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Equiparity induced path decomposition in trees

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Abstract

A *decomposition* of a graph G is a collection $\psi = \{H_1, H_2, \ldots, H_k\}$ of subgraphs of G such that every edge of G belongs to exactly one H_i. The decomposition ψ is called a *path decomposition* of G if each H_i is a path in G. Several studies have been undertaken on path decompositions by imposing certain conditions on the paths considered in the decomposition of the graph where the primary objective is to obtain the minimum number of paths required for a certain type of decomposition for a given graph. A path decomposition ψ such that the paths in ψ are induced as well as of same parity is defined as an *equiparity induced path decomposition*. The minimum number of paths in such a decomposition of a graph G is called the *equiparity induced path decomposition number* of G and is denoted by $\pi_{pi}(G)$. In this paper we determine the value of π_{pi} for trees of even size.

Keywords: Equiparity induced path decomposition, Equiparity induced path decomposition number.

AMS Subject Classification(2010): 05C.

1 Introduction

Graph decomposition problems is the most prominent area of research in graph theory and combinatorics and further it has numerous applications in various fields such as networking, engineering and DNA analysis.

A decomposition of a graph G is a collection of its subgraphs such that every edge of G lies in exactly one member of the collection. Various types of decompositions have been introduced and well-studied by imposing conditions on the members of the decomposition. Harary introduced the notion of path decomposition [5] which demands each member of a decomposition to be a path and it was further studied by Schwenk, Peroche, Stanton, Cowan and James ([6], [7], [9]). Unrestricted path cover [6], cycle decomposition [4], simple path cover [2] and graphoidal cover [1] are some variations of decomposition. In this direction the notion of equiparity induced path decomposition was introduced and some initial developments have been made in [8]. In this paper we determine the value of the equiparity induced path decomposition number for the families of trees of even size.

2 Definitions and preliminary results

By a graph $G = (V, E)$, we mean a non-trivial, finite, connected and undirected graph with neither loops nor multiple edges. For the terms not defined here we refer to [3]. Throughout the paper the order and size of G are denoted by n and m respectively.

Definition 2.1. [8] A *decomposition* of a graph G is a collection $\psi = \{H_1, H_2, \ldots, H_k\}$ of subgraphs of G such that every edge of G belongs to exactly one H_i. The decomposition ψ is called an *induced path decomposition* of G if each H_i is an induced path in G. Further, if the lengths of all the paths in ψ are of the same parity then it is called an *equiparity induced path decomposition* ($\mathcal{ED}$). The minimum cardinality of an $\mathcal{ED}$ for a graph G is called the *equiparity induced path decomposition number* and is denoted by $\pi_{pi}(G)$.

An $\mathcal{ED}$ of a graph G in which every path is of even length is called an *even parity induced path decomposition* ($\mathcal{EED}$) and similarly an *odd parity induced path decomposition* ($\mathcal{OED}$) is defined.

If P is a $u-v$ path in G, the vertices of P other than u and v are called the *internal vertices* of P while u and v are called respectively the *origin* and *terminal* of the path P. For an $\mathcal{ED}$ ψ of G, a vertex is said to be *interior* if it is an internal vertex of an induced path in ψ; otherwise it is called an *exterior vertex* to ψ. Further, we define $t_\psi = \sum_{p\in\psi} t(P)$ where $t(P)$ denotes the number of internal vertices of the path P and $t = \max t_\psi$, here the maximum is taken over all $\mathcal{ED}, \psi$ of G. An expression for π_{pi} in terms of the number of vertices of odd degree and the maximum number of interior vertices $t(G)$ is derived in [8].

Theorem 2.2. *[8] For any graph G, we have $\pi_{pi}(G) = m - t$.*

The following corollaries are immediate consequences of the above theorem.

Corollary 2.3. *[8] If G is a graph with k vertices of odd degree, then*

$$\pi_{pi}(G) = \frac{k}{2} + \sum_{v\in V(G)} \left\lfloor \frac{\deg v}{2} \right\rfloor - t.$$

Corollary 2.4. *[8] For any graph G, $\pi_{pi}(G) \geq \frac{k}{2}$.*

3 π_{pi} for trees

In this section we determine the equiparity induced path decomposition number for trees. First we prove the following lemma. For this we define the terms smooth path and almost smooth path as follows.

A $u-v$ path P such that $degu \geq 3$, $degv = 1$ and degree of each internal vertex of P is exactly 2 is said to be a *smooth path based at* u. If P is a $u-v$ path and w is an internal vertex of P such that the (w,u)-section and the (w,v)-section of P are smooth paths based at the vertex w, then P is called an *almost smooth path based at* w. Further, if P is a $u-v$ path and Q is a $v-w$ path, then the walk consisting of the path P followed by the path Q is denoted by $P \circ Q$. Also, by P^{-1} we mean the same path P but reading from the vertex v to u.

Lemma 3.1. *Let T be a tree other than a path. If v is a vertex of maximum eccentricity such that $degv \geq 3$, then there exists a smooth path of even length or an almost smooth path of even length based at v.*

Proof. Given that T is a tree other than a path. Then there exists a vertex of degree at least three. Among all the vertices of T with degree at least three, let v be a vertex having maximum eccentricity. Hence, as T has no cycle there exist at least two smooth paths based at v, say P_1 and P_2. Then either P_1 and P_2 are of odd length or at least one of them is of even length. If one of them is of even length there is nothing to prove, otherwise $P_1^{-1} \circ P_2$ will be a desired almost smooth path of even length based at v. ∎

Theorem 3.2. *Every tree of even size with k vertices of odd degree admits an $\mathcal{EED}$ of cardinality $\frac{k}{2}$.*

Proof. We prove the theorem by induction on l, the number of pendant vertices of T. If $l = 2$, then T is a path of even length so that the result holds true. Assume that $l \geq 3$ and the result is true for any tree with less that l pendant vertices. Let y be a vertex of degree greater than two such that the eccentricity of y is maximum, where the maximum is taken over all vertices of T with degree greater than two. Then by Lemma 3.1, there exists either a smooth path of even length or an almost smooth path of even length based at y.

Case 1. T contains a smooth path P of even length based at y.

Let T' be a tree obtained from T by truncating the smooth path P. That is, the tree T' is obtained from T by removing all the vertices lying on P other than the vertex y. Then T' is a tree of even size having $l - 1$ pendant vertices. Let k' denote the number of odd vertices in T'. Consider the following two possibilities.

Subcase 1.1 $\deg_T y$ is odd.

Then, $\deg_{T'} y$ is even and $k' = k - 2$. Therefore the induction hypothesis gives us an $\mathcal{EED}$ ψ' of T' with cardinality $\frac{k'}{2} = \frac{k}{2} - 1$. we have $\pi_{pi}(T') \leq \frac{k-2}{2} = \frac{k}{2} - 1$. Now, $\psi = \psi' \cup P$ will be a $\mathcal{EED}$ for T with cardinality $\frac{k}{2}$.

Subcase 1.2 $\deg_T y$ is even.

In this case $\deg_{T'} y$ is odd so that $k' = k$. Applying the induction hypothesis we get an $\mathcal{EED}$ ψ' for T' with cardinality $\frac{k'}{2} = \frac{k}{2}$. Choose a path P' in ψ' having y as an external vertex; this is possible as $\deg_{T'} y$ is odd. If we assume that the path P' ends at y, then $P' \circ P$ will be an induced path of even length in T and thus $\psi = \{\psi' \setminus \{P'\}\} \cup \{P' \circ P\}$ yields $\mathcal{EED}$ for T which of course is an $\mathcal{EED}$ with cardinality $\frac{k}{2}$.Thus we have an $\mathcal{EED}$ of cardinality $\frac{k}{2}$ as desired.

Case 2. T contains an almost smooth path of even length based at y.

Take P as such an almost smooth path based at y. As in Case 1, let T' be the tree obtained by removing the path P from T. Then T' is of even size having $l - 2$ pendant vertices and $k - 2$ odd vertices. Therefore using induction hypothesis find an $\mathcal{EED}$ ψ' of cardinality $\frac{k'}{2} = \frac{k}{2} - 1$. Hence ψ' together with P forms an $\mathcal{EED}$ for T of cardinality $\frac{k}{2}$. ∎

Corollary 3.3. *The value of* π_{pi} *for a tree of even size is* $\frac{k}{2}$.

Proof. For a tree T of even size , Theorem 3.2 guarantees the existence of an $\mathcal{EED}$ of cardinality $\frac{k}{2}$. This means $\pi_{pi}(T) \leq \frac{k}{2}$. Then Corollary 2.4 completes the proof. ∎

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