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Prime O-ideals of distributive lattices

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Abstract

In this paper we derive some sufficient conditions for a prime ideal of a distributive lattice to become an O-ideal. A set of equivalent conditions are established for every O-ideal to become an annihilator ideal. An equivalency is obtained between prime O-ideals and minimal prime ideals of a distributive lattice. Finally, the concept of co-maximality is introduced and obtained that any two distinct prime O-ideals of a distributive lattice are co-maximal.

Keywords: Prime O-ideal, filter, annihilator ideal, $\star$-lattice, co-maximality.
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1 Introduction

The notion of annihilators was introduced in lattices by Mark Mandelker [6] and the properties of these annihilators were extensively studied by many authors, particularly T.P. Speed in the papers [7] and [8]. Later the class of annulets played a vital role in characterizing many algebraic structures like normal lattices [2], quasi-complemented lattices [3]. These quasi-complemented lattices are equivalent to $\star$-lattices of T.P. Speed [8]. As an extension of the study of W.H. Cornish in [3], he introduced the concept of O-ideals in distributive lattices and studied their properties with the help of congruences in the paper [4].

This paper is to study the primeness of O-ideals in distributive lattices. In this paper, we observe that the class of O-ideals forms a distributive lattice on its own. We also derive some sufficient conditions for a prime ideal of a distributive lattice to become an O-ideal. An equivalency between the class of all prime O-ideals and class of all minimal prime ideals is obtained. A necessary and sufficient condition for every O-ideal of a distributive lattice to become a prime ideal is also derived in our preset work. In a $\star$-lattice, a set of equivalent conditions are established for every O-ideal to become an annihilator ideal.

Later the concept of co-maximality of O-ideals is introduced and a sufficient condition for two annulets to become co-maximal is obtained . Further, it is proved that any two distinct prime O-ideals of a distributive lattice are co-maximal.

2 Preliminaries

In this section, we recall certain definitions and important results which are useful for our preset work.

Definition 2.1. *[1] An algebra* $(L, \wedge, \vee)$ *of type* $(2, 2)$ *is called a distributive lattice if for all* $x, y, z \in L$, *it satisfies the following properties* $(1), (2), (3)$ *and* (4) *along with* (5) *or* $(5')$

(1). $x \wedge x = x, x \vee x = x$
(2). $x \wedge y = y \wedge x, x \vee y = y \vee x$

(3). $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$
(4). $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$
(5). $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5′). $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

A non-empty subset I of L is called an ideal(filter) of L if $a \vee b \in A(a \wedge b \in A)$ and $a \wedge x \in A(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. A proper ideal P of L is said to be prime if for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. It is clear that a subset P of L is a prime ideal if and only if $L - P$ is a prime filter. For any prime ideal P of L, the set $O(P) = \{ x \in L \mid x \wedge y = 0$ for some $y \notin P \}$ is an ideal [2] of L such that $O(P) \subseteq P$. A prime ideal P of L is called a minimal prime ideal[5] if there exists no prime ideal Q such that $Q \subset P$.

Theorem 2.2. *[1] If I is an ideal and F a filter of L with $I \cap F = \emptyset$, then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Theorem 2.3. *[5] A prime ideal P of L is a minimal prime ideal of L if and only if to each $x \in P$ there exists $y \notin P$ such that $x \wedge y = 0$.*

For any $A \subseteq L$, $A^* = \{ x \in L \mid a \wedge x = 0$ for all $a \in A \}$ is an ideal [8] of L. We write $(a]^*$ for $\{a\}^*$. Then clearly $(0]^* = L$ and $L^* = (0]$.

Lemma 2.4. *[8] For any two ideals I, J of L, we have the following:*

1). If $I \subseteq J$, then $J^* \subseteq I^*$
2). $I \subseteq I^{**}$
3). $I^{***} = I^*$
4). $I \cap J = (0] \Leftrightarrow I \subseteq J^*$
5). $(I \vee J)^* = I^* \cap J^*$

Definition 2.5. *[3] An ideal I of L is called an annihilator ideal of L if $I = I^{**}$ or equivalently $I = S^*$ for some non-empty subset S of L.*

An element $x \in L$ is called dense [8] if $(x]^* = (0]$. The set D of all dense elements of L forms a filter, whenever D is nonempty. An ideal I of L is called dense if $I^* = (0]$. Two ideals I, J of L are called co-maximal if $I \vee J = L$.

Definition 2.6. *[8] A lattice L is called a $\star$-lattice, if to each $x \in L$, $(x]^{**} = (x']^*$ for some $x' \in L$.*

Theorem 2.7. *[8] A lattice L is a $\star$-lattice if and only if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is dense.*

Definition 2.8. *[4] An ideal I of L is called an O-ideal if $I = O(F) = \bigcup_{x \in F} (x]^*$ for some filter F of L.*

3 Prime O-ideals

In this section, the properties of prime O-ideals are observed. Some characterization theorems are proved with the help of prime O-ideals of a distributive lattice.

Lemma 3.1. *For any distributive lattice L, we have the following:*

(a) For any filter F of L, $F \cap O(F) \neq \emptyset \Rightarrow F = O(F) = L$

(b) For any prime ideal P of L, $O(P) = O(L - P)$

Proof. (a) Suppose $F \cap O(F) \neq \emptyset$. Choose $x \in F \cap O(F)$. Then $x \in F$ and $x \wedge f = 0$ for some $f \in F$. Hence $0 = x \wedge f \in F$. Therefore $F = O(F) = L$.

(b) Let P be a prime ideal of L. Then we have

$$\begin{aligned} x \in O(P) &\Leftrightarrow x \wedge y = 0 \text{ for some } y \notin P \\ &\Leftrightarrow x \wedge y = 0 \text{ for some } y \in L - P \\ &\Leftrightarrow x \in O(L - P) \end{aligned}$$

Therefore $O(P) = O(L - P)$ for any prime ideal P of L. ■

Theorem 3.2. *For any distributive lattice L, we have the following conditions:*

(a) Every minimal prime ideal is an O-ideal

(b) Every non-dense prime ideal is an O-ideal

(c) If L is a $\star$-lattice, then every prime ideal P with $P \cap D = \emptyset$ is an O-ideal

Proof. (a). Let P be a minimal prime ideal of L. Then $L - P$ is a maximal filter of L. Since P is minimal, we get $P = O(P) = O(L - P)$. Hence P is an O-ideal.

(b). Let P be a non-dense prime ideal of L. Then there exists $0 \neq x \in L$, such that $x \in P^*$. Thus $P \subseteq P^{**} \subseteq (x]^*$. On the other hand, let $a \in (x]^*$. Then $a \wedge x = 0 \in P$ and $x \notin P$, because of $x \in P^*$. Hence $a \in P$. Thus $(x]^* \subseteq P$. Hence we get $P = (x]^* = O([x))$. Therefore P is an O-ideal of L.

(c). Let L be a $\star$-lattice and P a prime ideal of L such that $P \cap D = \emptyset$. We now show that $P = O(L - P)$. Clearly $O(L - P) = O(P) \subseteq P$. Conversely, let $x \in P$. Since L is a $\star$-lattice, there exists $y \notin P$ such that $x \wedge y = 0$ and $x \vee y \in D$. Hence $x \in (y]^*$ and $x \vee y \notin P$. Thus $x \in (y]^*$ and $y \notin P$. Therefore $x \in O(L - P)$. Thus P is an O-ideal. ■

In general, the converse of Theorem 3.2(a) is not true. That is, an O-ideal of a distributive lattice need not be a minimal prime ideal. In fact, it needs not even be a prime ideal which is evident from the following example.

Example 3.3. *Let $X = \{1, 2, 3, 4\}$ be a set and L the sublattice of the power set of X, which is generated by the sets $\{1\}, \{2\}$ and $\{3\}$. That is, $L = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\big\}$. Take $I = \big\{\emptyset, \{1\}\big\}$ and $F = \big\{\{2, 3\}, \{1, 2, 3\}\big\}$. Clearly I is an ideal and F a filter of L. Now*

$O(F) = (\{2,3\}]^* \cup (\{1,2,3\}]^* = \{\emptyset, \{1\}\} \cup \{\emptyset\} = \{\emptyset, \{1\}\} = I$. Hence I is an O-ideal of L. But I is not prime, because of $\{2\} \notin I$ and $\{3\} \notin I$ but $\{2\} \cap \{3\} = \emptyset \in I$.

Theorem 3.4. *Let I be a proper O-ideal of a distributive lattice L. Then I is prime if and only if I contains a prime ideal.*

Proof. The necessary part is obvious. To prove the sufficient part, assume that I contains a prime ideal, say P. Since I is an O-ideal of L, we get that $I = O(F)$ for some filter F of L. Choose $a, b \in L$ such that $a \notin I$ and $b \notin I$. Then $a \notin P$ and $b \notin P$. Hence $a \wedge b \notin P$. Thus $(a \wedge b]^* \subseteq P \subseteq I = O(F)$. Suppose $a \wedge b \in I = O(F)$. Then $a \wedge b \wedge f = 0$ for some $f \in F$. Hence $f \in (a \wedge b]^* \subseteq O(F)$. Thus $f \in F \cap O(F)$. Therefore $F \cap O(F) \neq \emptyset$. Thus $I = O(F) = F = L$, which is a contradiction. Therefore I is prime. ∎

Theorem 3.5. *Every prime O-ideal of a distributive lattice L is a minimal prime ideal.*

Proof. Let P be a prime O-ideal of L. Then $P = O(F)$ for some filter F of L. Let $x \in P = O(F)$. Then $x \wedge y = 0$ for some $y \in F$. Suppose $y \in P$. Then $y \in F \cap O(F)$. Hence $F \cap O(F) \neq \emptyset$. By Lemma 3.1(a), $P = O(F) = F = L$, which is a contradiction. Hence $y \notin P$. Therefore P is a minimal prime ideal. ∎

Lemma 3.6. *The class of all O-ideals of L forms a distributive lattice on its own*

Proof. For any two filters F, G of L, define

$$O(F) \cap O(G) = O(F \cap G) \text{ and } O(F) \sqcup O(G) = O(F \vee G)$$

$O(F \cap G)$ is the infimum of the O-ideals $O(F)$ and $O(G)$ and $O(F \vee G)$ is an upper bound of $O(F)$ and $O(G)$. Suppose $O(F) \subseteq O(H)$ and $O(G) \subseteq O(H)$ for some filter H of L. Let $x \in O(F \vee G)$. Then $x \wedge f \wedge g = 0$ for some $f \in F$ and $g \in G$. Hence, $x \wedge f \in O(G) \subseteq O(H)$. Thus, $x \wedge f \wedge h_1 = 0$ for some $h_1 \in H$. Hence $x \wedge h_1 \in O(F) \subseteq O(H)$. Thus $x \wedge h_1 \wedge h_2 = 0$ for some $h_2 \in H$. Since $h_1 \wedge h_2 \in H$, we get $x \in O(H)$. Hence $O(F \vee G)$ is the supremum of $O(F)$ and $O(G)$. Now, it can be easily observed that the class of all O-ideals forms a distributive lattice with respect to the operations $\cap$ and $\sqcup$. ∎

Theorem 3.7. *Let L be a $\star$-lattice. Then the following conditions are equivalent.*

(1) Every O-ideal is an annihilator ideal
(2) Every minimal prime ideal is an annihilator ideal
(3) Every prime O-ideal is of the form $(x]^{**}$ for some $x \in L$
(4) Every O-ideal is of the form $(x]^{**}$ for some $x \in L$
(5) Every minimal prime ideal is non-dense

Proof. $(1) \Rightarrow (2)$: Since every minimal prime ideal is an O-ideal, it is obvious.

$(2) \Rightarrow (3)$: Assume the condition (2). Let P be a prime O-ideal of L. Then by Theorem 3.5, P is a minimal prime ideal. Hence by condition (2), P is an annihilator ideal. Thus P is non-dense. Therefore $P = (x]^*$ for some $0 \neq x \in P^*$. Since L is a $\star$-lattice, there exists some $y \in L$ such that $P = (x]^* = (y]^{**}$.

$(3) \Rightarrow (4)$: Assume the condition (3). Let I be an O-ideal of L. Suppose I cannot be expressed in the form $(x]^{**}$ for some $x \in L$. Let us consider the set
$\Im = \{ \ J \mid J$ is an O-ideal of L which is not in the form of $(x]^{**}$ for some $x \in L \ \}$. Then clearly $\Im \neq \emptyset$. Let $\{J_i \mid i \in \Delta\}$ be a chain of O-ideals in $\Im$. Then clearly $\cup J_i$ is an O-ideal of L. Suppose $\cup J_i = (x]^{**}$ for some $x \in L$. We have always $J_i \subseteq \cup J_i = (x]^{**}$ for all $i \in \Delta$. Conversely, we have $x \in (x]^{**} = \cup J_i$. Hence $x \in J_i$ for some $i \in \Delta$. Thus $(x]^{**} \subseteq J_i$ for some $i \in \Delta$. Therefore $J_i = (x]^{**}$ for some $i \in \Delta$ and $x \in L$, which is a contradiction. Let M be a maximal element of $\Im$. Choose $a, b \in L$ such that $a \notin M$ and $b \notin M$. Since L is a $\star$-lattice, there exists $a', b' \in L$ such that $(a]^{**} = (a']^*$ and $(b]^{**} = (b']^*$. Then $M \subset M \vee (a] \subseteq M \vee (a]^{**} = M \vee (a']^* \subseteq M \sqcup (a']^*$. Similarly we get $M \subset M \sqcup (b']^*$. By the maximality of M, we can get $M \sqcup (a']^* = (x]^{**}$ and $M \sqcup (b']^* = (y]^{**}$ for some $x, y \in L$. Now

$$\begin{aligned} M \sqcup (a \wedge b]^{**} &= M \sqcup \{(a]^{**} \cap (b]^{**}\} \\ &= M \sqcup \{(a']^* \cap (b']^*\} \\ &= \{M \sqcup (a']^*\} \cap \{M \sqcup (b']^*\} \\ &= (x]^{**} \cap (y]^{**} \\ &= (x \wedge y]^{**} \end{aligned}$$

If $a \wedge b \in M$, then $(a \wedge b]^{**} \subseteq M$(since M is an O-ideal). Hence $M = (x \wedge y]^{**}$, which is a contradiction to $M \in \Im$. Therefore M is a prime O-ideal which cannot be expressed in the form of $(x]^{**}$ for some $x \in L$, which is a contradiction. Thus every O-ideal is of the form $(x]^{**}$ for some $x \in L$.

$(4) \Rightarrow (5)$: Assume the condition (4). Let P be a minimal prime ideal of L. Since every minimal prime ideal is an O-ideal, we get $P = (x]^{**}$ for some $x \in L$. Suppose $P^* = (0]$. Then $x \in (x]^{**}$ is a dense element in P, which is a contradiction. Therefore P is non-dense.

$(5) \Rightarrow (1)$: Assume that every minimal prime ideal is non-dense. Let I be an O-ideal of L. Then $I = O(F)$ for some filter F of L. We have always $I \subseteq I^{**}$. Let $c \in I^{**}$. We first prove that $(I \vee I^*) \cap D \neq \emptyset$. Suppose $(I \vee I^*) \cap D = \emptyset$. Then there exists a prime ideal P(say) such that $(I \vee I^*) \subseteq P$ and $P \cap D = \emptyset$. Let $x \in P$. Since L is a $\star$-lattice, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D$. Hence $x \vee y \notin P$. Thus $y \notin P$. Therefore P is a minimal prime ideal in L. By condition (5), P is non-dense. But $P^* \subseteq (I \vee I^*)^* = I^* \cap I^{**} = (0]$, which is a contradiction. Hence $(I \vee I^*) \cap D \neq \emptyset$. Choose $d \in (I \vee I^*) \cap D$. Then $d = a \vee b$ for some $a \in I$ and $b \in I^*$. Hence $(a]^* \cap (b]^* = (a \vee b]^* = (d]^* = (0]$. Thus $(b]^* \subseteq (a]^{**}$. Since $b \in I^*$ and $c \in I^{**}$, we get $b \wedge c = 0$. Also $a \in I = O(F)$ implies that $a \wedge f = 0$ for some $f \in F$. Now $c \in (b]^* \subseteq (a]^{**} \subseteq (f]^* \subseteq O(F) = I$. Hence we get $I^{**} \subseteq I$. Therefore I is an annihilator ideal of L. ■

4 Comaximality of prime O-ideals

Let us recall that two ideals I, J of a lattice L are comaximal if $I \vee J = L$. We now introduce $\sqcup$-comaximality of O-ideals of a distributive lattice.

Definition 4.1. *Two O-ideals I, J of a distributive lattice L are called $\sqcup$-comaximal if $I \sqcup J = L$.*

Any two comaximal O-ideals are $\sqcup$-comaximal. But the converse is not true. This can be illustrated in the following example.

Example 4.2. *Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in Figure 4.1.*

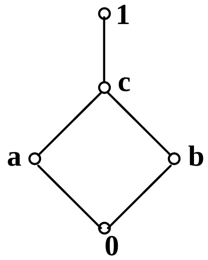

Figure 4.1: Hasse diagram of the distributive lattice $L = \{0, a, b, c, 1\}$.

Consider the ideals $I = \{0, a\}$ and $J = \{0, b\}$. Clearly $F = \{b, c, 1\}$ and $G = \{a, c, 1\}$ are filters in L. It can be easily verified that $I = O(F)$ and $J = O(G)$. Thus I and J are two distinct O-ideals of L. Now $I \sqcup J = O(F) \sqcup O(G) = O(F \vee G) = O(L) = L$. Hence I and J are $\sqcup$-comaximal. But $I \vee J = \{0, a\} \vee \{0, b\} = \{0, a, b, c\} \neq L$. Therefore I and J are not comaximal in L.

Lemma 4.3. *Let L be a distributive lattice. Then, for any $x, y \in L$ with $x \wedge y = 0$, $(x]^*$ and $(y]^*$ are $\sqcup$-comaximal in L.*

Proof. Choose $x, y \in L$ such that $x \wedge y = 0$. Then we have

$$\begin{aligned} (x]^* \sqcup (y]^* &= O([x)) \sqcup O([y)) \\ &= O([x) \vee [y)) \\ &= O([x \wedge y)) \\ &= (x \wedge y]^* \\ &= (0]^* \\ &= L \end{aligned}$$

Therefore, $(x]^*$ and $(y]^*$ are $\sqcup$-comaximal in L. ∎

Theorem 4.4. *Let L be a distributive lattice. Then any two distinct prime O-ideals of L are $\sqcup$-comaximal.*

Proof. Let P and Q be two distinct prime O-ideals of L. Then by Theorem 3.5, P and Q are minimal prime ideals. Choose $a \in P - Q$ and $b \in Q - P$. Since P, Q are minimal primes, there exists $x \notin P$ and $y \notin Q$ such that $a \wedge x = b \wedge y = 0$. Since $b \wedge x \notin P$ and P is a prime ideal, we can get $(b \wedge x]^* \subseteq P$. Similarly, we get $(a \wedge y]^* \subseteq Q$. Now

$$\begin{aligned}(b \wedge x) \wedge (a \wedge y) &= b \wedge a \wedge x \wedge y \\ &= a \wedge x \wedge b \wedge y \\ &= 0 \wedge 0 \\ &= 0\end{aligned}$$

By Lemma 4.3, we get $(b \wedge x]^* \sqcup (a \wedge y]^* = L$. Hence $L = (b \wedge x]^* \sqcup (a \wedge y]^* \subseteq P \sqcup Q$. Therefore P and Q are $\sqcup$-comaximal in L. ∎

In the above theorem, the primeness of O-ideals of a distributive lattice is inevitable to derive the $\sqcup$ -comaximality which can be observed from the distributive lattice given in Example 3.3. For, consider the two filters $F = \{\{2,3\}, \{1,2,3\}\}$ and $G = \{\{1,2\}, \{1,2,3\}\}$. Then clearly $O(F) = \{\emptyset, \{1\}\}$ and $O(G) = \{\emptyset, \{3\}\}$ are O-ideals. It can also be noted that they are not prime ideals in L. Now, $O(F) \sqcup O(G) = O(F \vee G) = O\big(\{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}\big) = \{\emptyset, \{1\}, \{3\}, \{1,3\}\} \neq L$. Therefore $O(F)$ and $O(G)$ are not $\sqcup$-comaximal in L.

Definition 4.5. *Let J be an ideal of a distributive lattice L. For any $x \in L$, define*

$$(x]^+ = \{\, a \in J \mid a \wedge x = 0 \,\}$$

Clearly $(x]^+ = J \cap (x]^*$ is an ideal in J. Moreover $(x]^+$ is an O-ideal whenever J is an O-ideal.

Theorem 4.6. *Let J be an O-ideal of a distributive lattice L. Then for any $x, y \in L$ with $x \wedge y = 0$, $(x]^+$ and $(y]^+$ are $\sqcup$-comaximal in J.*

Proof. Let $x, y \in L$ such that $x \wedge y = 0$. Then by Lemma 4.3, we get that $(x]^* \sqcup (y]^* = L$. Now

$$\begin{aligned} J &= J \cap L \\ &= J \cap \{(x]^* \sqcup (y]^*\} \\ &= \{J \cap (x]^*\} \sqcup \{J \cap (y]^*\} \\ &= (x]^+ \sqcup (y]^+\end{aligned}$$

Therefore $(x]^+$ and $(y]^+$ are $\sqcup$-comaximal in J. ∎

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