(1) , v_i(2) \}$, $1 \leq i \leq 2^{r-1}$. ■

In the following procedure, $u \rightarrow v$ denotes a traversal from vertex u to vertex v. Again, removing all the binding diamonds of $BF(r)$ leaves 4 copies of $BF(r – 2)$, two on the top and two at the bottom. Let $BF^{TL}(r – 2)$, $BF^{TR}(r – 2)$, $BF^{BL}(r – 2)$ and $BF^{BR}(r – 2)$ denote the top left, top right, bottom left and bottom right copies of $BF(r – 2)$.

Procedure CYCLECOVER *BF*(*r*)
Input: An r-dimensional butterfly network $BF(r)$.

Algorithm:
Step 1: Begin C_1 .
Initialize $i = 1$ and $j = 2^{r-1}$.
Step 2: $u_i(1) \rightarrow u_i(2)$ traversing 2 edges in HB_i and passing through u_i.
$u_i(2) \rightarrow v_j(1)$ traversing $r – 2$ edges in $BF^{TR}(r – 2)$.

$v_j(1) \rightarrow v_j(2)$ traversing 2 edges in VB_j and passing through v_j.
$v_j(2) \rightarrow u_j(2)$ traversing $r - 2$ edges in $BF^{BR}(r - 2)$.

Step 3: Increment or decrement i in HB_i and j in VB_j

Step 4: $u_j(2) \rightarrow u_j(1)$ traversing 2 edges in HB_j and passing through u_j.
$u_j(1) \rightarrow v_i(2)$ traversing $r - 2$ edges in $BF^{BL}(r - 2)$.
$v_i(2) \rightarrow v_i(1)$ traversing 2 edges in VB_i and passing through v_i .
Traverse $r - 2$ edges in $BF^{TL}(r - 2)$.

Step 5: Increment or decrement i in HB_i and j in VB_j.

Step 6: Repeat.

Output: A cycle C_1.

Proof of correctness: $BF(r)$ contains 2^{r-1} horizontal and 2^{r-1} vertical binding diamonds. C_1 traverses two edges in each of the 2^r binding diamonds. The number of edges used in 2^r binding diamonds is 2^{r+1}. C_1 traverses $(r - 2)$ edges in each of the butterflies of smaller dimension, namely $BF^{TL}(r - 2)$, $BF^{TR}(r - 2)$, $BF^{BL}(r - 2)$ and $BF^{BR}(r - 2)$ while moving from HB_i to VB_j, $1 \le i, j \le 2^{r-1}$ and vice-versa. The number of edges traversed in $BF(r - 2)$ is $2^r(r - 2)$.

To proceed we prove that the complement of C_1 in $BF(r)$ is another cycle C_2 isomorphic to C_1. We know that $BF(r)$ is the union of 4-cycles. By Procedure CYCLECOVER $BF(r)$, which generates a cycle C_1, the subgraph of C_1 in any 4-cycle C^* in $BF(r)$ is isomorphic to its complement in C^*. Thus the subgraph induced by the complementary subgraphs of all the four cycles in $BF(r)$ induce a cycle C_2 isomorphic to C_1. Hence, $| E(C_1) | = | E(C_2) | = 2^{r+1} + 2^r(r - 2) = r\,2^r$.

Therefore, $| E(C_1) | + | E(C_2) | = r\,2^{r+1} = | E(BF(r)) |$. The procedure is illustrated in Figure 6.

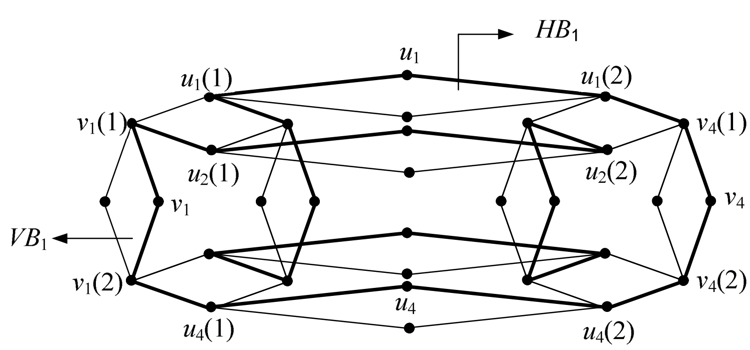

Figure 6: Two edge-disjoint cycles of BF_3.

4 Conclusion

In this paper, minimum vertex-disjoint cycle cover number for odd dimensional butterfly networks $BF(2r+1)$, $r \ge 1$ is obtained. It is also proved that a vertex-disjoint cycle cover does not exist for even dimensional butterfly networks and benes networks $BB(r)$, $r \ge 1$. Further, we obtain minimum edge-disjoint cycle cover number for $BF(r)$.

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