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Secondary orthogonal similarity of a real matrix and its secondary transpose

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Abstract

In this paper we present some extended results of Vermeer in the context of secondary orthogonal (respectively complex secondary unitary) Matrices.

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1 Introduction

Throughout this paper we use the following notation:

Notation 1.1. Let F be a field and $\mathcal{M}_n(F)$ be the algebra of $n \times n$ matrices. Let $\mathcal{GL}_n(F)$ denote the set of invertible $n \times n$ matrices. Let $\mathcal{O}_n^s(F)$ and $\mathcal{U}_n^s(F)$ denote the set of s-orthogonal and s-unitary matrices respectively.

Notation 1.2. The secondary transpose (conjugate secondary transpose) of A is defined by $A^s = VA^TV \;\; (A^\Theta = VA^*V)$, where "V" is the fixed disjoint permutation matrix with units in its secondary diagonal.

Definition 1.3. [4] Let $A \in \mathcal{M}_n(F)$.

i. The matrix A is called *s-symmetric* if $A^s = A$.

ii. The matrix A is called *s-skew symmetric* if $A^s = -A$.

iii. The matrix A is called *s-normal* if $AA^\Theta = A^\Theta A$

iv. The matrix A is called *s-orthogonal* if $AA^s = A^sA = I$.

v. The matrix A is called *s-unitary* if $AA^\Theta = A^\Theta A = I$.

Definition 1.4. [4] Let $A \in \mathcal{M}_n(F)$ and $B \in \mathcal{M}_n(F)$. A is said to be *s-orthogonally similar* (respectively *s-unitarily*) to B if there exists a s-orthogonal matrix $Q \in \mathcal{M}_n(F)$ such that $A = Q^sBQ (A = Q^\Theta BQ)$.

Theorem 1.5. *[1] Every square complex matrix is similar to a s-symmetric matrix.*

Theorem 1.6. *[6] Let* $A \in \mathcal{M}_n(F)$*. Then*

i. *there exists an* $X \in \mathcal{GL}_n(F)$ *such that* $XAX^{-1} = A^s$*.*

ii. *there exists a s-symmetric* $X \in \mathcal{GL}_n(F)$ *such that* $XAX^{-1} = A^s$*.*

iii. *every* $X \in \mathcal{GL}_n(F)$ *with* $XAX^{-1} = A^s$ *is s-symmetric if and only if the minimal polynomial of A is equal to its characteristic polynomial.*

2 Secondary Orthogonal similarity

Lemma 2.1 ([5]). *A s-unitary matrix U is a product of a s-symmetric s-unitary matrix and s-orthogonal matrix O. That is,* $U = e^{iS}O$*. It is also true that* $U = O'e^{iS'}$*, where* O' *is s-orthogonal and* S' *is real s-symmetric.*

Theorem 2.2. $A \in \mathcal{M}_n(\mathbb{C})$*. The following assertions are equivalent.*

i. *A is s-unitarily similar to a complex s-symmetric matrix.*

ii. *There exists a s-symmetric s-unitary matrix U such that* UAU^{Θ} *is s-symmetric.*

iii. *There exists a s-symmetric s-unitary matrix U and a s-symmetric matrix S such that* $A = SU$*.*

iv. *There exists a s-symmetric s-unitary matrix V such that* $VAV^{\Theta} = A^s$*.*

Proof. $(i) \Rightarrow (ii)$: Suppose that $V^{\Theta}AV = S$, where $V \in \mathcal{U}_n^s(F)$ and S is s-symmetric. By Lemma 2.1, $V = UO$ with U is s-symmetric s-unitary matrix and $O \in \mathcal{O}_n^s(F)$ We have $A = VSV^{\Theta} = U(OSO^s)U^{\Theta}$ and hence $U^{\Theta}AU = OSO^s$ is s-symmetric.

$(ii) \Rightarrow (iii)$: Suppose that $U^{\Theta}AU = S$ with U is s-symmetric s-unitary and S is s-symmetric. Then $A = USU^{\Theta} = (USU^s)(UU^s)^{\Theta}$ with USU^{Θ} s-symmetric and $(UU^s)^{\Theta}$ s-symmetric s-unitary.

$(iii) \Rightarrow (iv)$: Suppose that $A = SU$ with S s-symmetric and U s-symmetric s-unitary. Then $UAU^{\Theta} = US = U^sS^s = A^s$.

$(iv) \Rightarrow (i)$: Suppose $VAV^{\Theta} = A^s$ for some V s-symmetric s-unitary, then there exists $U \in \mathcal{U}_n^s(F)$ such that $V = U^sU$. Now $(U^sU)A(U^sU)^{\Theta} = A^s$ and hence $UAU^{\Theta} = (U^s)^{\Theta}A^sU^s = (UAU^{\Theta})^s$, which is s-symmetric. ∎

Theorem 2.3. *Let* $A \in \mathcal{M}_n(\mathbb{C})$ *be non-singular. Then there are matrices R and E such that*

i. *If A is s-unitary then R is s-orthogonal.*

ii. *If A is s-unitary and s-symmetric then R is s-orthogonal and s-symmetric.*

Proof. (i) If A is s-unitary then $\overline{A}^{-1}A = A^sA$ is s-symmetric and hence E is also s-symmetric. The property $E\overline{E} = I_n$ and the s-symmetry implies that E is s-unitary. Finally, $R = AE^{-1}$ is s-unitary and real, so it is s-orthogonal.
(ii) If A is s-unitary and s-symmetric then $\overline{A}A = I = A\overline{A}$ and so $A = ER = RE$. It follows that $R = \overline{E}A = A\overline{E}$ and since E (and A) are s-symmetric we see that $R^s = (\overline{E}A)^s = A^s\overline{E}^s = A\overline{E} = R$, That is, R is s-symmetric. ∎

Corollary 2.4. *Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be given.*

i. A,B are real s-orthogonally similar if and only if A,B are s-unitarily similar.

ii. A,B are similar via a real s-symmetric s-orthogonal matrix if and only if A, B are similar via a s-symmetric s-unitary matrix.

We assume that all matrices are real for the rest of the paper. We use the notation Q, Ψ for arbitrary matrices in $\mathcal{O}_n^s(\mathbb{R})$, Ω for s-symmetric s-orthogonal matrices, S for a s-symmetric matrix, T for a s-skew symmetric matrix, D for diagonal matrices and Σ for a real diagonal matrix with $\Sigma^2 = I_n$.

We say that A has type $T + D$ if it is the sum of a s-skew symmetric and a diagonal matrix; and A is of type QD if A is the product of an s-orthogonal and a diagonal matrix.

We write $A \simeq B$(respectively $A \simeq_S B$)if there exists an s-orthogonal (respectively s-symmetric and s-orthogonal) matrix Q with $QAQ^s = B$. The notation [A] denotes the equivalence class of A with respect to $\simeq$. The relation $\simeq_S$ is not an equivalence relation.

Lemma 2.5. *Let $A, B \in \mathcal{M}_n(\mathbb{R})$.*

i. If $A \simeq A^s$ and $A \simeq B$, then $B \simeq B^s$

ii. If $A \simeq_S A^s$ and $A \simeq B$, then $B \simeq_S B^s$

iii. If $Q \in \mathcal{O}_n(\mathbb{R})$ and $A = QA^sQ^s$, then $A = Q^sA^sQ$.

iv. If $A \simeq B$, then $A + A^s \simeq B + B^s$ and $AA^s \simeq BB^s$

Proof. (i) and (ii). If $A = Q_1BQ_1^s$ and $A = Q_2A^sQ_2(Q_1, Q_2$ are s-orthogonal matrices) then $A^s = Q_1B^sQ_1^s$ and so $B = Q_1^sAQ_1 = Q_1^sQ_2A^sQ_2Q_1 = Q_1^sQ_2Q_1B^sQ_1^sQ_2Q_1$, and we conclude $B \simeq B^s$. If Q_2 is s-symmetric then $Q_1^sQ_2Q_1$ and so $B \simeq_S B^s$.
(iii). $A = QA^sQ^s = QA^sQ^{-1} \Rightarrow Q^sAQ = A^s$, so $A = (A^s)^s = (Q^sAQ)^s = Q^sA^sQ$.
(iv). Obvious. ∎

Theorem 2.6. *Let $A \in \mathcal{M}_n(\mathbb{R})$. The following assertions are equivalent.*

i. A is s-unitarily similar to a complex s-symmetric matrix.

ii. $A \simeq_S A^s$.

iii. $A = \Omega S$, *where* $\Omega \in \mathcal{O}_n^s(\mathbb{R})$ *is s-symmetric and S is real s-symmetric.*

iv. $A \simeq \Omega' D$, *where* $\Omega' \in \mathcal{O}_n^s(\mathbb{R})$ *is s-symmetric and D is real diagonal.*

Proof. (i)$\Rightarrow$ (ii) : According to Theorem 2.1 there exists a s-unitary secondary symmetric matrix V with $VAV^{\Theta} = A^s$. Corollary 2.4 now ensures that $A \simeq_S A^s$.
The equivalence of (i), (ii) and (iii) can be proved as in Theorem 2.2.
(iii) $\Rightarrow (iv)$: If $A = \Omega S$, then write $S = QDQ^s$ and we obtain $A = \Omega QDQ^s \simeq Q^s\Omega QD = \Omega' D$.
(iv) $\Rightarrow$ (i) If $A \simeq \Omega' D = B$ then $B \simeq_S B^s$ and so by Lemma 2.5 we have $A \simeq_S A^s$. ■

Lemma 2.7. *Let* $A \in \mathcal{M}_n(\mathbb{R})$. *Then*

i. $A \simeq B$, *where B has type* $T + D$.

ii. $A \simeq B$, *where B has type* QD.

Proof. (i). Consider the Toeplitz decomposition $A = 1/2(A + A^s) + 1/2(A - A^s)$ and suppose $Q(A + A^s)Q^{-1} = 2D$. Then $Q^sAQ = D + 1/2Q^s(A - A^s)Q$ is the sum of a diagonal matrix and a secondary skew-symmetric matrix.
(ii). Consider the singular value decomposition $A = Q_1DQ_2^s$.
Then $A \simeq B = Q_2^sAQ_2 = (Q_2^sQ_1)D$. ■

Lemma 2.8. *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be a matrix of type* $T + D$, *where* $D = \bigoplus_{i=1}^r d_iI_i$ *with* $d_i \neq d_j$ *for* $i \neq j$ *and* I_i *the* $k_i \times k_i$ *identity matrix. If* $Q \in \mathcal{O}_n^s(\mathbb{R})$ *and* $QAQ^s = A^s$ *then* $Q = \bigoplus_{i=1}^r Q_i$, *with* $Q_i \in \mathcal{O}_n^s(\mathbb{R})$.

Proof. By Lemma 3.1(3), $QAQ^s = A^s$ implies $QA^sQ^s = A$ and so $Q(A + A^s)Q^s = A + A^s$, That is, $Q(2D)Q^s = 2D$. Hence Q commutes with the diagonal matrix 2D and this implies the result. ■

Theorem 2.9. *Assume that* $A \in \mathcal{M}_n(\mathbb{R})$ *satisfies one of the following properties*

i. A has n different real eigenvalues

ii. $A + A^s$ *has n different eigenvalues*

iii. A has n different singular values.

If there exists $Q \in \mathcal{O}_n^s(\mathbb{R})$ *with* $A = QA^sQ^s$, *then Q is s-symmetric and there exist at most* 2^n *such matrices.*

Proof. (i). The s-symmetry of Q is a direct corollary of Theorem 1.5. We have to check that there exist at most 2^n such matrices Q. Let $D \in \mathcal{M}_n(\mathbb{R})$ be a diagonal matrix whose diagonal entries are the eigen values of A. Suppose $P, R \in \mathcal{M}_n(\mathbb{R})$ are nonsingular and diagonalize A. That is, $A = PDP^{-1} = RDR^{-1}$.

Corresponding columns of P and R are non-zero vectors in the same one-dimensional eigen space of A so there is a diagonal matrix $C \in \mathcal{M}_n(\mathbb{R})$ such that $R = PC$. We fix one special matrix $P \in \mathcal{M}_n(\mathbb{R})$ with $A = PDP^{-1}$.
Now suppose $Q, \Psi \in \mathcal{O}_n^s(\mathbb{R})$ are such that $A = QA^sQ^s = \Psi A^s \Psi$, so
$A = Q(PDP^{-1})^sQ^s = (Q(P^s)^{-1})D(Q(P^s)^{-1})^{-1}$ and
$A = \Psi(PDP^{-1})^s\Psi^s = (\Psi(P^s)^{-1})D(\Psi(P^s)^{-1})^{-1}$.
Note that $Q(P^s)^{-1} = PC_1$ for some diagonal matrix $C_1 \in \mathcal{M}_n(\mathbb{R})$, so $Q = PC_1P^s$ is s-symmetric. Note that $Q(P^s)^{-1} = \Psi(P^s)^{-1}C_2$ for some diagonal matrix $C_2 \in \mathcal{M}_n(\mathbb{R})$, so $P^s(\Psi^{-1}Q)(P^s)^{-1} = C_2$ and we have obtained a real diagonalization of the real s-orthogonal matrix $\Psi^{-1}Q$ whose only possible real eigen values are ± 1. Thus $C_2 = diag(\pm 1, ..., \pm 1)$ which shows that there are at most 2^n choices for Ψ, since $\Psi = Q(P^s)^{-1}C_2^{-1}P^s$.
(ii). If $Q \in \mathcal{O}_n^s(\mathbb{R})$ and $A = Q^sA^sQ$ then by Lemma 2.5(iii),$A^s = QAQ^s$ and so $Q(A + A^s)Q^s = A + A^s = (A + A^s)^s$, the conclusion follows from part(i).
(iii). Fix $Q \in \mathcal{O}_n^s(\mathbb{R})$ with the property $A = QA^sQ^s$ and consider a singular value decomposition $A = UDV^s$ of A. Then $A = QA^sQ^s = (QV)D(QU)^s$, so we have two singular value decomposition of A. Distinct singular values and the uniqueness theorem for the singular value decomposition([3], Theorem 3.1.1) ensure that $QV = UP$ and $QU = VP$ for some diagonal real s-orthogonal matrix P. Then $Q = UPV^s = VPU^s = Q^s$ is s-symmetric and there are only 2^n choices for P. ∎

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