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Roughness in orthomodular lattices

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Abstract

In this paper, we define the rough approximation operators in an algebra using its congruence relations and study some of their properties. Further, we consider the rough approximation operators in orthomodular lattices. We introduce the notion of rough ideal (filter) with respect to a p-ideal in an orthomodular lattice. We show that the upper approximation of an ideal J with respect to a p-ideal I of an orthomodular lattice is the smallest ideal containing I and J. Further we study the homomorphic images of the rough approximation operators.

Keywords: Rough approximations, Orthomodular lattice, p-ideal, Rough ideal, Rough filter, Homomorphism.
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1 Introduction

The basic notions of rough set theory proposed by Pawlak [8] are approximation space (U, R), where R is an equivalence relation on U and rough approximations (lower and upper approximation). The lower and upper approximations of a subset X of U are respectively defined as follows:

$$\begin{aligned} X^{\blacktriangledown} &= \{x \in U / R(x) \subseteq X\} \\ X^{\blacktriangle} &= \{x \in U / R(x) \cap X \neq \phi\} \end{aligned} \quad (1)$$

where $R(x)$ is the equivalence class containing x.

When the rough approximation operators are defined in an algebraic system using its congruence relations, there arise some interesting properties in the algebraic systems. Biswas and Nanda [1] first defined rough substructures in groups. Following them, Davvaz and Kuroki applied the concept of rough approximation operators in algebras like semigroups[7], groups[1], rings[2, 4], modules[3] etc. In this paper, we apply the concept of rough approximation operators in an algebra (universal algebra). We consider the approximation space (A, θ), where A is an arbitrary algebra and θ is a congruence relation on A. In (A, θ), we define the lower and upper approximation operators with respect to the congruence relation θ and study some of their properties. There exists a one-one correspondence between the congruence relations and p-ideals in an orthomodular lattice. We consider the rough approximation operators defined with respect to p-ideals in orthomodular lattices. We introduce the notion of rough ideal and rough filter in orthomodular lattices and give their characterization in the lattice of ideals and the lattice of filters respectively.

2 Basic Concepts and Definitions

An n-ary operation α on a nonempty set A is a map from A^n into A. In other words, if $a_1, a_2, \cdots, a_n \in A$, then $\alpha(a_1, a_2, \cdots, a_n) \in A$.

Definition 2.1. *A type of algebras is a set $\mathcal{F}$ of function symbols such that a nonnegative integer n is assigned to each member α of $\mathcal{F}$. This integer is called the arity of α and α is said to be an n-ary function symbol.*

Definition 2.2. *If $\mathcal{F}$ is a language of algebras then an algebra **A** of type $\mathcal{F}$ is an ordered pair $\langle A, F\rangle$ where A is a nonempty set and F is a family of finitary operations on A indexed by the language $\mathcal{F}$ such that corresponding to each n-ary function symbol α in $\mathcal{F}$ there is an n-ary operation α^A on A. The set A is called the underlying set of **A** $= \langle A, F\rangle$ and the α^A's are called fundamental operations of **A**.*

Let **A** and **B** be two algebras of the same type. Then **B** is a subalgebra of **A** if $B \subseteq A$ and every fundamental operation of **B** is the restriction of the corresponding operation of **A**. Let $\langle A, F\rangle$ be an algebra of type $\mathcal{F}$ and θ be an equivalence relation on A. Then θ is a congruence on A if θ satisfies the following compatibility property : for each n-ary function symbol $\alpha \in \mathcal{F}$ and $a_i, b_i \in A$, $\alpha^A(a_1, a_2, \cdots, a_n)\theta\alpha^A(b_1, b_2, \cdots, b_n)$ holds if $a_i\theta b_i$ holds for $1 \leq i \leq n$.

Definition 2.3. *Let **A** and **B** be two algebras of the same type. Then a function $f : A \to B$ is said to be a homomorphism from A into B if $f(\alpha^A(a_1, a_2, \cdots, a_n)) = \alpha^B(f(a_1), f(a_2), \cdots, f(a_n))$ for every n-ary $\alpha \in \mathcal{F}$ where $a_1, a_2, \cdots, a_n \in A$.*

Let $f : A \to B$ be a homomorphism. Then the kernel of f, written as $ker(f)$, is defined by $ker(f) = \{(x, y) \in A \times A / f(x) = f(y)\}$. $ker(f)$ is a congruence relation on A.

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. An element b of L is said to be a complement of an element a in L if $a \wedge b = 0$ and $a \vee b = 1$. A bounded lattice $(L, \vee, \wedge, 0, 1)$ is said to be an ortholattice if there exists a unary operation $'$: $L \to L$ called as orthocomplement satisfying the conditions $x \vee x' = 1$, $x \wedge x' = 0$, $x \leq y \Rightarrow y' \leq x'$ and $x'' = x$. An orthomodular lattice is an ortholattice satisfying the orthomodular law $x \leq y \Rightarrow x \vee (x' \wedge y) = y$. Let $a, b \in L$. Then a is said to be perspective to b, $a \sim b$ if a and b have a common complement. A p-ideal in L is an ideal closed under perspectivity. A p-filter in L is a filter closed under perspectivity.

Proposition 2.4. *[6] If I is an ideal of an orthomodular lattice $(L, \vee, \wedge,', 0, 1)$, then the following statements are equivalent:*

i. *I is a p-ideal.*

ii. *$a \in I$ implies $x \wedge (x' \vee a) \in I$, for all $x \in L$.*

Theorem 2.5. *[6] If I is a p-ideal of an orthomodular lattice $(L, \vee, \wedge,', 0, 1)$, then the following statements are equivalent: For any $a, b \in L$,*

i. $(a \vee b) \wedge (a' \vee b') \in I$.

ii. *there exists* $t \in I$ *such that* $a \vee t = b \vee t$.

iii. $a' \wedge (a \vee b) \in I$ *and* $b' \wedge (a \vee b) \in I$.

For further definitions and results in lattice theory the reader can refer [5, 6].

3 Rough Approximations in algebraic systems

Let $\langle A, F\rangle$ be any algebra and θ be a congruence relation on A. Let X be a nonempty subset of A. Then we define the sets

$$\underline{\theta}(X) = \{x \in L/[x]_\theta \subseteq X\} and$$

$$\overline{\theta}(X) = \{x \in L/[x]_\theta \cap X \neq \phi\}$$

as the lower and upper approximations of X respectively with respect to a congruence relation θ of A, where $[x]_\theta = \{y \in A/x\theta y\}$ is the congruence class containing x. A subset X of A is said to be exact or definable if $\underline{\theta}(X) = X = \overline{\theta}(X)$.

Since the congruence relations are stronger than the equivalence relations, in any algebra $\langle A, F\rangle$ and for any congruence relation θ on A, the rough approximation operators $\underline{\theta}$ and $\overline{\theta}$ satisfy the following properties of rough approximation operators determined by an equivalence relation. For $X, Y \subseteq A$,

1. $\underline{\theta}(\phi) = \phi = \overline{\theta}(\phi)$.
2. $\underline{\theta}(A) = A = \overline{\theta}(A)$.
3. $X \subseteq Y$ implies $\underline{\theta}(X) \subseteq \underline{\theta}(Y)$ and $\overline{\theta}(X) \subseteq \overline{\theta}(Y)$.
4. $\underline{\theta}(\underline{\theta}(X)) = \underline{\theta}(X)$ and $\overline{\theta}(\overline{\theta}(X)) = \overline{\theta}(X)$.
5. $\underline{\theta}(\overline{\theta}(X)) = \overline{\theta}(X)$ and $\overline{\theta}(\underline{\theta}(X)) = \underline{\theta}(X)$.
6. $\underline{\theta}(X \cap Y) = \underline{\theta}(X) \cap \underline{\theta}(Y)$.
7. $\overline{\theta}(X \cup Y) = \overline{\theta}(X) \cup \overline{\theta}(Y)$.
8. $\overline{\theta}(A) = \cup_{x \in A}[x]_\theta$

Remark 3.1.

i. If θ *is an identity relation on* A*, then every subset of* A *is definable.*

ii. If $\overline{\theta}(A) = A$ *or* $\underline{\theta}(A) = A$*, then* A *is definable.*

iii. $\overline{\theta}(A)$ *and* $\underline{\theta}(A)$ *are definable. Therefore, if* $x \in \overline{\theta}(A)$*, then* $[x]_\theta \subseteq \overline{\theta}(A)$.

Proposition 3.2. *Let* $\langle A, F\rangle$ *be an algebra and* θ *be a congruence relation on* A*. Then for any subalgebra* B *of* A*,* $\overline{\theta}(B)$ *is a subalgebra of* A*.*

Proof. Let B be a subalgebra of A. Let $a_1, a_2, \cdots, a_n \in \overline{\theta}(B)$ and $\alpha \in F$. Then for each $i = 1, 2, \cdots, n$, $[a_i]_\theta \cap B \neq \phi$. Therefore for each $i = 1, 2, \cdots, n$, there exists $b_i \in A$ such that $a_i \theta b_i$ and $b_i \in B$. Then $\alpha(a_1, a_2, \cdots, a_n)\theta\alpha(b_1, b_2, \cdots, b_n)$ and $\alpha(b_1, b_2, \cdots, b_n) \in B$. Therefore $[\alpha(a_1, a_2, \cdots, a_n)]_\theta \cap B \neq \phi$. Then $\alpha(a_1, a_2, \cdots, a_n) \in \overline{\theta}(B)$ and hence $\overline{\theta}(B)$ is a subalgebra of A. ∎

Proposition 3.3. *Let $\langle A, F\rangle$ be an algebra and θ be a congruence relation on A. If $\alpha \in F$ is a unary operator such that $\alpha(\alpha(a)) = a$ for all $a \in A$, then for any $B \subseteq A$, the following hold.*

i. $\overline{\theta}(\alpha(B)) = \alpha(\overline{\theta}(B))$.

ii. $\underline{\theta}(\alpha(B)) = \alpha(\underline{\theta}(B))$.

Proof. Let $B \subseteq A$.

i.
$$\begin{aligned} x \in \alpha(\overline{\theta}(B)) &\Longleftrightarrow \alpha(x) \in \overline{\theta}(B) \\ &\Longleftrightarrow [\alpha(x)]_\theta \cap B \neq \phi \\ &\Longleftrightarrow \exists\ y \in B \text{ such that } \alpha(x)\theta y \\ &\Longleftrightarrow \exists\ \alpha(y) \in \alpha(B) \text{ such that } x\theta\alpha(y) \\ &\Longleftrightarrow [x]_\theta \cap \alpha(B) \neq \phi \\ &\Longleftrightarrow x \in \overline{\theta}(\alpha(B)). \end{aligned}$$

Hence $\overline{\theta}(\alpha(B)) = \alpha(\overline{\theta}(B))$.
ii. Let $x \in \alpha(\underline{\theta}(B))$. Then $\alpha(x) \in \underline{\theta}(B)$ if and only if $[\alpha(x)]_\theta \subseteq B$, if and only if $\alpha([\alpha(x)]_\theta) \subseteq \alpha(B)$. We have $\alpha([\alpha(x)]_\theta) = [x]_\theta$. Therefore, $[x]_\theta \subseteq \alpha(B)$ if and only if $x \in \underline{\theta}(\alpha(B))$. Hence, $\underline{\theta}(\alpha(B)) = \alpha(\underline{\theta}(B))$. ∎

Proposition 3.4. *Let $\langle A, F\rangle$ and $\langle B, F\rangle$ be any two algebras of the same type. Let $f : A \to B$ be a homomorphism. Then for any subset $X \subseteq A$, $f(\overline{ker(f)}(X)) = f(X)$. Further, if f is one-one, then $f(\underline{ker(f)}(X)) = f(X)$.*

Proof. Let $x \in f(\overline{ker(f)}(X))$. Then $x = f(y)$, for some $y \in \overline{ker(f)}(X)$. This implies that, there exists $z \in X$ such that $y\, ker(f)\, z$. Then $f(y) = f(z)$ and $x = f(z) \in f(X)$. Therefore, $f(\overline{ker(f)}(X)) \subseteq f(X)$. Also, $X \subseteq \overline{ker(f)}(X)$ implies $f(X) \subseteq f(\overline{ker(f)}(X))$. Hence we have $f(\overline{ker(f)}(X)) = f(X)$. If f is one-one, then $ker(f)$ is an identity relation and consequently we have $f(\underline{ker(f)}(X)) = f(X)$. ∎

4 Rough Approximations in Orthomodular Lattices

Let L be an orthomodular lattice and I be a p-ideal of L. Then

$$\theta_I = \{(a, b) \in L^2 / (a \vee b) \wedge (a' \vee b') \in I\}$$

is a congruence relation on L [6]. For $a, b \in L$, we say a is congruent to b with respect to I, $a\theta_I b$, if $(a \vee b) \wedge (a' \vee b') \in I$. For any $a \in L$. let $[a]_I = \{b \in L/(a \vee b) \wedge (a' \vee b') \in I\}$ denote the congruence class of θ_I containing a. Let A be a nonempty subset of L. We denote the lower and the upper approximation operators $\underline{\theta_I}$ and $\overline{\theta_I}$ defined with respect to the congruence relation θ_I by $\underline{I}$ and $\overline{I}$. That is,

$$\underline{I}(A) = \{x \in L/[x]_I \subseteq A\} \text{ and}$$
$$\overline{I}(A) = \{x \in L/[x]_I \cap A \neq \phi\}.$$

For any subset A of L, $A' = \{a'/a \in A\}$ where a' is the orthocomplement of a in L and the annihilator of A is $A^{\perp} = \{x \in L/x \wedge y = 0, \text{ for all } y \in A\}$. If $A, B \subseteq L$ such that $A \subseteq B$, then $B^{\perp} \subseteq A^{\perp}$.

Proposition 4.1. *Let I be a p-ideal of an orthomodular lattice L and A be a nonempty subset of L. Then*

i. $\underline{I}(A') = \underline{I}(A)'$ *and* $\overline{I}(A') = \overline{I}(A)'$.

ii. $\underline{I}(A^{\perp}) \subseteq \underline{I}(A)^{\perp}$ *and* $\overline{I}(A)^{\perp} \subseteq \overline{I}(A^{\perp})$.

iii. *If A is a subalgebra of L, then $\overline{I}(A)$ is a subalgebra of L.*

iv. *If A is an ideal of L, then $\overline{I}(A)$ is also an ideal of L.*

v. *If A is a filter of L, then $\overline{I}(A)$ is also a filter of L.*

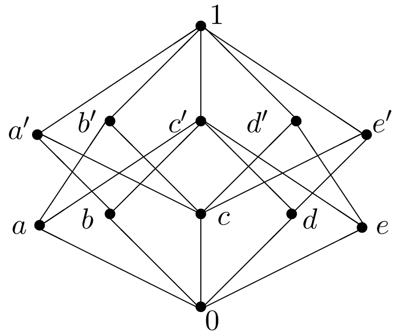
vi. *If A is a p-ideal of L, then $\overline{I}(A)$ is a p-ideal of L.*

vii. *If A is a p-filter of L, then $\overline{I}(A)$ is a p-filter of L.*

Proof. i. The proof follows from Proposition 3.3.
ii. We have $\underline{I}(A) \subseteq A$ which implies $A^{\perp} \subseteq \underline{I}(A)^{\perp}$. Therefore $\underline{I}(A^{\perp}) \subseteq A^{\perp} \subseteq \underline{I}(A)^{\perp}$. Dually, we have $\overline{I}(A)^{\perp} \subseteq \overline{I}(A^{\perp})$.
iii. The proof follows from Proposition 3.2.
iv. Let A be an ideal of L. It is enough to show that if $x, y \in L$ such that $x \leq y$ and $y \in \overline{I}(A)$, then $x \in \overline{I}(A)$. Let $x, y \in L$ be such that $x \leq y$ and $y \in \overline{I}(A)$. Then $[y]_I \cap A \neq \phi$ and there exists $a \in L$ such that $a \in [y]_I$ and $a \in A$. Since θ_I is a congruence relation and A is an ideal, $(a \wedge x)\theta_I(y \wedge x) = x$ and $a \wedge x \in A$. This implies that $[x]_I \cap A \neq \phi$ and so $x \in \overline{I}(A)$. Hence, $\overline{I}(A)$ is an ideal of L.
v. The proof follows by the duality of (iv).
vi. Let A be a p-ideal of L. It is enough to show that $a \in \overline{I}(A)$ implies $x \wedge (x' \vee a) \in \overline{I}(A)$, for any $x \in L$. If $a \in \overline{I}(A)$, then $[a]_I \cap A \neq \phi$. There exists $b \in L$ such that $b \in [a]_I$ and $b \in A$. Since θ_I is a congruence relation on L, $(x' \vee a)\theta_I(x' \vee b)$ for any $x \in L$. This implies that $[x \wedge (x' \vee a)]\theta_I[x \wedge (x' \vee b)]$ for any $x \in L$. Thus, we have $x \wedge (x' \vee b) \in [x \wedge (x' \vee a)]_I$. Since A is a p-ideal and $b \in A$, by Proposition 2.4 we have $x \wedge (x' \vee b) \in A$ for any $x \in L$. Thus $[x \wedge (x' \vee a)]_I \cap A \neq \phi$, which implies that $x \wedge (x' \vee a) \in \overline{I}(A)$. Hence, $\overline{I}(A)$ is a p-ideal of L.
vii. The proof follows by the duality of (vi). ■

Example 4.2. *$I = \{0, c\}$ is a p-ideal of the orthomodular lattice given in Figure 1. Then their corresponding congruence classes of θ_I are* $[0]_I = [c]_I = \{0, c\}, [a]_I = [b']_I = \{a, b'\}$, $[b]_I = [a']_I =$

$\{a', b\}, [d]_I = [e']_I = \{d, e'\}$, $[e]_I, [d']_I = \{e, d'\}$, $[c']_I = [1]_I = \{c', 1\}$. *Let* $A = \{a, b', a'\}$. *Then* $A^{\perp} = \{0, d, e\}$. *This implies* $\overline{I}(A^{\perp}) = \{0, c, d, e, d', e'\}$ *and* $\overline{I}(A)^{\perp} = \{0, d, e\}$. *Similarly, we have* $\underline{I}(A^{\perp}) = \phi$ *and* $\underline{I}(A)^{\perp} = \{0, b, d, e\}$. *Hence,* $\overline{I}(A^{\perp}) \nsubseteq \overline{I}(A)^{\perp}$ *and* $\underline{I}(A)^{\perp} \nsubseteq \underline{I}(A^{\perp})$.



$Figure\ 1$

Note that the subset $M = \{c, c'\}$ is not a subalgebra of L where as $\overline{I}(M) = \{0, c, c', 1\}$ is a subalgebra of L. The subset $A = \{c\}$ is not an ideal of L where as $\overline{I}(A) = \{0, c\}$ is an ideal, in fact a p-ideal of L.

On the other hand, for all the ideals of L, its lower approximation is not an ideal of L. For instance, in the above example consider the ideal $A = \{0, a\}$ of L. We have $\underline{I}(A) = \phi$, which is not an ideal of L. Hence we have the following characterization.

Lemma 4.3. *Let I be a p-ideal of an orthomodular lattice L. Then for an ideal A of L, the following are equivalent:*

i. $\underline{I}(A)$ *is an ideal of* L

ii. $\underline{I}(A) \neq \phi$

iii. $I \subseteq A$

Moreover, $\underline{I}(A) = A$.

Proof. (i)⇒(ii) Since $\underline{I}(A)$ is an ideal, $0 \in \underline{I}(A)$. Thus $\underline{I}(A) \neq \phi$.
(ii)⇒(iii) Since $\underline{I}(A) \neq \phi$, there exists $x \in L$ such that $x \in \underline{I}(A)$. This implies that $[x]_I \subseteq A$. Let $y \in I = [0]_I$. Since θ_I is a congruence relation, $(x \vee y)\theta_I(x \vee 0) = x$ and consequently $x \vee y \in [x]_I \subseteq A$. So, we have $x \vee y \in A$. Since A is an ideal and $y = y \wedge (x \vee y) \leq x \vee y$, $y \in A$. Thus $I \subseteq A$.
(iii)⇒(i) Suppose $I \subseteq A$. Since A is an ideal, it is enough to show that $\underline{I}(A) = A$. We have $\underline{I}(A) \subseteq A$. Let $x \in A$ and $y \in [x]_I$. By Theorem 2.5, we have $x \vee t = y \vee t$, for some $t \in I$. Since $I \subseteq A$ and $x \in A, x \vee t \in A$, we get $y \vee t \in A$. Since A is an ideal and $y = y \wedge (y \vee t) \leq y \vee t \in A$, $y \in A$. Thus $[x]_I \subseteq A$ and so $x \in \underline{I}(A)$. Therefore, $A \subseteq \underline{I}(A)$ and hence $\underline{I}(A)$ is an ideal of L. ■

Dually, we have the following result for filter. If I is an ideal of an orthomodular lattice L, then I' is a filter of L.

Lemma 4.4. *Let I be a p-ideal of an orthomodular lattice L. Then for any filter A of L, the following are equivalent:*

i. $\underline{I}(A)$ *is a filter of* L.

ii. $\underline{I}(A) \neq \phi$.

iii. $I' \subseteq A$.

Moreover, $\underline{I}(A) = A$.

5 Rough Ideals and Rough Filters

Definition 5.1. *Let I be a p-ideal of an orthomodular lattice L and A be a nonempty subset of L. Then A is said to be a rough ideal (filter) with respect to the p-ideal I, if both $\underline{I}(A)$ and $\overline{I}(A)$ are ideals (filters) of L.*

Lemma 5.2. *Let I be a p-ideal and A be an ideal of an orthomodular lattice L. Then A is a rough ideal with respect to a p-ideal I of L if and only if $I \subseteq A$.*

Proof. Since A is an ideal, $\overline{I}(A)$ is also an ideal. Hence A is a rough ideal with respect to the p-ideal I if and only if $\underline{I}(A)$ is an ideal, if and only if $I \subseteq A$. ■

Let L be a lattice and $a \in L$. Then $(a] = \{x \in L/x \leq a\}$ and $[a) = \{x \in L/a \leq x\}$ are called the principal ideal and principal filter respectively generated by a . The set of all ideals and filters of L denoted by $Id(L)$ and $F(L)$ respectively form a lattice with respect to the set inclusion. They are called the lattice of ideals and the lattice of filters of L respectively.

Theorem 5.3. *Let I be a p-ideal of an orthomodular lattice L. Then for $A \in Id(L)$, the following are equivalent:*

i. $A = \overline{I}(B)$ *for some* $B \subseteq L$.

ii. $A \in [I)$ *in* $Id(L)$.

iii. A *is definable with respect to the p-ideal* I.

iv. A *is a rough ideal of* L *with respect to the p-ideal* I.

Proof. Let A be an ideal of L.
(i)⇒ (ii) Suppose $A = \overline{I}(B)$ for some $B \subseteq L$. Since A is an ideal, $0 \in A$. Then by (iii) of Remark 3.1, $0 \in \overline{I}(B)$ implies $I = [0]_I \subseteq \overline{I}(B) = A$. Therefore $I \subseteq A$ and hence $A \in [I)$ in $Id(L)$.
(ii)⇒ (iii) Suppose $A \in [I)$ in $Id(L)$, then $I \subseteq A$. By Lemma 4.3, we have $\underline{I}(A) = A$. Then by (ii) of Remark 3.1, we have A is definable with respect to the p-ideal I.
(iii)⇒ (iv) Suppose A is definable with respect to the p-ideal I, then $\underline{I}(A) = A = \overline{I}(A)$. Since A is an ideal of L, $\underline{I}(A)$ and $\overline{I}(A)$ are also ideals of L. Hence A is a rough ideal of L with respect to the p-ideal

I.
(iv)⇒ (i) Suppose A is a rough ideal of L with respect to the p-ideal I. By Lemma 5.2, we have $I \subseteq A$. Then $A = \overline{I}(A)$. ∎

From the above theorem, we have the set of all rough ideals with respect to a p-ideal I of L forms a principal filter generated by the p-ideal I in $Id(L)$. Dually, we have the following results for filter.

Theorem 5.4. *Let I be a p-ideal of an orthomodular lattice L. Then for $A \in F(L)$, the following are equivalent:*

i. $A = \overline{I}(B)$ *for some* $B \subseteq L$.

ii. $A \in [I')$ *in* $F(L)$.

iii.A *is definable with respect to the p-ideal* I.

iv. A *is a rough filter of* L *with respect to the p-ideal* I.

From the above theorem, we have the set of all rough filters with respect to a p-ideal I of L forms a principal filter generated by the p-filter I' in $F(L)$.

Proposition 5.5. *Let I be a p-ideal of an orthomodular lattice L and $A \subseteq L$. Then A is a rough ideal of L if and only if A' is a rough filter of L.*

Proof. A is a rough ideal of $L \Leftrightarrow \underline{I}(A)$ and $\overline{I}(A)$ are ideals of L
$\Leftrightarrow \underline{I}(A)'$ and $\overline{I}(A)'$ are filters of L
$\Leftrightarrow \underline{I}(A')$ and $\overline{I}(A')$ are filters of L (By Proposition 4.1)
$\Leftrightarrow A'$ is a rough filter of L. ∎

Remark 5.6. *In an orthomodular lattice L, $\{0\}$ is a p-ideal. Therefore $\theta_{\{0\}}$ is the null congruence on L and $[a]_{\{0\}} = \{a\}$, for all $a \in L$. Then every subset of an orthomodular lattice L is definable with respect to $\{0\}$. So, every ideal of an orthomodular lattice L is a rough ideal of L and every filter of L is a rough filter of L.*

Theorem 5.7. *Let I be a p-ideal of an orthomodular lattice L and A be a subset of L. If $\overline{I}(A)$ is an ideal, then $\overline{I}(A)$ is the smallest ideal containing the ideal I and the set A in L.*

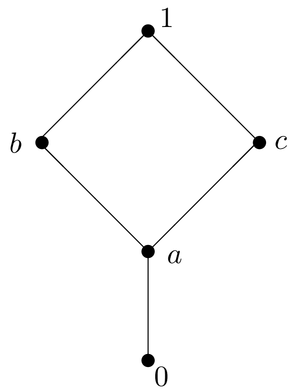
Proof. If $\overline{I}(A)$ is an ideal, then $I \subseteq \overline{I}(A)$. Thus $\overline{I}(A)$ is an ideal containing I and A. Suppose B is any other ideal containing I and A. Then $A \subseteq B$ implies $\overline{I}(A) \subseteq \overline{I}(B)$. Since $I \subseteq B$, by Theorem 5.3 we have $\overline{I}(B) = B$. Therefore $\overline{I}(A) \subseteq B$. Hence $\overline{I}(A)$ is the smallest ideal containing the ideal I and the set A in L. ∎

Corollary 5.8. *Let I be a p-ideal of an orthomodular lattice L and J be an ideal of L. Then $\overline{I}(J)$ is the smallest ideal containing the ideals I and J in L. That is, $\overline{I}(J)$ is the join of the ideals I and J in the lattice of ideals $Id(L)$.*

Proof. By Proposition 4.1, we have $\overline{I}(J)$ is an ideal of L. Then by Theorem 5.7, we have $\overline{I}(J)$ is the smallest ideal containing the ideals I and J in L. ■

The above theorem 5.7 is not true in general for any lattice. That is, for any ideal A and a congruence relation θ of an arbitrary lattice, $\overline{\theta}(A)$ needs not be the smallest ideal containing A.

Remark 5.9. *Consider the lattice L given in Figure 2. Let θ be a congruence relation on L such that $[a]_\theta = [b]_\theta = [c]_\theta = [1]_\theta = \{a, b, c, 1\}$ and $[0]_\theta = \{0\}$. $A = \{0, a\}$ is an ideal of L. But $\overline{\theta}(A) = L$. Hence $\overline{\theta}(A)$ is not the smallest ideal containing A.*



$Figure\ 2$

Corollary 5.10. *Let I, J be two p-ideals of an orthomodular lattice L. Then $\overline{I}(J) = \overline{J}(I)$.*

Proof. By Theorem 5.7, $\overline{I}(J)$ is the smallest ideal containing both I and J. Similarly, we have $\overline{J}(I)$ is also the smallest ideal containing the ideals I and J. Hence $\overline{I}(J) = \overline{J}(I)$. ■

Corollary 5.11. *Let I be a p-ideal of an orthomodular lattice L and J be an ideal of L. Then the smallest rough ideal containing the ideals I and J is $\overline{I}(J)$.*

Theorem 5.12. *Let I be a p-ideal of an orthomodular lattice L and A be a nonempty subset of L. If $\overline{I}(A)$ is a filter, then $\overline{I}(A)$ is the smallest filter containing the filter I' and the set A in L.*

Proof. If $\overline{I}(A)$ is a filter, then $I' \subseteq \overline{I}(A)$. Thus $\overline{I}(A)$ is a filter containing I' and A. Suppose B is any other filter containing I' and A. Then $\overline{I}(A) \subseteq \overline{I}(B)$ and since $I' \subseteq B$, by Theorem 5.4 we have $\overline{I}(B) = B$. Therefore $\overline{I}(A) \subseteq B$. Hence $\overline{I}(A)$ is the smallest filter containing the filter I' and the set A in L. ■

Corollary 5.13. *Let I be a p-ideal of an orthomodular lattice L and J be a filter of L. Then $\overline{I}(J)$ is the smallest filter containing the filters I' and J in L. That is, $\overline{I}(J)$ is the join of the filters I' and J in the lattice of filters $F(L)$.*

Proof. By Proposition 4.1, we have $\overline{I}(J)$ is a filter. Then by the above theorem, we have $\overline{I}(J)$ is the smallest filter containing the filters I' and J in L. ■

Corollary 5.14. *Let I be a p-ideal of an orthomodular lattice L and J be a filter of L. Then the smallest rough filter containing the filters I' and J is $\overline{I}(J)$.*

Proposition 5.15. *Let f be a homomorphism from an orthomodular lattice L_1 onto an orthomodular lattice L_2. Let I_2 be a p-ideal of L_2 and $I_1 = \{x \in L_1 / f(x) \in I_2\}$. Then for any $A \subseteq L_1$, the following hold:*

i. *I_1 is a p-ideal of L_1.*

ii. $f(\overline{I_1}(A)) = \overline{I_2}(f(A))$.

iii. $f(\underline{I_1}(A)) \subseteq \underline{I_2}(f(A))$ *and if f is 1-1, then* $f(\underline{I_1}(A)) = \underline{I_2}(f(A))$.

Proof. i. Let $x, y \in I_1$. Then $f(x), f(y) \in I_2$. Since I_2 is a p-ideal, $f(x \vee y) = f(x) \vee f(y) \in I_2$. So, $x \vee y \in I_1$. Now let $x, y \in L_1$ be such that $x \in I_1$ and $y \leq x$ in L_1. Then $f(y) \leq f(x)$ in L_2. Since I_2 is a p-ideal and $f(x) \in I_2$, $f(y) \in I_2$. So, $y \in I_1$. Thus I_1 is an ideal of L_1. Let $a \in I_1$ and $x \in L_1$. Then $f(a) \in I_2$ and $f(x) \in L_2$. Since I_2 is a p-ideal, $f(x \wedge (x' \vee a)) = f(x) \wedge (f(x)' \vee f(a)) \in I_2$. Therefore $x \wedge (x' \vee a) \in I_1$. Hence I_1 is a p-ideal of L_1.
ii. Let $x \in f(\overline{I_1}(A))$. Then $x = f(y)$, where $y \in \overline{I_1}(A)$ and so $[y]_{I_1} \cap A \neq \phi$. Then there exists $a \in L_1$ such that $a \in [y]_{I_1}$ and $a \in A$, imply $f(a) \in [f(y)]_{I_2}$ and $f(a) \in f(A)$. So, we have $x = f(y) \in \overline{I_2}(f(A))$. Therefore $f(\overline{I_1}(A)) \subseteq \overline{I_2}(f(A))$. Conversely, let $x \in L_1$ be such that $f(x) \in \overline{I_2}(f(A))$. So, $[f(x)]_{I_2} \cap f(A) \neq \phi$. Then there exists $a \in A$ such that $f(a) \in [f(x)]_{I_2}$ which implies $a \in [x]_{I_1}$. Therefore $x \in \overline{I_1}(A)$. That is, $f(x) \in f(\overline{I_1}(A))$. Thus $\overline{I_2}(f(A)) \subseteq f(\overline{I_1}(A))$. Hence $f(\overline{I_1}(A)) = \overline{I_2}(f(A))$.

iii. Let $x \in f(\underline{I_1}(A))$. Then $x = f(y)$, where $y \in \underline{I_1}(A)$. Then $[y]_{I_1} \subseteq A$ and hence $[f(y)]_{I_2} \subseteq f(A)$. So, $f(y) \in \underline{I_2}(f(A))$. That is, $x \in \underline{I_2}(f(A))$. Therefore, $f(\underline{I_1}(A)) \subseteq \underline{I_2}(f(A))$. Now suppose f is 1-1. Let $x \in \underline{I_2}(f(A))$. Since f is surjective, $x = f(y)$, where $y \in L_1$. Then $[f(y)]_{I_2} \subseteq f(A)$. Since f is 1-1, $[f(y)]_{I_2} \subseteq f(A)$ implies $[y]_{I_1} \subseteq A$. Then $y \in \underline{I_1}(A)$ and $x = f(y) \in f(\underline{I_1}(A))$. Hence $f(\underline{I_1}(A)) = \underline{I_2}(f(A))$. ∎

Corollary 5.16. *Let f be a homomorphism from an orthomodular lattice L_1 onto an orthomodular lattice L_2. Let I_2 be a p-ideal of L_2 and $I_1 = \{x \in L_1 / f(x) \in I_2\}$. Then for a subset A of L_1,*

i. $\overline{I_1}(A)$ is an ideal of L_1 if and only if $\overline{I_2}(f(A))$ is an ideal of L_2,

ii. $\underline{I_1}(A)$ is an ideal of L_1 if and only if $\underline{I_2}(f(A))$ is an ideal of L_2 if f is 1-1.

Proof. Since f is a bijection, B is an ideal in L_1 if and only if $f(B)$ is an ideal in L_2. Hence the proof follows from the Proposition 5.15. ∎

Corollary 5.17. *Let f be an isomorphism from an orthomodular lattice L_1 onto an orthomodular lattice L_2. Let I_2 be a p-ideal of L_2 and $I_1 = \{x \in L_1 / f(x) \in I_2\}$. Then for $A \subseteq L_1$, A is a rough ideal of L_1 with respect to the p-ideal I_1 if and only if $f(A)$ is a rough ideal of L_2 with respect to the p-ideal I_2.*

Proof. The proof follows from Proposition 5.16. ∎

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