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Partial orders induced by convolutions

U.M. Swamy
Department of Mathematics, G.V.P.College of Engineering,
Vizianagaram,INDIA.

Sagi Sankar
Department of Mathematics, M.V.G.R.College of Engineering,
Vizianagaram,INDIA.
E-mail: sagi-sankar@yahoo.co.in

Abstract

A convolution is a mapping $\mathcal{C}$ of the set $\mathcal{Z}^+$ of positive integers into the set $\mathcal{P}(\mathcal{Z}^+)$ of all subsets of $\mathcal{Z}^+$ such that for any $n \in \mathcal{Z}^+$, each member of $\mathcal{C}(n)$ is a divisor of n. If $D(n)$ is the set of all divisors of n, for any n, then D is called the Dirichlet's convolution. If $U(n)$ is the set of all unitary(square free) divisors of n, for any n, then U is called unitary(square free) convolution. Corresponding to any general convolution $\mathcal{C}$, we can define a binary relation $\leq_{\mathcal{C}}$ on $\mathcal{Z}^+$ by ' $m \leq_{\mathcal{C}} n$ if and only if $m \in \mathcal{C}(n)$ '. In this paper, we characterize convolutions $\mathcal{C}$ for which $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ and discuss the various properties of the partial ordered set $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ in terms of the convolution $\mathcal{C}$.

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1 Introduction

A convolution is a mapping $\mathcal{C}$ of the set $\mathcal{Z}^+$of positive integers into the set $\mathcal{P}(\mathcal{Z}^+)$ of subsets of $\mathcal{Z}^+$ such that for any $n \in \mathcal{Z}^+$, $\mathcal{C}(n)$ is a nonempty set of divisors of n. If $\mathcal{C}(n)$ is the set of all divisors of n for each $n \in \mathcal{Z}^+$, then $\mathcal{C}$ is the classical Dirichlet convolution[5]. If $\mathcal{C}(n)$= $\{d \;/\; d|n \text{ and } (d, \frac{n}{d}) = 1\}$, then $\mathcal{C}$ is the Unitary convolution[2]. If $\mathcal{C}(n)$=$\{d \;/\; d|n$ and m^k doesnot divide d for any $m \in \mathcal{Z}^+\}$ then $\mathcal{C}$ is called the k-free convolution.

Corresponding to any convolution $\mathcal{C}$ we can define a binary relation $\leq_{\mathcal{C}}$ in a natural way by '$m \leq_{\mathcal{C}} n$ if and only if $m \in \mathcal{C}(n)$.' $\leq_{\mathcal{C}}$ is called the binary relation on $\mathcal{Z}^+$ induced by the convolution $\mathcal{C}$[6].

In this paper we consider convolutions $\mathcal{C}$ for which $\leq_{\mathcal{C}}$ becomes a partial order on $\mathcal{Z}^+$ and prove some results on the partial orders induced by convolutions. Throughout this paper, $\mathcal{Z}^+$ denotes the set of all positive integers and $\mathcal{N}$ denotes the set of all nonnegative integers.

2 Preliminaries

For any nonempty set X, a binary relation $\leq$ on X which is reflexive, transitive and antisymmetric is called a partial order on X and the pair $(X,\leq)$ is called a partially ordered set (poset). If A is a subset of X and $x \in X$, x is called a lower bound (upper bound) of A if $x \leq a$ ($a \leq x$ respectively) for all $a \in A$. A lower bound (upper bound) x of A is called the greatest lower bound (least upper bound) of A if $y \leq x$ ($x \leq y$ respectively) of all lower bounds (upper bounds) y of A and is denoted by glb A (lub A

respectively). If A is a finite set $\{a_1, a_2 \cdots, a_n\}$ then the glb A and lub A are denoted respectively by

$$\text{glb } A = a_1 \wedge a_2 \wedge \cdots \wedge a_n = \bigwedge_{i=1}^{n} a_i$$
$$\text{lub } A = a_1 \vee a_2 \vee \cdots \vee a_n = \bigvee_{i=1}^{n} a_i$$

A partially ordered set $(X,\leq)$ is called a *meet semi-lattice* if any two-element subset and hence any nonempty finite subset of X has the greatest lower bound and is called a *join semi-lattice* if any two-element subset of X has the least upper bound in X.

If $(X,\leq)$ is a *meet semi-lattice* and $a \wedge b$ is the greatest lower bound of $\{a,b\}$ for any $a,b \in X$, then $\wedge$ is an idempotent, associative and commutative binary operation on X. Conversely, if $\wedge$ is a binary operation on a set X and if we define $a \leq b \Leftrightarrow a = a \wedge b$ for any a, $b \in X$, then $(X, \leq)$ is a *meet semi-lattice* in which $a \wedge b$ is the greatest lower bound of $\{a, b\}$.

A partially ordered set $(X, \leq)$ is called a lattice if any two-element subset of X has both the greatest lower bound and least upper bound in X. A nonempty subset Y of X is called a sublattice of X if $a \wedge b$ and $a \vee b \in Y$ for all $a, b \in Y$.

The binary relation $|$ defined on $\mathcal{Z}^+$ by '$a|b$ if and only if a divides b' is a partial order on $\mathcal{Z}^+$ called the division order on $\mathcal{Z}^+$ and the binary relation $\leq$ defined on $\mathcal{N}$ by '$a \leq b$ if and only if $b - a \in \mathcal{N}$' is a partial order on $\mathcal{N}$ called the usual order. Clearly $(\mathcal{Z}^+, |)$ is a lattice in which the least upper bound $a \vee b$ and the greatest lower bound $a \wedge b$ are given by

$$a \vee b = \text{The least common multiple of } a \text{ and } b$$
$$\text{and } a \wedge b = \text{The greatest common divisor of } a \text{ and } b$$

Definition 2.1. *Let P be the set of all prime numbers. For any mapping $\alpha : P \to \mathcal{N}$, the support of α is the set defined by $|\alpha| = \{p \in P \ / \ \alpha(p) \neq 0\}$.*

The set of all mappings from P into $\mathcal{N}$ whose supports are finite is denoted by $\sum_P \mathcal{N}$; that is ,$\sum_P \mathcal{N}$ = $\{\alpha \in \mathcal{N}^P \ / \ |\alpha| \text{ is finite}\}$.

Theorem 2.2. *$\mathcal{N}^P$ is a lattice under the point-wise ordering with respect to the usual order on $\mathcal{N}$ in which, for any $\alpha, \beta \in \mathcal{N}^P$, $(\alpha \vee \beta)(p)$ = maximum of $\{\alpha(p), \beta(p)\}$ and $(\alpha \wedge \beta)(p)$ = minimum of $\{\alpha(p), \beta(p)\}$ for any $p \in P$. Also, $\sum_P \mathcal{N}$ is a sublattice of $(\mathcal{N}^P, \leq)$.*

Definition 2.3. *Let $\{(L_i, \vee)\}_{i \in I}$ be a class of join semi-lattices indexed by a set I. A pair $(L, \{\theta_i\}_{i \in I})$ is called a direct sum of $\{L_i\}_{i \in I}$ if the following conditions are satisfied.*

(1) L is a join semi-lattice.

(2) For each $i \in I$, $\theta_i : L_i \to L$ is a homomorphism of join semi-lattices.

and *(3) If M is any* join semi-lattice *and for each $i \in I$, $\phi_i : L_i \to M$ is a homomorphism, then there exists a unique homomorphism $\phi : L \to M$ such that $\phi o \theta_i = \phi_i$ for all $i \in I$.*

Theorem 2.4. *Let* $(L, \{\theta_i\}_{i\in I})$ *be the direct sum of* $\{L_i\}_{i\in I}$ *and* M *be any* join semi-lattice *which is isomorphic to* L *. If* $\phi : L \to M$ *is an isomorphism, then* $(M, \{\phi o\theta_i\}_{i\in I})$ *is also a direct sum of* $\{L_i\}_{i\in I}$ *. Conversely, if* $(M, \{\phi_i\}_{i\in I})$ *is a direct sum of* $\{L_i\}_{i\in I}$*, then there exists a unique isomorphism* $\phi : L \to M$ *such that* $\phi o\theta_i = \phi_i$ *for all* $i \in I$*.*

A *join semi-lattice* $(L, \vee)$ is said to be with zero if there is an element, called zero denoted by o such that $o \vee a = a$ for all $a \in L$. Note that $(\mathcal{Z}^+, \vee)$ is a *join semi-lattice* with zero, where $a \vee b =$l.c.m$\{a, b\}$ and the integer 1 is the zero element. Also, $(\mathcal{N}, \vee)$ is a *join semi-lattice* with zero, where $a \vee b =$max$\{a, b\}$ and the integer 0 is the zero element. If $(L, \vee)$ and $(M, \vee)$ are join semi lattices with zero, then a mapping $f : L \to M$ is called a homomorphism if $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in L$ and $f(o) = o$.

Theorem 2.5. *Let* P *be the set of all prime numbers and* $\mathcal{N}_P = \mathcal{N}$ *for all* $p \in P$*. For any* $p \in P$*, define* $\theta_p : \mathcal{N} \to \mathcal{Z}^+$ *by* $\theta_p(n) = p^n$ *for all* $n \in \mathcal{N}$*. Then* $(\mathcal{Z}^+, \{\theta_p\}_{p\in P})$ *is the direct sum of* $\{\mathcal{N}_p\}_{p\in P}$*, where* $\mathcal{Z}^+$ *and* $\mathcal{N}_P$ *are join semi-lattices with zero with respect to the division order and the usual order respectively.*

Proof. For any non-negative integers n, m and a prime number p,

$$\begin{aligned} \theta_p(n \vee m) &= p^{n\vee m} \\ &= p^{max\{n,m\}} \\ &= l.c.m\{p^n, p^m\} \\ &= \theta_p(n) \vee \theta_p(m) \end{aligned}$$

and $\theta_p(0) = p^0 = 1$, the zero element in $\mathcal{Z}^+$.
Therefore $\theta_p : \mathcal{N} \to \mathcal{Z}$ is a homomorphism for each $p \in P$.
Let $(L, \vee)$ be any *join semi-lattice* with zero and for each $p \in P$, $\phi_p : \mathcal{N} \to L$ be a homomorphism.
Define $\phi : \mathcal{Z}^+ \to L$ by

$$\phi(a) = \begin{cases} 0 & \text{if } a = 1 \\ \bigvee\limits_{i=1}^{k} \phi_{p_i}(n_i) & \text{if } a = p_1^{n1} p_2^{n_2} \cdots p_k^{n_k}, p_i \in P, n_i \in \mathcal{N} \end{cases}$$

Then ϕ is well defined and $\phi o\theta_p = \phi_p$ for all $p \in P$. Also $\phi(a \vee b) = \phi(a) \vee \phi(b)$ for any a, $b \in \mathcal{Z}^+$ and $\phi(1) = 0$, the least element in L. Therefore ϕ is a homomorphism.

If $\psi : \mathcal{Z}^+ \to L$ is any other homomorphism of join semi lattices with zero such that $\psi o\theta_p = \phi_p$ for all $p \in P$, then it can be easily shown that $\psi(a) = \phi(a)$ for all $a \in \mathcal{Z}^+$. This proves the uniqueness of ϕ. ∎

Theorem 2.6. *The lattices* $(\mathcal{Z}^+, /)$ *and* $(\sum\limits_P \mathcal{N}, \leq)$ *are isomorphic .*

Proof. Define $\theta : \mathcal{Z}^+ \to \sum\limits_P \mathcal{N}$ by
$\theta(a)(p) =$The largest n in $\mathcal{N}$ such that p^n divides a for any $a \in \mathcal{Z}^+$ and $p \in P$. Then it can be easily verified that θ is an isomorphism. ∎

For any subsets A and B of $\mathcal{Z}^+$, let AB denote the set of all products ab with $a \in A$ and $b \in B$.

Definition 2.7. *A mapping $\mathcal{C} : \mathcal{Z}^+ \to \mathcal{P}(\mathcal{Z}^+)$ is called a convolution if for any $m \in \mathcal{Z}^+$, $\mathcal{C}(m)$ is a nonempty set of positive divisors of m.*

A convolution $\mathcal{C}$ is said to be multiplicative if for any m and $n \in \mathcal{Z}^+$, $\mathcal{C}(mn) = \mathcal{C}(m).\mathcal{C}(n)$ whenever $(m,n) = 1$.

$\mathcal{C}$ is said to be completely multiplicative if $\mathcal{C}(mn) = \mathcal{C}(m).\mathcal{C}(n)$ for all m and $n \in \mathcal{Z}^+$.

We observe that

- the Dirichlet convolution D defined by $D(n) = \{d \ / \ d$ is a divisor of $n\}$ is completely multiplicative
- the Unitary convolution U defined by $U(n) = \{d \ / \ d$ is a divisor of n and $(d, \frac{n}{d}) = 1\}$ is multiplicative but not completely multiplicative
- the k-free convolution F_k, defined by $F_k(n) = \{d \ / \ d$ is a divisor of n and m^k does not divide d for any $n \in \mathcal{Z}^+\}$ is multiplicative but not completely multiplicative.
- The Trivial convolution T defined by $T(n) = \{1, n\}$ is not multiplicative.

Theorem 2.8. *A Convolution $\mathcal{C}$ is multiplicative if and only if*

$$\mathcal{C}(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}) = \mathcal{C}(p_1^{a_1}).\mathcal{C}(p_2^{a_2}) \cdots \mathcal{C}(p_r^{a_r})$$

for any distinct primes $p_1, p_2, \cdots, p_r$ and non-negative integers $a_1, a_2, \cdots, a_r$.

Theorem 2.9. *Let $\mathcal{C}$ be a multiplicative convolution. For each prime p and for each non-negative integer a, let $\mathcal{C}_p(a) = \{b \in \mathcal{N} \ / \ p^b \in \mathcal{C}(p^a)\}$ Then, for any $n = p_1^{a_1}.p_2^{a_2} \cdots p_r^{a_r}$ where $p_1 < p_2 < \cdots < p_r$ are primes and $a_i^{'}s$ are non-negative integers,*
$\mathcal{C}(n) = \{p_1^{b_1}.p_2^{b_2} \cdots p_r^{b_r} \ / \ b_i \in \mathcal{C}_{p_i}(a_i)$ for all $1 \leq i \leq r\}$.

Theorem 2.10. *For each prime p, let $\mathcal{C}_p : \mathcal{N} \to \mathcal{P}(\mathcal{N})$ be a mapping such that $\mathcal{C}_p(a)$ is non-empty and $b \leq a$ for any $a \in \mathcal{N}$ and $b \in \mathcal{C}_p(a)$. Define $\mathcal{C} : \mathcal{Z}^+ \to \mathcal{P}(\mathcal{Z}^+)$ by $\mathcal{C}(1) = \{1\}$ and $\mathcal{C}(\prod_{i=1}^{r} p_i^{a_i}) = \{\prod_{i=1}^{r} p_i^{b_i} \ / \ b_i \in \mathcal{C}_{p_i}(a_i)$ for all $1 \leq i \leq r\}$ for any distinct primes $p_1, p_2, \cdots, p_r$ and $a_1, a_2, \cdots, a_r \in \mathcal{N}$. Then $\mathcal{C}$ is a multiplicative convolution.*

3 Main Results

For any convolution $\mathcal{C}$ we can define a binary relation $\leq_{\mathcal{C}}$ on $\mathcal{Z}^+$ by

$$m \leq_{\mathcal{C}} n \text{ if and only if } m \in \mathcal{C}(n).$$

$\leq_{\mathcal{C}}$ is called the relation induced by $\mathcal{C}$ on $\mathcal{Z}^+$. If $m \leq_{\mathcal{C}} n$ and $n \leq_{\mathcal{C}} m$, then m and n are divisors of each other and hence $m = n$. Therefore, the relation $\leq_{\mathcal{C}}$ is always antisymmetric.

Theorem 3.1. *Let $\mathcal{C}$ be a convolution and $\leq_{\mathcal{C}}$ be the relation on $\mathcal{Z}^+$ induced by $\mathcal{C}$. Then $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ if and only if the following conditions are satisfied for any $n \in \mathcal{Z}^+$.*

(1) $n \in \mathcal{C}(n)$ and (2) $\bigcup\limits_{m \in \mathcal{C}(n)} \mathcal{C}(m) \subseteq \mathcal{C}(n)$.

The proof of the theorem is trivial. The above two conditions are independent to each other. In the presence of the first condition (1), the second condition (2) can be replaced by $(2^{'})$ $\bigcup\limits_{m \in \mathcal{C}(n)} \mathcal{C}(m) = \mathcal{C}(n)$.

However, the condition $(2^{'})$ is not sufficient for $\leq_{\mathcal{C}}$ to be a partial order on $\mathcal{Z}^+$. The square-free convolution F_2 defined by $F_2(n) = \{d \in \mathcal{Z}^+ \ / \ d$ is a square-free divisor of $n\}$ satisfies $(2^{'})$ and since $4 \notin F_2(4)$, $\leq_{F_2}$ is not a partial order.

Theorem 3.2. *Let $\mathcal{C}$ be a convolution such that $\bigcup\limits_{m \in \mathcal{C}(n)} \mathcal{C}(m) = \mathcal{C}(n)$for all $n \in \mathcal{Z}^+$. For any $n \in \mathcal{Z}^+$, define $\mathcal{C}^{'}(n) = \mathcal{C}(n) \cup \{n\}$. Then $\mathcal{C}^{'}$ is a convolution and $\leq_{\mathcal{C}^{'}}$ is a partial order on $\mathcal{Z}^+$.*

Since we are interested only in convolutions which induce partial orders on $\mathcal{Z}^+$, we assume that a convolution is a mapping $\mathcal{C} : \mathcal{Z}^+ \to \mathcal{P}(\mathcal{Z}^+)$ that satisfies the following conditions for any $n \in \mathcal{Z}^+$.

(1) Every member of $\mathcal{C}(n)$ is a positive divisor of n

(2) $n \in \mathcal{C}(n)$

and (3) $\mathcal{C}(n) = \bigcup\limits_{m \in \mathcal{C}(n)} \mathcal{C}(m)$

Non-trivial convolutions satisfying the above three conditions are given in Examples 3.3 and 3.4.

Example 3.3. *Let $\mathcal{C}(n) = \{1, 5, 10\}$, then the convolution $\mathcal{C}$ satisfies the above three conditions.*

Example 3.4. *For each $n \in Z^+$, let $\mathcal{T}(n) = \{n\}$. Then $\mathcal{T}$ is a convolution and $\leq_{\mathcal{T}}$ is precisely the trivial ordering defined by $n \leq_{\mathcal{T}} m$ if and only if $n = m$.*

Definition 3.5. *Let $\mathcal{C}$ be a convolution and p a prime. For any a and b in $\mathcal{N}$, define $a \leq^p_{\mathcal{C}} b$ if and only if $p^a \leq_{\mathcal{C}} p^b$. For any α and β in $\sum\limits_P \mathcal{N}$, define $\alpha \leq^p_{\mathcal{C}} \beta$ if and only if $\alpha(p) \leq_{\mathcal{C}} \beta(p)$ for all $p \in P$.*

Since $\sum\limits_P \mathcal{N}$ and $\mathcal{Z}^+$ are bijective by Theorem 2.6, there is no ambiguity in using the symbol $\leq_{\mathcal{C}}$ to denote the relation defined above on $\sum\limits_P \mathcal{N}$ as well as the relation induced by the convolution on $\mathcal{Z}^+$.

Theorem 3.6. *For any convolution $\mathcal{C}$ and for each prime p, $\leq^p_{\mathcal{C}}$ is a partial order on $\mathcal{N}$ and $\leq_{\mathcal{C}}$ is a partial order on $\sum\limits_P \mathcal{N}$.*

Definition 3.7. *Let X and Y be non-empty sets and R and S be binary relations on X and Y respectively. A bijection $f : X \to Y$ is said to be a relation isomorphism of (X, R) into (Y, S) if aRb in X if and only if $f(a)Sf(b)$ in Y for all $a,\ b \in X$.*

Theorem 3.8. *Let $\theta : \mathcal{Z}^+ \to \sum\limits_P \mathcal{N}$ be the bijection defined by*

$$\theta(n)(p) = \text{The largest } a \text{ in } \mathcal{N} \text{ such that } p^a \text{ divides } n.$$

Then a convolution $\mathcal{C}$ is multiplicative if and only if θ is a relation isomorphism of $(\mathcal{Z}^+, \leq_\mathcal{C})$ onto $(\sum\limits_P \mathcal{N}, \leq_\mathcal{C})$.

Proof. Suppose that $\mathcal{C}$ is multiplicative.
To prove that θ is a relation isomorphism, we have to prove that

$$m \leq_\mathcal{C} n \Longleftrightarrow \theta(m) \leq_\mathcal{C} \theta(n) \text{ for any } n, m \in \mathcal{Z}^+.$$

Let m and $n \in \mathcal{Z}^+$ such that $m \leq_\mathcal{C} n$.
We can write $n = p_1^{a_1}.p_2^{a_2} \cdots p_r^{a_r}$ where $p_1 < p_2 < \cdots < p_r$ are primes and $a_1, a_2, \cdots, a_r$ are non-negative integers. Then $|\theta(n)| \subseteq \{p_1, p_2, \cdots, p_n\}$ and $\theta(n)$ is defined by

$$\theta(n)(p) = \begin{cases} a_i & \text{if } p = p_i \\ 0 & \text{if } p \neq p_i,\ 1 \leq i \leq r \end{cases}$$

Now, since $m \leq n$ we have $m \in \mathcal{C}(n) = \mathcal{C}(\prod\limits_{i=1}^{r} p_i^{a_i}) = \prod\limits_{i=1}^{r} \mathcal{C}(p_i^{a_i})$ and hence $m = m_1.m_2 \cdots m_r$, for some $m_i \in \mathcal{C}(p_i^{a_i})$. So we have, $m = \prod\limits_{i=1}^{r} p_i^{b_i}$ for some $p_i^{b_i} \in \mathcal{C}(p_i^{a_i})$.
Therefore $\theta(m)$ is defined by $\theta(m)(p) = \begin{cases} b_i & \text{if } p = p_i \\ 0 & \text{if } p \neq p_i, 1 \leq i \leq r \end{cases}$
Now, if $p \neq p_i, 1 \leq i \leq r$, then $\theta(m)(p) = 0 = \theta(n)(p)$ and if $p = p_i$, then $\theta(m)(p_i) = b_i \leq_\mathcal{C}^p \theta(n)(p)$ for all $p \in P$.
Therefore, $\theta(m)(p) \leq_\mathcal{C}^p \theta(n)(p)$ for all $p \in P$ and hence $\theta(m) \leq_\mathcal{C} \theta(n)$ in $\sum\limits_P \mathcal{N}$.
On the other hand, suppose that $\theta(m) \leq_\mathcal{C} \theta(n)$ in $\sum\limits_P \mathcal{N}$.
Then $\theta(m)(p) \leq_\mathcal{C}^p \theta(n)(p)$ for all primes $p \in P$.
In particular, $\theta(m)(p) = 0$ whenever $\theta(n)(p) = 0$, that is $|\theta(m)| \subseteq |\theta(n)|$.
Let $|\theta(n)| = \{p_1, p_2, \cdots, p_r\}$ and $\theta(n)(p_i) = a_i$.
Then, $p_i^{a_i}$ is the largest power of p_i dividing n and $\theta(n)(p) = 0$ for all $p \neq p_i, 1 \leq i \leq r$. Therefore, $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. Since $\theta(m)(p) \leq_\mathcal{C} \theta(n)(p)$ for all $p \in P$, we have $\theta(m)(p) = 0$ for all $p \neq p_i$, $1 \leq i \leq r$.
Let $\theta(m)(p_i) = b_i$. Then $m = p_1^{b_1}.p_2^{b_2} \cdots p_r^{b_r}$. Since $\theta(m) \leq_\mathcal{C} \theta(n)$, we get
$b_i = \theta(m)(p_i) \leq_\mathcal{C}^{p_i} \theta(n)(p_i)$ for all $1 \leq i \leq r$ and hence $b_i \leq_\mathcal{C}^{p_i} a_i$. Therefore $p_i^{b_i} \in \mathcal{C}(p_i^{b_i})$.
This implies that

$$m = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} \in \mathcal{C}(p_1^{a_1})\mathcal{C}(p_2^{a_2}) \cdots \mathcal{C}(p_r^{a_r}) = \mathcal{C}(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}) = \mathcal{C}(n)$$

and hence $m \leq n$ in $\mathcal{Z}^+$.
Thus, $m \leq_\mathcal{C} n$ in $\mathcal{Z}^+$ if and only if $\theta(m) \leq_\mathcal{C} \theta(n)$ in $\sum\limits_P \mathcal{N}$.
Hence, θ is a relation isomorphism of $(\mathcal{Z}^+, \leq_\mathcal{C})$ onto $(\sum\limits_P \mathcal{N}, \leq_\mathcal{C})$.

Conversely, suppose that θ is a relation isomorphism. Let $p_1, p_2, \cdots, p_r$ be distinct primes and $a_1, a_2, \cdots, a_n$ non-negative integers. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. Then

$$\begin{aligned} m \in \mathcal{C}(n) &\iff m \leq_{\mathcal{C}} n \text{ in } \mathcal{Z}^+ \\ &\iff \theta(m) \leq_{\mathcal{C}} \theta(n) \text{ in } \sum_P \mathcal{N} \\ &\iff \theta(m)(p) \leq_{\mathcal{C}}^{p} \theta(n)(p) \text{ for all } p \in P \\ &\iff \theta(m)(p_i) \leq_{\mathcal{C}}^{p_i} \theta(n)(p_i) \text{ for all } 1 \leq i \leq r \\ & (\text{since } \theta(m)(p) = 0 = \theta(n)(p) \text{ for all } p \neq p_i, 1 \leq i \leq r) \\ &\iff b_i \leq_{\mathcal{C}}^{p_i} a_i \text{ for all } 1 \leq i \leq r, \text{ where } b_i = \theta(m)(p_i) \\ &\iff m = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} \in \mathcal{C}(p_1^{a_1}) \cdots \mathcal{C}(p_r^{a_r}) \end{aligned}$$

Thus, $\mathcal{C}(\prod_{i=1}^{a_i}) = \prod_{i=1}^{r} \mathcal{C}(p_i^{a_i})$ and hence by Theorem 2.8, $\mathcal{C}$ is multiplicative. ∎

Definition 3.9. *A poset is said to satisfy the Descending Chain Condition (D.C.C) if every non-empty subset has a minimal element. Dually a poset is said to satisfy the Ascending Chain Condition (A.C.C) if every non-empty subset has a maximal element.*

Theorem 3.10. *For any convolution $\mathcal{C}$, the poset $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ satisfies the Descending Chain Condition.*

Proof. Let $\mathcal{C}$ be a convolution and $\leq_{\mathcal{C}}$ be the partial order induced by $\mathcal{C}$ on $\mathcal{Z}^+$. Let X be a non-empty subset of $\mathcal{Z}^+$. Choose $x_1 \in X$. If X has no minimal element, then we get a sequence $\{x_n\}$ of elements in X such that

$$\cdots\cdots <_{\mathcal{C}} x_n <_{\mathcal{C}} x_{n-1} <_{\mathcal{C}} \cdots <_{\mathcal{C}} x_3 <_{\mathcal{C}} x_2 <_{\mathcal{C}} x_1$$

which is a contradiction, since all x_i's must be in $\mathcal{C}(x_1)$, which is a finite set.
Therefore, X has a minimal element. ∎

The dual of Theorem 3.8 is not true. For, consider the Dirichlet convolution D, defined by $D(n) =$ The set of all positive divisors of n. Then for any $n \in \mathcal{Z}^+$, we have $n <_D 2n$ and hence $(\mathcal{Z}^+, \leq_{\mathcal{D}})$ has no maximal elements.

There are convolutions $\mathcal{C}$ such that $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ satisfy both the Ascending Chain Condition and Descending Chain Condition. The trivial convolution T defined by $T(n) = \{n\}$ is an example for that.

Now, we study certain elementary properties of the poset $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ using the notion of an initial segment in posets.

Definition 3.11. *Let $(X, \leq)$ be a poset. A non-empty subset I of X is called an initial segment if $a \in I,\ x \in X$ and $x \leq a \Rightarrow x \in I$.*

Clearly, the class $\mathcal{I}n(X)$ of initial segments of any poset $(X, \leq)$ is closed under arbitrary unions in the sense that for any class $\{I_\alpha\}_{\alpha \in \triangle}$ of initial segments, $\bigcup_{\alpha \in \triangle} I_\alpha$ is also an initial segment. However,

$\mathcal{I}n(X)$ need not be closed under arbitrary intersections. In fact, it can be easily proved that $\mathcal{I}n(X)$ is closed under arbitrary intersections if and only if the poset $(X, \leq)$ has a least element.

For any element a in a poset $(X, \leq)$, let us define the set $(a]$ as $(a] = \{x \in X \mid x \leq a\}$. Clearly $(a]$ is an initial segment of $(X, \leq)$; in fact, it is the smallest initial segment containing a. It can be easily proved that a subset I of X is an initial segment of $(X, \leq)$ if and only if I is an union of sets of the form $(a]$, $a \in Y$ for some non-empty subset Y of X.

Though an intersection of initial segments may be empty in a general poset, it becomes an initial segment if the intersection is non-empty. In the following, we obtain several equivalent conditions on the poset $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ to satisfy the property that the intersection of any two (and hence any finite) initial segments is non-empty. First, let us recall that a poset $(X, \leq)$ is said to be directed below if, for any a and $b \in X$, there exists $x \in X$ such that $x \leq a$ and $x \leq b$.

Theorem 3.12. *Let $\mathcal{C}$ be a convolution and $\leq_{\mathcal{C}}$ be the partial order induced by $\mathcal{C}$ on $\mathcal{Z}^+$. Then the following are equivalent to each other.*

(1) $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ is directed below

(2) $1 \in \mathcal{C}(n)$ for all $n \in \mathcal{Z}^+$

(3) $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ has least element

(4) Intersection of any class of initial segments in $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ is an initial segment

(5) $I \cap J \neq \emptyset$ for any initial segments I and J of $(\mathcal{Z}^+, \leq_{\mathcal{C}})$

(6) $\mathcal{C}(m) \cap \mathcal{C}(n) \neq \emptyset$ for all m and $n \in \mathcal{Z}^+$.

Proof. (1)$\Rightarrow$ (2) Suppose that $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ is directed below .
Let $n \in \mathcal{Z}^+$. Choose a prime p that does not divide n. There exists $m \in \mathcal{Z}^+$ such that $m \leq_{\mathcal{C}} n$ and $m \leq_{\mathcal{C}} p$. Then m is a divisor of p. Since p is a prime, either $m = 1$ or $m = p$. Since p does not divide n and m divides n, we get $m \neq p$. Therefore, $m = 1$ and hence $1 \leq_{\mathcal{C}} n$ so that $1 \in \mathcal{C}(n)$.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(4) follows from the fact that any initial segment is non-empty and hence contains the least element of the poset, so that the intersection of any class of initial segments is non-empty.

(4)$\Rightarrow$(5) also follows from the fact that every initial segment is non-empty.

(5)$\Rightarrow$(6) is obvious since $\mathcal{C}(n)$ $(= (n])$ is an initial segment of $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ for any $n \in \mathcal{Z}^+$.

(6)$\Rightarrow$(1) is a consequence of the definition of $\leq_{\mathcal{C}}$. ∎

References

[1] G. Birkhoff, *Lattice Theory*, American Mathematical Society, 1967.

[2] E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math.Z.,74(1960),66-80.

[3] B.A. Davey,Priestly,H.A. *Introduction to lattices and order*, Cambridge University Press,2002.

[4] G.A. Gratzer, *General Lattice Theory*, Birkhauser,1971.

[5] W. Narkiewicz, *On a class of arithmetical convolutions*, Colloq. Math.,10(1963),81-94.

[6] U.M. Swamy,G.C. Rao, Sita V. Ramaiah, *On a conjecture in a ring of arithmetic functions*, Indian J.Pure Appl.Math., 14(12)(1983),1519-1530.