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d-ideals and injective ideals in a distributive lattice

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Abstract

The concepts of d-ideals and injective ideals are introduced in a distributive lattice with respect to derivations. d-ideals are characterized in terms of principal ideals of a distributive lattice. Further, an equivalent condition is derived for a d-ideal to become an injective ideal. Also the Stone's theorem for ideals of a distributive lattice is extended to the case of injective ideals.

Keywords: Derivation, kernel, d-ideal, injective ideal, d-prime ideal.
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1 Introduction

For a ring $(R, +, \cdot)$ where $+$ and $\cdot$ denote two binary operations, we recall the derivation of R as a mapping $f : R \longrightarrow R$ satisfying the following properties:

$$f(x \cdot y) = (x \cdot f(y)) + (f(x) \cdot y)$$
$$f(x + y) = f(x) + f(y)$$

H.E. Bell, L.C. Kappe [3] and K. Kaya [5] have studied derivations in rings and prime rings after Posner [6] had given the definition of the derivation in ring theory. Szasz have introduced and developed the theory of derivations in lattice structure. In a series of papers [7] and [8] he established the main properties of derivations of lattices. L. Ferrari [4] extended these concepts to lattices and he embedded any lattice having some additional properties into the lattice of its derivations.

G. Birkhoff [2], George Grätzer, G. Szász and many authors have studied about various types of ideals and congruences all intimated to some extent the behavior of ideals in a distributive lattice.

The aim of this paper is to study the structure of certain classes of ideals in a distributive lattice with respect to a derivation. On this way, the notions of d-ideals and injective ideals are introduced and their preliminary properties are studied in a distributive lattice. These classes of ideals are then characterized in terms of principal ideals. A necessary and sufficient condition is established for a d-ideal to become an injective ideal. The concept of d-prime ideals is introduced and the relations between the class of all injective ideals and the class of all d-prime ideals are obtained. Finally, the famous and crucial result of M.H. Stone is extended to the case of injective ideals and d-prime ideals.

2 Preliminaries

We give some elementary aspects and important results which are used in the sequel of this paper.

Definition 2.1. *[1] An algebra* $(L, \wedge, \vee)$ *of type* $(2, 2)$ *is called a distributive lattice if for all* $x, y, z \in L$, *it satisfies the following properties* $(1), (2), (3)$ *and* (4) *along with* (5) *or* $(5')$

(1) $x \wedge x = x, x \vee x = x$
(2) $x \wedge y = y \wedge x, x \vee y = y \vee x$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z)$
(4) $(x \wedge y) \vee x = x, (x \vee y) \wedge x = x$
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Remark 2.2. *The least element of a distributive lattice is denoted by* 0. *Throughout this article* L *stands for a distributive lattice with* 0, *unless otherwise mentioned.*

Definition 2.3. *[1] Let* $(L, \wedge, \vee)$ *be a lattice. A partial ordering relation* $\leq$ *is defined on* L *by* $x \leq y$ *if and only if* $x \wedge y = x$ *and* $x \vee y = y$. *In this case, the pair* $(L, \leq)$ *is called a Partially Ordered set or simply POset.*

Definition 2.4. *[2] A non-empty subset* A *of* L *is called an ideal(filter) of* L *if* $a \vee b \in A (a \wedge b \in A)$ *and* $a \wedge x \in A (a \vee x \in A)$ *whenever* $a, b \in A$ *and* $x \in L$.

Remark 2.5. *The set* $\mathcal{I}(L)$ *of all ideals of a distributive lattice* L *is a complete distributive lattice with least element* $\{0\}$ *and the greatest element* L *under set inclusion in which, for any* $I, J \in \mathcal{I}(L),\ I \cap J$ *is the infimum of* I, J *and the supremum is given by* $I \vee J = \{\ i \vee j \mid i \in I, j \in J\ \}$. *For any* $a \in L,\ (a] = \{\ x \mid x \leq a\ \}$ *is the principal ideal generated by* a. *The set* $\mathcal{PI}(L)$ *of all principal ideals of* L *is a sublattice of the distributive lattice* $\mathcal{I}(L)$.

Theorem 2.6. *[2] Let* I *be an ideal and* F *a filter of* L *such that* $I \cap F = \emptyset$. *Then there exists a prime ideal* P *such that* $I \subseteq P$ *and* $P \cap F = \emptyset$.

Definition 2.7. *[2] Let* L *be a lattice. The mapping* $f : L \longrightarrow L$ *is called a homomorphism if it satisfies the following conditions for all* $x, y \in L$:

(1) $f(x \wedge y) = f(x) \wedge f(y)$
(2) $f(x \vee y) = f(x) \vee f(y)$

A homomorphism is called injective if it is one-one.

Definition 2.8. *[9] A self-map* $d : L \longrightarrow L$ *is called a derivation of* L *if it satisfies the following conditions:*

(1) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$
(2) $d(x \vee y) = d(x) \vee d(y)$

Remark 2.9. *In [4], Ferrari observed the condition* (1) *is redundant and is equivalent to*

$$(1') \quad d(x \wedge y) = d(x) \wedge y = x \wedge d(y)$$

A function satisfying the above condition $(1')$ is called a translation and these translations are extensively studied by Szasz in [7] and [8].

Lemma 2.10. *[4] Let d be a derivation of L. Then for any $x, y \in L$, we have*

(1) $d(0) = 0$
(2) $d(x) \leq x$
(3) $x \leq y \Rightarrow d(x) \leq d(y)$

3 Main Results

In this section, the concepts of d-ideals and injective ideals are introduced in a distributive lattice. Further, d-ideals are characterized in terms of principal ideals. An equivalent condition is obtained for a d-ideal to become an injective ideal.

Definition 3.1. *A self-mapping $d : L \longrightarrow L$ is called a derivation of L if it satisfies the following properties:*

(i) $d(x \wedge y) = d(x) \wedge y$
(ii) $d(x \vee y) = d(x) \vee d(y)$ *for all* $x, y \in L$

The kernel of a derivation is defined as the set $Ker\ d = \{x \in L \mid d(x) = 0\}$.

Proposition 3.2. *For any derivation d of L, $Ker\ d$ is an ideal of L.*

Proof. Clearly $0 \in Ker\ d$. Let $x, y \in Ker\ d$. Then $d(x \vee y) = d(x) \vee d(y) = 0$. Hence $x \vee y \in Ker\ d$. Again, let $x \in Ker\ d$ and $r \in L$. Then $d(x \wedge r) = d(x) \wedge r = 0 \wedge r = 0$. Hence $x \wedge r \in Ker\ d$. Thus $Ker\ d$ is an ideal of L. ■

Theorem 3.3. *Let d be a derivation and I an ideal of L. Then we have*

(i) $d(I)$ *is an ideal of* L *such that* $d(I) \subseteq I$.
(ii) $d^{-1}(I)$ *is an ideal of* L *such that* $Ker\ d \subseteq d^{-1}(I)$.

Proof. (i). Clearly $0 = d(0) \in d(I)$. Let $a, b \in d(I)$. Then $a = d(x)$ and $b = d(y)$ for some $x, y \in I$. Now $a \vee b = d(x) \vee d(y) = d(x \vee y) \in d(I)$. Again, let $c \in d(I)$ and $x \in L$. Then $c = d(z)$ for some $z \in I$. Now $c \wedge x = d(z) \wedge x = d(z \wedge x) \in d(I)$. Therefore $d(I)$ is an ideal of L. Let $x \in d(I)$. Then $x = d(y)$ for some $y \in I$. Since $d(y) \leq y$, we get $d(y) = y \wedge d(y) \in I$. Hence $x \in I$.

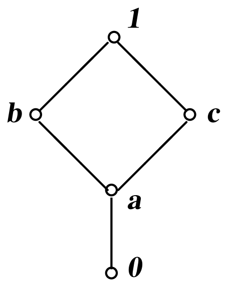
(ii). Since $d(0) = 0 \in I$, we get $0 \in d^{-1}(I)$. Let $a, b \in d^{-1}(I)$. Then we have $d(a), d(b) \in I$. Since I is an ideal, we can get $d(a \vee b) = d(a) \vee d(b) \in I$. Hence $a \vee b \in d^{-1}(I)$. Again, let $x \in d^{-1}(I)$ and $r \in L$. Then we get $d(x) \in I$. Since I is an ideal of L, we get $d(x \wedge r) = d(x) \wedge r \in I$. Thus $x \wedge r \in d^{-1}(I)$. Therefore $d^{-1}(I)$ is an ideal of L. Since $0 \in I$, we get $Ker\ d = d^{-1}(\{0\}) \subseteq d^{-1}(I)$. ■

Remark 3.4. *One can easily observe that the condition of onto which is necessary for getting the image of an ideal under a homomorphism of distributive lattices to became again an ideal is not required in* *case of a derivation.*

Definition 3.5. *An ideal I of L is called a d-ideal if $I = d(I)$.*

Remark 3.6. *Since $d(0) = 0$, it can be easily observed that the zero ideal $\{0\}$ is a d-ideal of L. Furthermore, if d is onto, then $d(L) = L$ and hence L is also a d-ideal. However, a proper d-ideal is given in the following example.*

Example 3.7. *Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.*



Define a self map $d : L \longrightarrow L$ such that $d(0) = 0, d(a) = d(c) = a$ and $d(b) = d(1) = b$. Then clearly d is a derivation on L. Consider the ideal $I = \{0, a, b\}$. It can be verified that $d(I) = I$. Therefore, I is a d-ideal of L.

Lemma 3.8. *Let d be a derivation of L and I, J any two ideals of L. Then we have*

- *(a) $I \subseteq J$ implies that $d(I) \subseteq d(J)$.*
- *(b) $d(I \cap J) = d(I) \cap d(J)$.*
- *(c) $d(I \vee J) = d(I) \vee d(J)$.*

Proof. (a) Suppose that $I \subseteq J$. Let $x \in d(I)$. Then we get that $x = d(y)$ for some $y \in I \subseteq J$. Hence we get $x = d(y) \in d(J)$. Therefore, $d(I) \subseteq d(J)$.

(b) Clearly $d(I \cap J) \subseteq d(I) \cap d(J)$. Conversely, let $x \in d(I) \cap d(J)$. Then $x = d(a)$ for some $a \in I$ and $x = d(b)$ for some $b \in J$. Since $b \in J$ and $d(b) \leq b$, we get that $d(b) \in J$ and hence $a \wedge d(b) \in I \cap J$. Thus $x = d(a) \wedge d(b) = d(a \wedge d(b)) \in d(I \cap J)$. Therefore $d(I) \cap d(J) \subseteq d(I \cap J)$.

(c) Clearly $d(I) \vee d(J) \subseteq d(I \vee J)$. Conversely, let $x \in d(I \vee J)$. Then $x = d(z)$ for some $z \in I \vee J$. Hence $z = a \vee b$ for some $a \in I$ and $b \in J$. Thus $x = d(z) = d(a \vee b) = d(a) \vee d(b) \in d(I) \vee d(J)$. Therefore, $d(I \vee J) \subseteq d(I) \vee d(J)$. ∎

For any derivation d of L, let us denote the class of all d-ideals of L by $\mathcal{I}_d(L)$.

Theorem 3.9. *Let d be a derivation of L. Then $\mathcal{I}_d(L)$ is a complete distributive lattice with respect to set inclusion. Moreover, $\mathcal{I}_d(L)$ has greatest element if and only if the map d is onto.*

Proof. Define an order $\leq$ on $\mathcal{I}_d(L)$ by $I \leq J$ if and only if $I \subseteq J$ for any two $I, J \in \mathcal{I}_d(L)$. Then clearly $(\mathcal{I}_d(L), \leq)$ is a partially ordered set. Since $\{0\}$ is a d-ideal, it can be easily obtained that

$(\mathcal{I}_d(L), \leq)$ is a complete lattice. Again by the above lemma, it yields that $\langle \mathcal{I}_d(L), \cap, \vee \rangle$ is a sublattice of $\mathcal{I}(L)$ of all ideals of L. Hence $\langle \mathcal{I}_d(L), \cap, \vee, \{0\} \rangle$ is a complete distributive lattice. The remaining part is clear by the observation that d is onto if and only if $d(L) = L$. ∎

Theorem 3.10. *Let d be a derivation of L. Then for any ideal I of L, the following conditions are equivalent.*

(1) I is a d-ideal.
(2) $I = \bigcup\limits_{x \in I} (d(x)]$.
(3) For any $x \in I$, there exists $y \in I$ such that $x = d(y)$.

Proof. $(1) \Rightarrow (2)$: Since $d(x) \leq x$, we have always $\bigcup\limits_{x \in I} (d(x)] \subseteq \bigcup\limits_{x \in I} (x] = I$. Conversely, let $x \in I$. Then there exists $a \in I$ such that $x = d(a)$. Hence $x \in (d(a)] \subseteq \bigcup\limits_{t \in I} (d(t)]$. Thus we get $I \subseteq \bigcup\limits_{t \in I} (d(t)]$. Therefore $I = \bigcup\limits_{t \in I} (d(t)]$.
$(2) \Rightarrow (3)$: Assume the condition (2). Let $x \in I$. Then $x \in (d(a)]$ for some $a \in I$. Hence $x = d(a) \wedge x = d(a \wedge x)$. Therefore $x = d(a \wedge x)$ and $a \wedge x \in I$.
$(3) \Rightarrow (1)$: Assume the condition (3). We have always $d(I) \subseteq I$. Now let $x \in I$. Then there exists $y \in I$ such that $x = d(y) \in d(I)$. Therefore $I = d(I)$. ∎

Definition 3.11. *Let d be a derivation of L. An ideal I of L is called an injective ideal with respect to d if for $x, y \in L$, $d(x) = d(y)$ and $x \in I$ implies that $y \in I$.*

Evidently $Ker\ d$ is an injective ideal of L. Though the zero ideal $\{0\}$ is a d-ideal, there is no guarantee that it is an injective ideal. However, a set of equivalent conditions are established for $\{0\}$ to become an injective ideal.

Proposition 3.12. *Let d be a derivation of L. Then the following conditions are equivalent.*

(a) $\{0\}$ is injective with respect to d.
(b) $Ker\ d = \{0\}$.
(c) $d(x) = 0$ implies that $x = 0$ for all $x \in L$.

Proof. $(a) \Rightarrow (b)$: Assume that $\{0\}$ is injective with respect to d. Let $x \in Ker\ d$. Then $d(x) = 0 = d(0)$. Since $\{0\}$ is injective, we can get that $x \in \{0\}$. Therefore $Ker\ d = \{0\}$.
$(b) \Rightarrow (c)$: The proof is trivial.
$(c) \Rightarrow (a)$: Assume the condition (c). Let $d(x) = d(y)$ and $x \in \{0\}$. Hence $d(y) = d(x) = d(0) = 0$. Therefore condition (c) yields that $y = 0 \in \{0\}$. ∎

Theorem 3.13. *Let d be a derivation of L. Then the following conditions are equivalent:*

(1) d is injective.
(2) Every ideal is injective with respect to d.

(3) Every prime ideal is injective with respect to d.

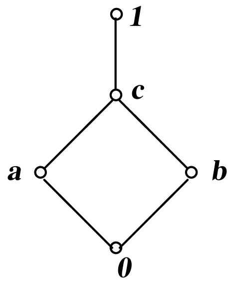
Proof. $(1) \Rightarrow (2)$: It is clear by the definition of an injective ideal.

$(2) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (1)$: Assume that every prime ideal of L is an injective ideal. Let $x, y \in L$ be such that $d(x) = d(y)$. Suppose that $x \neq y$. Without loss of generality, we can assume that $(x] \cap [y) = \emptyset$. Then there exists a prime ideal P such that $x \in P$ and $y \notin P$, which is a contradiction to P is an injective ideal. ∎

The independency between the class of all d-ideals and that of injective ideals are demonstrated in the following examples.

Example 3.14. *Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.*



Define a self-map $d : L \longrightarrow L$ as follows:

$$d(x) = \begin{cases} a & \text{if } x = a, c, 1 \\ 0 & \text{Otherwise} \end{cases}$$

Then it can be easily verified that d is a derivation of L. $I = \{0, b\}$ is an injective ideal of L. Now $d(I) = \{0\}$ and hence $I \neq d(I)$. Therefore, I is an injective ideal with respect to d but not a d-ideal.

Example 3.15. *Consider the distributive lattice $L = \{0, a, b, c, 1\}$ given in example 2.7. Define a self map $d : L \longrightarrow L$ as follows:*

$$d(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x = a, c \\ b & \text{if } x = b, 1 \end{cases}$$

It can be easily verified that d is a derivation of L. Now consider the ideal $I = \{0, a, b\}$. Clearly $d(I) = I$ and hence I is a d-ideal. But $d(a) = d(c), a \in I$ and $c \notin I$. Therefore, I is a d-ideal but not injective with respect to d.

Theorem 3.16. *Let d be a derivation of L. Then a d-ideal I of L is injective with respect to d if and only if for any $x \in L$, $d(x) \in I$ implies $x \in I$.*

Proof. Let I be a d-ideal of L. Assume that I is an injective ideal with respect to d. Let $x \in L$. Suppose $d(x) \in I = d(I)$. Then $d(x) = d(a)$ for some $a \in I$. Since I is injective and $a \in I$, we get that $x \in I$. Conversely, let $x, y \in L, d(x) = d(y)$ and $x \in I$. Since $x \in I = d(I)$, we get $x = d(a)$ for some $a \in I$. Hence, $d(y) = d(x) = d(d(a))$ and $d(a) \leq a \in I$ which implies that $y \in I$. Therefore, I is an injective ideal of L with respect to d. ■

Definition 3.17. *Let d be a derivation of L. Then for any ideal I of L, define an extension of I as $I' = \{x \in L \mid d(x) \in (d(a)]$ for some $a \in I\}$.*

The following lemma is a routine verification.

Lemma 3.18. *Let d be a derivation of L. Then for any two ideals I, J of L, we have the following:*

(1) I' is an ideal of L.
(2) $I \subseteq I'$.
(3) $I \subseteq J$ implies $I' \subseteq J'$.
(4) $I' \cap J' = (I \cap J)'$.

Proposition 3.19. *Let d be a derivation of L. Then for any ideal I of L, I' is the smallest injective ideal with respect to d such that $I \subseteq I'$.*

Proof. By the above lemma, I' is an ideal containing I. We now show that I' is an injective ideal with respect to d. Let $x, y \in L, d(x) = d(y)$ and $x \in I'$. Then $d(y) = d(x) \in (d(a)]$ for some $a \in I$. Thus $y \in I'$. Let J be an injective ideal with respect to d such that $I \subseteq J$. Let $t \in I'$. Then $d(t) \in (d(a)]$ for some $a \in I \subseteq J$. Hence, $d(t) = d(a) \wedge d(t) = d(a \wedge d(t))$ and $a \wedge d(t) \in J$. Since J is injective with respect to d, we get that $t \in J$. Thus $I' \subseteq J$ and hence I' is the smallest injective ideal with respect to d such that $I \subseteq I'$. ■

Corollary 3.20. *If I is an injective ideal, then $I = I'$.*

Theorem 3.21. *The set $\mathcal{I}^d(L)$ of all injective ideals of L, with respect to a given derivation of d of L, forms a distributive lattice on their own.*

Proof. For $I, J \in \mathcal{I}^d(L)$, define the operations $\wedge$ and $\sqcup$ such that $I \wedge J = I \cap J$ and $I \sqcup J = (I \vee J)'$. Clearly $(\mathcal{I}^d(L), \wedge, \sqcup)$ is a lattice. Now for any $I, J, K \in \mathcal{I}^d(L)$ we have $I \sqcup (J \cap K) = \{I \vee (J \cap K)\}' = \{(I \vee J) \cap (I \vee K)\}' = (I \vee J)' \cap (I \vee K)' = (I \sqcup J) \cap (I \sqcup K)$. Therefore, $(\mathcal{I}^d(L), \wedge, \sqcup)$ is a distributive lattice. ■

Definition 3.22. *Let d be a derivation of L. A proper ideal P of L is called a d-prime ideal if for any $x, y \in L$, $x \wedge y \in Ker\ d$ implies either $x \in P$ or $y \in P$.*

By a maximal injective ideal, we mean an injective ideal which is maximal in the class of all proper injective ideals with respect to a given derivation. Then the following series of propositions establish some useful relations between the class of all injective ideals with respect to a derivation d and the class of all d-prime ideals. For this, the following lemma is required.

Lemma 3.23. *Let d be a derivation of L. Then for any injective ideal I of L with respect to d , $Ker\ d \subseteq I$. In other words, $Ker\ d$ is the smallest injective ideal of L.*

Proof. Let I be an injective ideal with respect to d of L. Suppose $x \in Ker\ d$. Then $d(x) = 0 = d(0)$. Since I is injective with respect to d and $0 \in I$, it yields that $x \in I$. Since $Ker\ d$ is also an injective ideal with respect to d of L, it is the smallest injective ideal in L with respect to the derivation d. ∎

Proposition 3.24. *Let d be a derivation of L. Then every maximal injective ideal with respect to the derivation d of L is a d-prime ideal.*

Proof. Let M be a maximal injective ideal of L. Choose $x, y \in L$ such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq \{M \vee (x]\}'$ and $M \subset M \vee (y] \subseteq \{M \vee (y]\}'$. Since M is maximal, we can get $\{M \vee (x]\}' = L$ and $\{M \vee (y]\}' = L$. Hence

$$\begin{aligned}\{M \vee (x \wedge y]\}' &= \{(M \vee (x]) \bigcap (M \vee (y])\}' \\ &= \{M \vee (x]\}' \bigcap \{M \vee (y]\}' \\ &= L\end{aligned}$$

If $x \wedge y \in Ker\ d$, then $L = \{M \vee (x \wedge y]\}' \subseteq \{M \vee Ker\ d\}' = M' = M$, which is a contradiction to the fact that M is proper. Hence M is a d-prime ideal. ∎

Proposition 3.25. *Let d be a derivation of L. Then every d-prime ideal P is an injective ideal with respect to d if for each $a \in P$ there exists $b \notin P$ such that $a \wedge b \in Ker\ d$.*

Proof. Let P be a d-prime ideal which satisfies the given property. Let $x, y \in L$ be such that $d(x) = d(y)$. Suppose $x \in P$. Then there exists $x' \notin P$ such that $x \wedge x' \in Ker\ d$. Hence $d(x) \wedge x' = d(x \wedge x') = 0$, which yields that $d(y \wedge x') = d(y) \wedge x' = 0$. Therefore $y \wedge x' \in Kerd$. Since P is a d-prime ideal and $x' \notin P$, we get that $y \in P$. Hence, P is injective with respect to d. ∎

We generalise the celebrated result of M.H. Stone [1936] which is meant for ideals, filters and prime ideals of a distributive lattice to the case of injective ideals, filters and d-prime ideals.

Theorem 3.26. *Let d be a derivation of L. Suppose I is an injective ideal with respect to d and F a filter of L such that $I \cap F = \emptyset$. Then there exists a d-prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Let I be an injective ideal with respect to d and F a filter of L such that $I \cap F = \emptyset$. Consider $\mathfrak{F} = \{J \in I^d(L) \mid I \subseteq J \text{ and } J \cap F = \emptyset\}$. Clearly $I \in \mathfrak{F}$. Let $\{J_\alpha\}$ be a chain of elements of $\mathfrak{F}$. Then clearly $\cup J_\alpha$ is an upper bound for $\{J_\alpha\}$ in $\mathfrak{F}$. Hence by Zorn's lemma, $\mathfrak{F}$ has a maximal element, say M. We now prove that M is d-prime. Let $x, y \in L$ such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq \{M \vee (x]\}'$ and $M \subset M \vee (y] \subseteq \{M \vee (y]\}'$. By the maximality of M, we get that

$\{M \vee (x]\}' \cap F \neq \emptyset$ and $\{M \vee (y]\}' \cap F \neq \emptyset$. Choose $a \in \{M \vee (x]\}' \cap F$ and $b \in \{M \vee (y]\}' \cap F$. Then

$$\begin{aligned} a \wedge b \ &\in \ \{M \vee (x]\}' \bigcap \{M \vee (y]\}' \\ &= \ \{(M \vee (x]) \bigcap (M \vee (y])\}' \\ &= \ \{M \vee (x \wedge y]\}' \end{aligned}$$

Suppose $x \wedge y \in Ker\ d$. Then it reflects that $a \wedge b \in \{M \vee (x \wedge y]\}' = \{M \vee Ker\ d\}' = M' = M$(Since M is injective). Thus, $a \wedge b \in M \cap F$, which is a contradiction to $M \cap F = \emptyset$. Therefore, M is a d-prime ideal. ∎

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