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Harvesting and Hopf Bifurcation in a prey-predator model with Holling Type IV Functional Response

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Abstract.

This paper aims to study the effect of Harvesting on predator species with time-delay on a Holling type-IV prey-predator model. Harvesting has a strong impact on the dynamic evolution of a population. Two delays are considered in the model of this paper to describe the time that juveniles of prey and predator take to mature. Dynamics of the system is studied in terms of local and Hopf bifurcation analysis. Finally, numerical simulation is done to support the analytical findings.

Keywords: Prey-Predator model, Harvesting, Time-delay, Hopf bifurcation, Chaotic oscillations, Periodic orbit.

AMS Subject Classification(2010): 92B05.

1 Introduction

The predator-prey model is a kind of "pursuit and evasion" system in which the prey try to evade the predator and the predator try to catch the prey if they interact. Two species continuous time models of predator (natural enemy)–prey (pest) type of interaction with functional response of the predator to its prey have been extensively discussed in the literature [2-6]. Andrews [1] suggested a function

$$\varphi(x) = \frac{mx}{a + bx + x^2},$$

called the Monod-Haldane function and also called a Holling type- IV function. Sokol and Howell [11] proposed a simplified Holling type-IV function of the form

$$\varphi(x) = \frac{mx}{a + x^2}.$$

Many authors researched Holling type IV population models [8]. Keeping this in mind, we consider a prey-predator model with Holling type-IV functional response and harvesting of predator (natural enemy) species:

$$\begin{aligned} \frac{dX}{dt} &= rX\left(1 - \frac{X(t-\tau_1)}{K}\right) - \frac{\alpha XY}{1 + e_1 X^2}, \\ \frac{dY}{dt} &= b_0 Y(t-\tau_2) - (d_0 - \frac{\beta X}{1 + e_2 X^2})Y - qE_0 Y. \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ denote the population densities of prey (pest) and predator (natural enemy) at time t, respectively. All coefficients are positive constants. Specially, r and K represent the intrinsic growth rate and carrying capacity of prey in the absence of predation, respectively. $e_i, i = 1,2$ is the half-saturation constant. The predator consumes the prey with functional response $\frac{\alpha XY}{1 + e_1 X^2}$, known as Holling type-IV response and contributes to its growth rate $\frac{\beta XY}{1 + e_2 X^2}$. b_0

is a constant birth rate and d_0 is the maximum death rate in the absence of food. All the metabolic energy a predator obtains from its food goes into physiological maintenance, rather than into enhancing reproductive effort. In addition, we assume that the juveniles of prey and predator take τ_1 and τ_2 units of time to mature, respectively. Moreover, the catch rate function qE_0Y is based on the catch-per-unit-effort (CPUE) hypothesis. Here, q is the catchability coefficient of the predator species and E_0 is the harvesting effort.

The rest of this paper is organized as follows: In section 2, we investigate the stability of positive equilibrium and occurrence of Hopf bifurcation. In section 3, some numerical simulations are carried out to justify the analytic results obtained in the manuscript. Section 4 deals with the conclusions of the paper.

2 Stability And Local Hopf Bifurcation

Let (x^*, y^*) be the only interior equilibrium of the system (1). Then the linearization of Equation (1) at (x^*, y^*) is given by

$$\begin{aligned}\frac{dx}{dt} &= \left\{r\left(1-\frac{x^*}{K}\right)-\frac{\alpha y^*}{1+e_1x^{*2}}\right\}x-\frac{\alpha yx^*}{1+e_1x^{*2}}-\frac{rx^*x(t-\tau_1)}{K}\\ \frac{dy}{dt} &= \frac{\beta xy^*}{1+e_2x^{*2}}+\left(\frac{\beta x^*}{1+e_2x^{*2}}-d_0-qE_0\right)y+b_0y(t-\tau_2)\end{aligned} \tag{2}$$

whose characteristic equation is

$$\lambda^2+A\lambda+(B_1\lambda+B_2)e^{-\lambda\tau_1}+(C_1\lambda+C_2)e^{-\lambda\tau_2}+D_1e^{-\lambda(\tau_1+\tau_2)}+D_2=0 \tag{3}$$

where

$$A=\frac{\alpha y^*}{1+e_1x^{*2}}-r\left(1-\frac{x^*}{K}\right)+d_0+qE_0-\frac{\beta x^*}{1+e_2x^{*2}},\qquad B_1=\frac{rx^*}{K},\qquad B_2=\frac{rx^*}{K}\left(d_0+qE_0-\frac{\beta x^*}{1+e_2x^{*2}}\right),$$

$$C_1=-b_0,\ C_2=-b_0\left\{\frac{\alpha y^*}{1+e_1x^{*2}}-r\left(1-\frac{x^*}{K}\right)\right\},\quad D_1=\frac{-b_0rx^*}{K},$$

$$D_2=r\left(1-\frac{x^*}{K}\right)\left\{\frac{\beta x^*}{1+e_2x^{*2}}-d_0-qE_0\right\}+\frac{\alpha y^*(d_0+qE_0)}{1+e_1x^{*2}}$$

Case 1. $\tau_1=\tau_2=0$.

In this case, Equation (3) becomes

$$\lambda^2+(A+B_1+C_1)\lambda+B_2+C_2+D_1+D_2=0, \tag{4}$$

All roots of Equation (4) have negative real parts if and only if

(H_1) $A+B_1+C_1>0,\ B_2+C_2+D_1+D_2>0$

So the equilibrium point (x^*, y^*) is locally asymptotically stable when (H_1) holds and unstable if either of these conditions is not satisfied. Hence, in the absence of time delay, the system is locally asymptotically stable if and only if $A+B_1+C_1>0$ and $B_2+C_2+D_1+D_2>0$ hold simultaneously.

Case 2. $\tau_1=0, \tau_2>0$.

Here, Equation (3) becomes

$$\lambda^2+(A+B_1)\lambda+B_2+D_2+(C_1\lambda+C_2+D_1)e^{-\lambda\tau_2}=0 \tag{5}$$

Let $\lambda=iw$ be one such root. Substituting this in equation (5) and equating real and imaginary parts, we get

$$(C_2+D_1)\cos w\tau_2+C_1w\sin w\tau_2=w^2-(B_2+D_2) \tag{5a}$$

$$C_1 w\cos w\tau_2 - (C_2 + D_1)\sin w\tau_2 = -w(A+B_1) \quad (5b)$$

Squaring and adding the equations (5a) and (5b) we get

$$w^4 + \{(A+B_1)^2 - C_1^2 - 2(B_2+D_2)\} + (B_2+D_2)^2 - (C_2+D_1)^2 = 0 \quad (6)$$

From (6), it follows that if

(H_2) $(A+B_1)^2 - C_1^2 - 2(B_2+D_2) > 0, \qquad (B_2+D_2)^2 - (C_2+D_1)^2 > 0$

holds, then Equation (6) has no positive roots. Hence, all roots of (5) have negative real parts when $\tau_2 \in (0,\infty)$ under (H_1) and (H_2).

If (H_1) and (H_3) $(B_2+D_2)^2 - (C_2+D_1)^2 < 0$ hold, then (6) has a unique positive root w_0^2. Substituting w_0^2 into (5a) and (5b), we have

$$\tau_{2_n} = \frac{1}{w_0}\cos^{-1}\left[\frac{w_0^2\{C_2+D_1-C_1(A+B_1)\} - (C_2+D_1)(B_2+D_2)}{C_1^2 w_0^2 + (C_2+D_1)^2}\right] + \frac{2n\pi}{w_0}, n = 0,1,2,.....$$

Again, if

(H_4) $C_1^2 + 2(B_2+D_2) - (A+B_1)^2 > 0, (B_2+D_2)^2 - (C_2+D_1)^2 > 0$ and

$$\left[C_1^2 + 2(B_2+D_2) - (A+B_1)^2\right]^2 > 4\left[(B_2+D_2)^2 - (C_2+D_1)^2\right]$$

hold, then there are two positive solutions $w_\pm^2$. Substituting $w_\pm^2$ into (5a) and (5b), we have

$$\tau_{2_k}^{\pm} = \frac{1}{w_\pm}\cos^{-1}\left[\frac{w_\pm^2\{C_2+D_1-C_1(A+B_1)\} - (C_2+D_1)(B_2+D_2)}{C_1^2 w_\pm^2 + (C_2+D_1)^2}\right] + \frac{2k\pi}{w_\pm}, \quad (6a)$$

$$k = 0,1,2,.....$$

Differentiating equation (5) with respect to τ_2, we obtain

$$\left\{2\lambda + A + B_1 - \tau_2 e^{-\lambda\tau_2}(C_1\lambda + C_2 + D_1) + C_1 e^{-\lambda\tau_2}\right\}\frac{d\lambda}{d\tau_2} - \lambda e^{-\lambda\tau_2}(C_1\lambda + C_2 + D_1) = 0 \quad (7a)$$

Therefore,

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{2\lambda + A + B_1}{-\left\{\lambda^3 + \lambda^2(A+B_1) + (B_2+D_2)\lambda\right\}} + \frac{C_1}{\lambda(C_1\lambda + C_2 + D_1)} - \frac{\tau_2}{\lambda} \quad (7b)$$

by using $e^{-\lambda\tau_2} = \dfrac{-\left\{\lambda^2 + \lambda(A+B_1) + B_2 + D_2\right\}}{(C_1\lambda + C_2 + D_1)}$.

Thus, $sign\left\{\dfrac{d(\mathrm{Re}\,\lambda)}{d\tau_2}\right\}_{\lambda=iw} = sign\left\{\mathrm{Re}\left(\dfrac{d\lambda}{d\tau_2}\right)^{-1}\right\}_{\lambda=iw}$

$$= sign\left[\frac{(A+B_1)^2 + 2(1-B_2-D_2)}{(A+B_1)+(1-B_2-D_2)^2} - \frac{C_1^2}{C_1^4 w_0^2 + (C_2+D_1)^2}\right] \quad (7c)$$

Theorem 2.1. If (H_1) and (H_3) hold, then the equilibrium (x^*, y^*) is unstable for $\tau_2 < \tau_{2_0}$ and asymptotically stable for $\tau_2 > \tau_{2_0}$. Further, as τ increases through τ_{2_0}, (x^*, y^*) bifurcates into small amplitude periodic solutions where $\tau_{2_0} = \tau_{2_n}$ as $n = 0$.

Proof. For $\tau_2 = 0$, (x^*, y^*) is unstable if condition (H_1) holds. Hence, by Butler's lemma, (x^*, y^*) remains unstable for $\tau_2 < \tau_{2_0}$. Now we show that

$$\left[\frac{d(\mathrm{Re}\,\lambda)}{d\tau_2}\right]_{\tau=\tau_0, w=w_0} > 0 \quad (8)$$

This signifies that there exists at least one eigen value with positive real part for $\tau_2 < \tau_{2_0}$. Moreover, the conditions of Hopf bifurcation (Hale and Lunel, (1993)) are then satisfied yielding the required periodic solution.

From (7c), it follows that

$$\text{sign}\left\{\frac{\text{d}(\text{Re}\,\lambda)}{d\tau_2}\right\}_{\lambda=\text{iw}_0} = sign\left[\frac{(A+B_1)^2+2(1-B_2-D_2)}{(A+B_1)+(1-B_2-D_2)^2} - \frac{C_1^2}{C_1^4 w_0^2+(C_2+D_1)^2}\right].$$

Therefore, $\left[\frac{d(\text{Re}\,\lambda)}{d\tau_2}\right]_{\tau=\tau_0, w=w_0} > 0$.

Therefore, the transversability conditions holds and hence Hopf bifurcation occurs at $w = w_0, \tau_2 = \tau_{2_0}$. This completes the proof. ■

Theorem 2.2. Let $\tau_{2_k}^{\pm}$ be defined in (6a). If (H_1) and (H_4) hold, then there exists a positive integer *m* such that there are *m* switches from stability to instability and from instability to stability. In other words, when $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), \cdots, (\tau_{m-1}^-, \tau_m^+)$, the equilibrium (x^*, y^*) is unstable and when $\tau \in [\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), \cdots, (\tau_{m-1}^+, \tau_m^-)$, (x^*, y^*) is stable. Therefore, there are bifurcations at (x^*, y^*) when $\tau = \tau_k^{\pm}, k = 0,1,2,...$

Proof. If conditions (H_1) and (H_4) hold, then to prove the theorem we need only to verify the transversability conditions.

$$\left[\frac{d(\text{Re}\,\lambda)}{d\tau_2}\right]_{\tau=\tau_{2_k}^+} < 0 \quad \text{and} \quad \left[\frac{d(\text{Re}\,\lambda)}{d\tau_2}\right]_{\tau=\tau_{2_k}^-} > 0$$

From (7c), it follows that

$$\text{sign}\left\{\frac{\text{d}(\text{Re}\,\lambda)}{d\tau_2}\right\}_{\lambda=\text{iw}+} = sign\left[\frac{(A+B_1)^2+2(1-B_2-D_2)}{(A+B_1)+(1-B_2-D_2)^2} - \frac{C_1^2}{C_1^4 w_+^2+(C_2+D_1)^2}\right]$$

and therefore $\left[\frac{d(\text{Re}\,\lambda)}{d\tau_2}\right]_{w=w+, \tau=\tau_{2_k}^+} < 0$.

Again,

$$\text{sign}\left\{\frac{\text{d}(\text{Re}\,\lambda)}{d\tau_2}\right\}_{\lambda=\text{iw}-} = sign\left[\frac{(A+B_1)^2+2(1-B_2-D_2)}{(A+B_1)+(1-B_2-D_2)^2} - \frac{C_1^2}{C_1^4 w_-^2+(C_2+D_1)^2}\right]$$

and therefore $\left[\frac{d(\text{Re}\,\lambda)}{d\tau_2}\right]_{w=w-, \tau=\tau_{2_k}^-} > 0$.

Hence, the transversality conditions hold.

Case 3. $\tau_1 > 0, \tau_2 > 0$.

We consider Equation (3) with τ_2 in its stable interval. Regard τ_1 as a parameter. Without loss of generality, we consider system (1) under the assumptions (H_1) and (H_3). Let $\lambda = iw$ be one such root. Substituting this in equation (3) and equating the real and imaginary parts we get

$$\cos w\tau_1 \{B_2 + D_1 \cos w\tau_2\} + \sin w\tau_1 \{B_1 w - D_1 \sin w\tau_2\} = w^2 - D_2 - C_1 w \sin w\tau_2 - C_2 \cos w\tau_2 \quad \text{(9a)}$$

$$\cos w\tau_1 \{B_1 w - D_1 \sin w\tau_2\} - \sin w\tau_1 \{B_2 + D_1 \cos w\tau_2\} = C_2 \sin w\tau_2 - C_1 w \cos w\tau_2 - Aw \quad \text{(9b)}$$

Squaring and adding equation (9a) and (9b), we get

$$w^4 + \tilde{A}w^2 + D_2^2 + C_2^2 - D_1^2 - B_2^2 + 2\tilde{B}\sin w\tau_2 + 2\tilde{C}\cos w\tau_2 = 0 \quad \text{(10)}$$

where $\tilde{A} = C_1^2 + A^2 - B_1^2 - 2D_2$, $\tilde{B} = -C_1 w^3 + D_2 C_1 w - AwC_2 + B_1 D_1 w$ and $\tilde{C} = -w^2 C_2 + D_2 C_2 + Aw^2 C_1 - B_2 D_1$.

We define $F(w) = w^4 + \tilde{A}w^2 + D_1^2 + C_2^2 - D_1^2 - B_2^2 + 2\tilde{B}\sin w\tau_2 + 2\tilde{C}\cos w\tau_2$ and assume that:

(H_4) $\{(D_2 + C_2)^2 - (D_1 + B_2)^2\} < 0$ holds.

It is easy to check that $F(0) < 0$ and $F(\infty) > 0$. We can obtain that (10) has finite positive roots $w_1, w_2, ...w_k$. For every fixed $w_i, i = 1,2,......k$, there exists a sequence $\{\tau_{1_i}^j \mid j = 1,2,3,...\}$ such that (10) holds. Let $\tau_{1_0} = \min\{\tau_{1_i}^j \mid i = 1,2,......,k;\ j = 1,2,3......\}$. When $\tau_1 = \tau_{1_0}$. Equation (3) has a pair of purely imaginary roots $\pm iw^0$ for $\tau_2 \in [0, \tau_{2_0})$ .In the following we assume that

(H_6) $\left[\dfrac{d(\mathrm{Re}\,\lambda)}{d\tau_1}\right]_{\lambda = iw^0} \neq 0$.

Therefore, by the general Hopf bifurcation theorem for FDEs in Hale [7], we have the following result on stability and bifurcation in system (1). ■

Theorem 2.3. For the system (1), suppose (H_1), (H_3) and (H_5) are satisfied and $\tau_2 \in [0, \tau_{2_0})$. Then the equilibrium (x^*, y^*) is asymptotically stable when $\tau_1 \in (0, \tau_{1_0})$ and system (1) undergoes a Hopf bifurcation at (x^*, y^*) when $\tau_1 = \tau_{1_0}$.

3 Numerical Simulation and Discussion

In this section, we present numerical simulation to illustrate the results obtained in the previous sections. The system (1) is integrated using the fourth order Runge – Kutta Method with the help of MATLAB software package under the following set of parameters.

(a) $r = 20, K = 120, \alpha = 10, e_1 = 2, b_0 = 2.6, d_0 = 2.5, \beta = 4, e_2 = 2, q = 0.4, E_0 = 0.5$

The positive equilibrium point of Equation (1) with data (a) is $X^* = 0.0250313, Y^* = 2.00209$.

Then, we can easily obtain that is (H_1) and (H_4) to be satisfied. By computation, we have $w_0 = 2.1415, \tau_{2_0} = 0.245831$. Here we show that the transversal condition (8) is satisfied as: $\left[\dfrac{d(\mathrm{Re}\,\lambda)}{d\tau_2}\right]_{\tau = \tau_0} = 1.20339 > 0$. Therefore, (X^*, Y^*) is unstable for $\tau_2 < \tau_{2_0} = 0.245831$ and asymptotically stable for $2.5 > \tau_2 > \tau_{2_0} = 0.245831$.When $\tau_2 = \tau_{2_0}$, Equation(1) with data (a) undergoes a Hopf bifurcation at (X^*, Y^*), that is there is a small amplitude periodic solution around (X^*, Y^*) when $\tau_1 = 0$ and τ_2 is close to τ_{2_0} which is shown in Figure (1a) and Figure(1b).

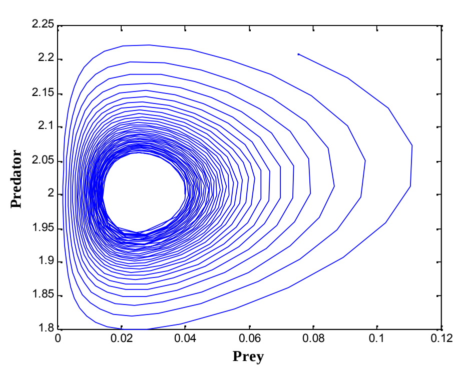

Figure 1a: when $\tau_1 = 0, \tau_2 = 0.2$.

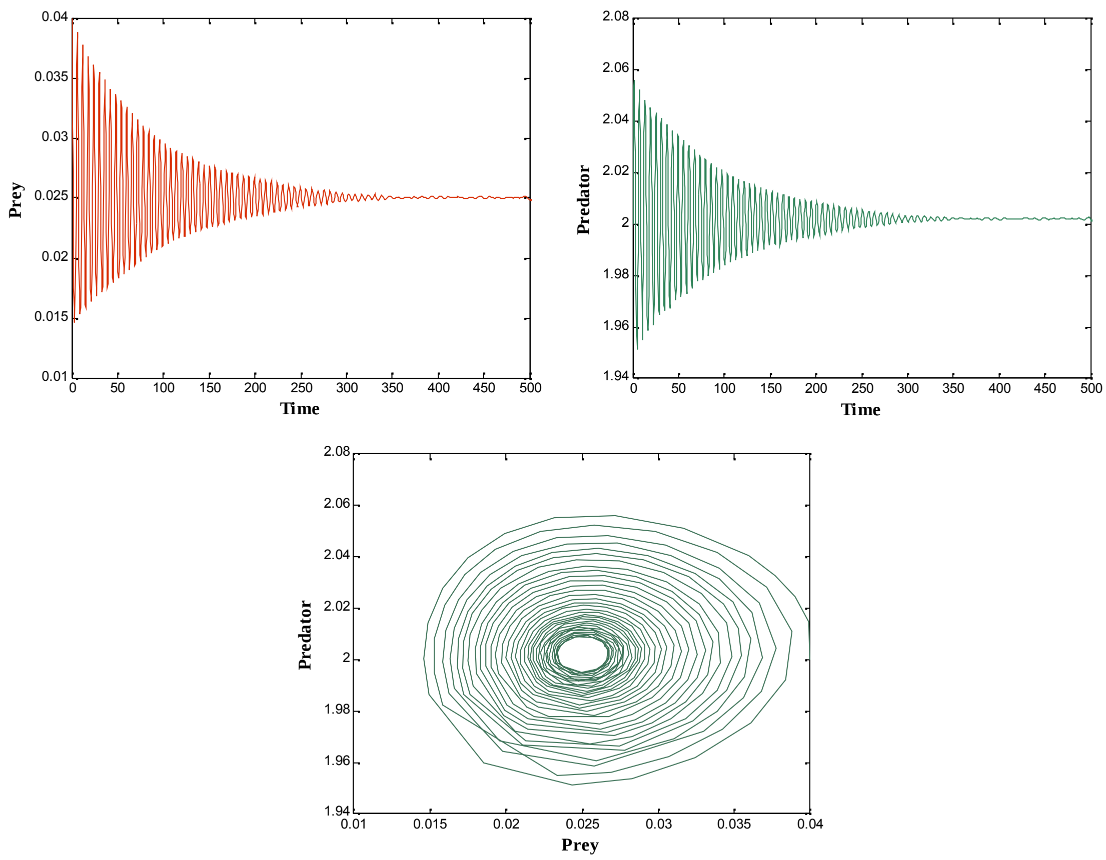

Figure 1b: when $\tau_1 = 0, \tau_2 = 0.3$

Regard τ_1 as a parameter and let $\tau_1 = 0.034$. From Figures (2a) and (2b) we can easily see that when $\tau_1 = 0.034, \tau_2 = 0.2 < \tau_{2_0}$ the system is unstable and when $\tau_1 = 0.034, \tau_2 = 0.3 > \tau_{2_0}$ the system becomes stable that is there is a small amplitude periodic solution around (X^*, Y^*) when $\tau_1 = 0.034, \tau_2 = \tau_{2_0}$. But when τ_2 reaches 2.5 the system becomes unstable, it means stability of the system lies in $[0.245831,\ 2.5]$. Thus, delay plays an important role in Hopf bifurcation.

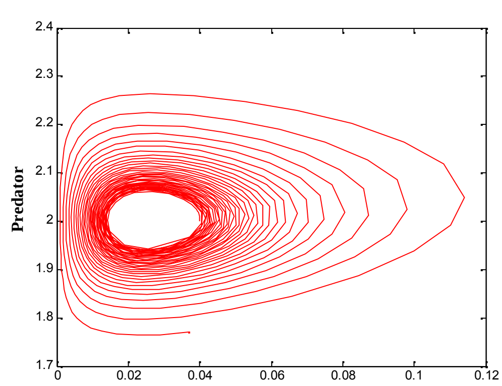

Figure 2a: when $\tau_1 = 0.034, \tau_2 = 0.2$

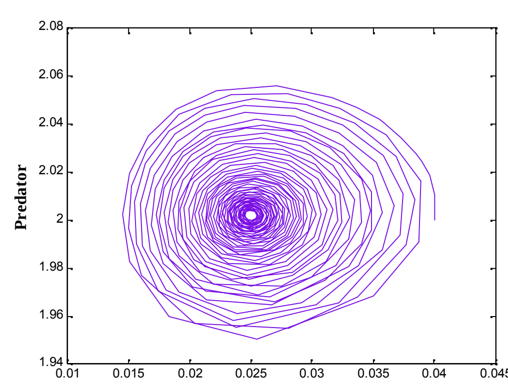

Figure 2b: when $\tau_1 = 0.034, \tau_2 = 0.3$

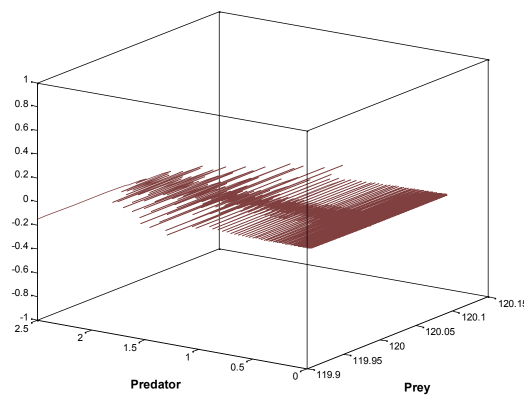

Figure 2c: when $\tau_1 = 0, \tau_2 = 0$

When there is no delay, the system shows chaotic behavior that means the system is unstable, refer Figure (2c). Thus, from numerical simulation we have observed that when population of prey is very small, maturation delay in predator plays an important role in stability of the system. When maturation time of predator lies between 0.245831 and 2.5, the system remains stable and the stability of the system is lost when $\tau_2 \geq 2.5$, refer Figure 3. Here, we can show ecological interpretation of maturation delay. Some insects, such as ladybirds are very helpful to farmers, because they eat pest species such as greenfly, whitefly and mealy bug. Ladybirds are natural enemies of pests. If maturation delay of natural enemies is $\tau_2 \geq 2.5$, the population of pests will increase and this is harmful for crops. For good production, it is necessary maturation time of natural enemies (predator) lies between 0.245831 and 2.5.

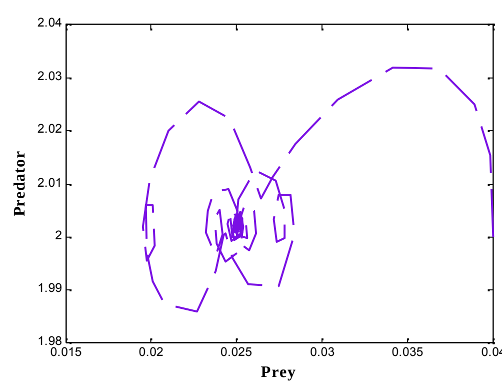

Figure 3: when $\tau_1 = 0.034, \tau_2 = 2.5$

Now, we see effect of Harvesting on the system. Harvesting has a strong impact on the dynamic evolution of a population. To a certain extent, it can control the long-term stationary density of population efficiently. However, it can also lead to the incorporation of a positive extinction probability and therefore to potential extinction in finite time. We take harvesting of predator species. If the population of ladybirds increases out of limit, it will create problem for the human beings. So we had taken harvesting of ladybirds (predator). Figure (4) is the plots of prey population and predator population against time for different value of E_0 at $\tau_2 = 0.2$. Figure 4 shows that predator population decreases with E_0 increases and becomes extinct if $E_0 \geq 0.5$. Here, we can also note that the prey population increases with the increase of E_0 and reaches to the peak level.

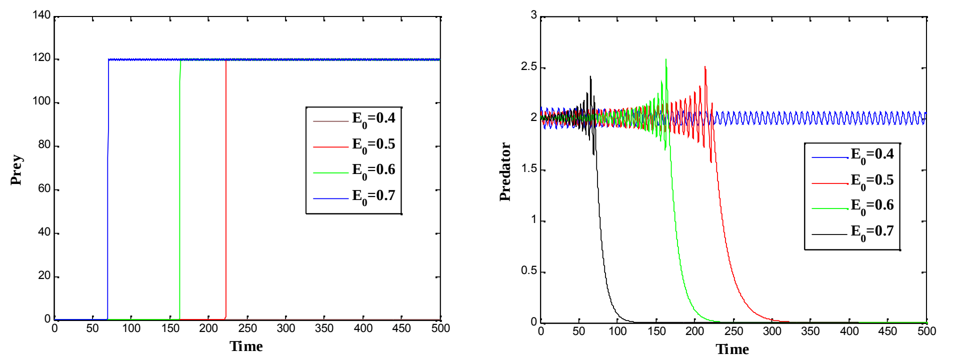

Figure 4: when $\tau_1 = 0, \tau_2 = 0.2$

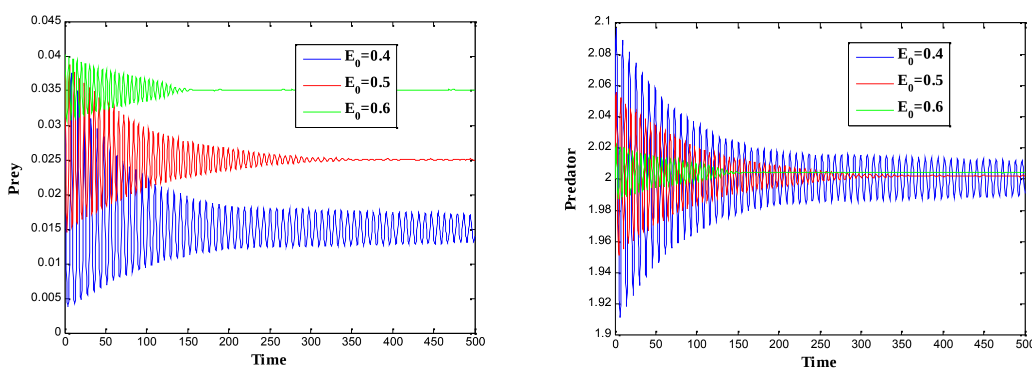

Figure 5: when $\tau_1 = 0, \tau_2 = 0.3$.

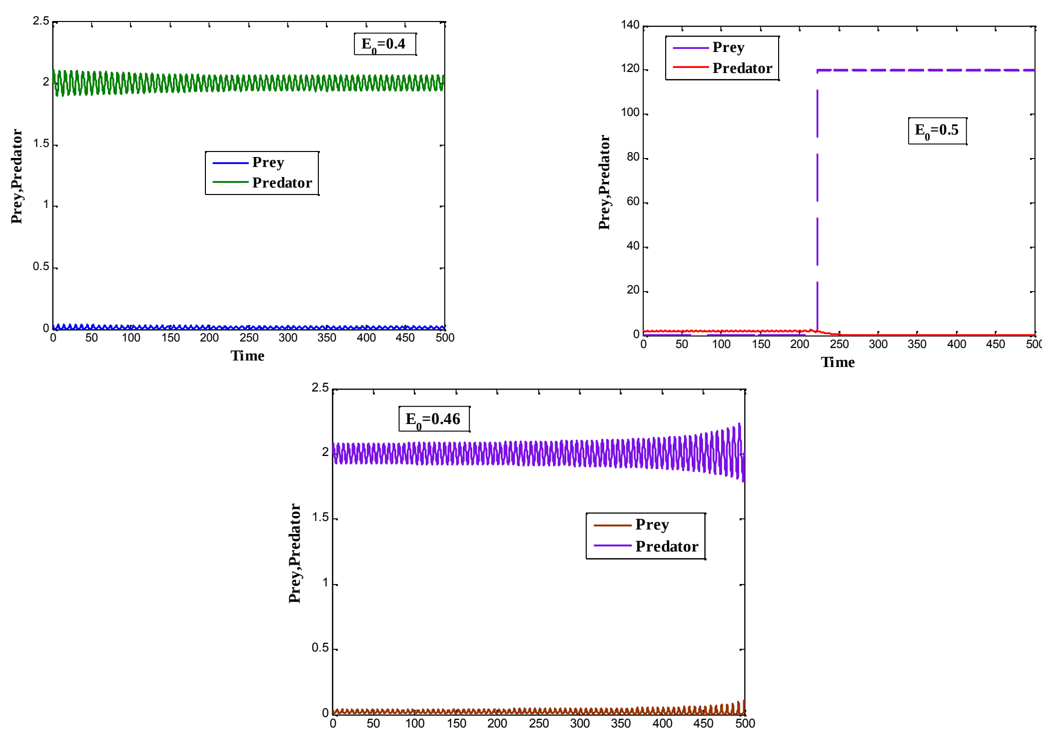

Figure 6: when $\tau_1 = 0, \tau_2 = 0.2$

From Figures 4 and 6 we can easily see that when the system is unstable that is. when $\tau_2 = 0.2$, the range of Harvesting is $(0.4,0.5)$. Here we can note that when $E_0 < 0.4$ the prey population becomes extinct and however for $E_0 \geq 0.5$ the predator population becomes extinct. Both of these cases do not have any biological meaning. Figure 5 is the plots of the prey population and predator population against time for different value of E_0 at $\tau_2 = 0.3$. Figure 5 shows that predator population decreases with E_0 increases and the prey population increases with E_0 increases. We can also note that the prey and predator population will not tend to extinction if $0.4 < E_0 < 0.7$. For good production it is necessary E_0 lies between 0.4 and 0.7.

4 Conclusion

In this paper we carried out simulations and found that when the system is unstable, that is when $\tau_2 = 0.2$, range of Harvesting is $(0.4,0.5)$. We had taken ladybirds as natural enemy (predator) and pests as prey. If the maturation delay of ladybirds is $\tau_2 \geq 2.5$, the population of pests will increase and it will be harmful for crops. So the maturation delay of ladybirds should less than 2.5. We had taken harvesting of ladybirds for controlling the population of them. If the population of ladybirds increases out of limit, it will create problems for humans. For good production it is necessary E_0 lies between 0.4 and 0.7. Hence, we conclude from our analysis that the stability properties of the system could switch with the Harvesting with time delay that is incorporated on different densities in the model.

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