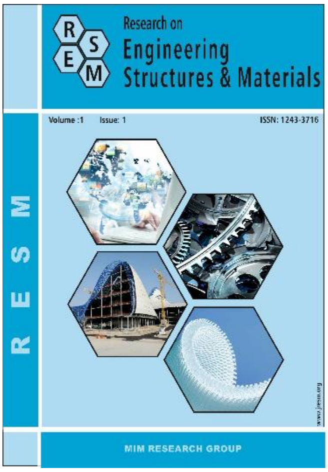
Research on Engineering Structures & Materials
journal homepage: http://jresm.org
RESEARCH GROUP



Solution of model equation of completely passive natural convection by improved differential transform method

Mustafa Ekici, Fatma Ayaz

Online Publication Date: 30 May 2016
URL: http://www.jresm.org/archive/resm2015.10me0818.html
DOI: http://dx.doi.org/10.17515/resm2015.10me0818

Journal Abbreviation: *Res. Eng. Struct. Mat.*

Research on Engineering Structures & Materials
journal homepage: http://jresm.org
MIM RESEARCH GROUP

Research Article

Solution of model equation of completely passive natural convection by improved differential transform method

Mustafa Ekici[1]*, Fatma Ayaz[2]

[1]*Faculty of Education, Usak University, Usak, Turkey*
[2]*Department of Mathematics, Gazi University, Ankara, Turkey*

Article Info

Article history:
Received 18 Aug 2015
Revised 17 May 2016
Accepted 27 May 2016

Keywords:
Differential transform method,
Natural convection,
Least square method,
Improved differential transform method

Abstract

In this paper we investigate the natural convection which exists under the Boussinesq approximation and completely passive boundary conditions. This phenomenon is applicable to the cooling of a heated body and heating and cooling of rooms and buildings and etc. The model equations of momentum and energy are solved by using improved differential transform method. This method is an iterative way for obtaining Taylor series coefficients. The convergence of the Taylor series is obtained by combining the method with the Least Square method for the problem first time here. The combination of these two powerful methods yields semi analytical solutions in the form of a polynomial through a straightforward manner. The efficiency and applicability of the algorithm are assessed and tested and error calculations have been presented.

Introduction

The heat transfers with convection involving fluid flow is generally divided into two basic processes: the forced convection and the natural convection. There are some differences between natural and forced convection but the main difference is the mechanism by which flow is generated. The forced convection where the motion of the fluid arises externally, for instance, by a fan, a blower, the wind, or the motion of the heated objects itself, and flow is driven by the pressure difference [1,2]. On the other hand, the natural convection of the fluid arises naturally by the effect of density difference which exists from a temperature or concentration difference in a body force field such as gravity. Also, velocities and the pressure differences in natural convection are usually much smaller than those in forced convection. Some of the examples of natural convection are the cooling of the heated body in ambient air, heating and cooling of rooms and buildings, recirculating flow driven by temperature and salinity differences in oceans as well as electronic cooling systems and further information can be obtained from [3,4].

High temperature for enclosures used in electronics has been the major problem. Reducing the operating temperature of components does not completely solve the problem, because integrated circuits are degraded by temperature cycling and surging. Heat conduction is also added to the overall cooling problem, since PCBs and other components are mounted using conducting materials such as bakelite and ceramics, Power dissipation by such electronic equipment is an important problem and will become more difficult due to future miniaturization and increased complexity, as a result, there is further need for more experimental and numerical research to improve electronic cooling techniques. As it is emphasized in [5], a need for thermal control in electronic 50% increases the component lifetime for operating temperatures at least 10 °C below the recommended manufacturer's

*Corresponding author: mustafa.ekici@usak.edu.tr
DOI: http://dx.doi.org/10.17515/resm2015.10me0818

maximum operating temperature. In [6], it is emphasized the role of the thermal analysis in the design of electronic packaging and the need for experimental heat transfer coefficient measurements to validate mathematical and numerical analyses. In [7] significant progresses of computational tools used to analyse cooling in electronic systems have been reported,

As it is stated in [2,8], the analytical and experimental study of processes involving natural convection much more complicated than those involving forced convection. The governing equations of the process are in nonlinear nature therefore; we need to use some approximate or numerical methods in order to overcome these difficulties. Consequently, here we use an approximate technique which is called Improved Differential Transform Method (IDTM) to solve these types of problems.

2. Preliminaries

We can denote the Differential Transform Method as shortly by DTM. Let us consider a function $w(x)$, the differential transform of the function is defined as follows,

$$W(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k}w(x)\right]_{x=0} \tag{1}$$

where $w(x)$ is the original function and $W(k)$ is the transformed function. Here $\frac{d^k}{dx^k}$ means the k th derivative with respect to x. The differential inverse transform of $W(k)$ is defined as

$$w(x) = \sum_{k=0}^{N} W(k)x^k \tag{2}$$

where N indicates the upper limit of the sum. Now combining Eqs. (1) and (2), we obtain

$$w(x) = \sum_{k=0}^{N} \frac{1}{k!}\left[\frac{d^k}{dx^k}w(x)\right]_{x=0} x^k \tag{3}$$

Consequently, Eq. (3) indicates the concept of DTM which overlaps with Taylor series expansion. However, this method differs from the way that the evaluation of the series coefficients. With this technique we overcome the difficulties due to obtaining higher order derivatives of the unknown function in Taylor series [9].

2.1. One-Dimensional Differential Transform Method

One-dimensional DTM has the following main properties [9,10]:

1. If $w(x) = u(x) \pm v(x)$ then $W(k) = U(k) \pm V(k)$
2. If $w(x) = \alpha\, u(x)$ then $W(k) = \alpha\, U(k) \quad (\alpha \in \mathbb{R})$
3. If $w(x) = \frac{d}{dx}u(x)$ then $W(k) = (k+1)U(k)$
4. If $w(x) = \frac{d^m}{dx^m}u(x)$ then $W(k) = \frac{(k+m)!}{k!}U(k)$
5. If $w(x) = x^m (m \in \mathbb{Z})$ then

   $W(k) = \delta(k-m) = \begin{cases} 1 & , \quad k = m \\ 0 & , \quad k \neq m \end{cases}$
6. If $w(x) = u(x)v(x)$ then $W(k) = \sum_{r=0}^{k} U(r)V(k-r)$

7. If $w(x) = u(x)v(x)s(x)$, $W(k) = \sum_{r=0}^{k} \sum_{t=0}^{k-r} U(r)V(t)S(k-r-t)$

2.2. Improved DTM by Using Least Square Correction Method

The first step of the solution technique is to find Taylor series approximation to the problem as it is defined in Eq. (2). Therefore, we start to the solution with the initial condition and generate the other coefficients in the series solution iteratively by the recurrence formula which is obtained from the transformed form of the given differential equation or a system of differential equations. Since the problem has also involved boundary conditions, in order to embodying these coefficients into the solution, we use extra unknowns (as the number of boundary conditions) in the Taylor series. Then, these unknowns are solved by substituting the given boundary conditions into the solution. However, the solution still may not converge to the exact solution due to using limited number of terms in infinite series. Generation of the other coefficients of the series by using initial condition(s) may also result in poor approximation for boundary value problems. To overcome these difficulties, in this study, we aimed adding some extra correction terms in Eq. (2) by using Least square method. Combining the Least Square Method with Differential Transform method or in other word, truncated Taylor polynomial which is first considered here. As a result, now we are able to solve more complicated problems easily without any discretization or linearization. By using these two powerful techniques together has built the new powerful method that is mentioned here.

The Least Square method is simple and straightforward and involves calculation of derivate of squares of residuals with respect to unknown coefficients that appears as extra terms in the series solution of the differential equation or a system of differential equations.

To apply the Least Square method, we first denote any ordinary differential equation in the following form:

$$Ly(x) + Ny(x) = F(x) \qquad a \le x \le b \tag{4}$$

where L, N and F denote the linear, nonlinear and nonhomogeneous parts respectively. Our solution is assumed to be in the following form:

$$y(x) = \sum_{i=0}^{N} c_i \, x^i + k_1 x^{N+1} + k_2 x^{N+2} + \cdots + k_m x^{N+m} \tag{5}$$

In this solution, we evaluate c_i's from the DTM and k_i's are obtained from the Least square method. The Least square method aims to minimize the integral of the square of residuals. To evaluate the values of k_i's we first need to know residuals, which are obtained from the following equation:

$$R(x) = Ly(x) + Ny(x) - F(x) \tag{6}$$

After evaluating the residuals for each k_i then, we can minimize the error. Therefore, we can set the following m equations as:

$$\frac{\partial}{\partial k_1} \int_a^b R^2(x) dx = 0$$

$$\frac{\partial}{\partial k_2}\int_a^b R^2(x)dx = 0 \tag{7}$$

$\vdots$

$$\frac{\partial}{\partial k_m}\int_a^b R^2(x)dx = 0$$

The above m equations altogether constitute a system of m unknowns where the unknowns are $k_1, k_2, \cdots, k_m$. Solving this system and substituting in

$$y(x) = c_0 + c_1x + c_2x^2 + \cdots + c_Nx^N + k_1x^{N+1} + k_2x^{N+2} + \cdots + k_mx^{N+m} \tag{8}$$

gives the desired polynomial solution of the problem.

3. Parallel Plates

Now, we first consider the completely passive natural convection problem between parallel plates. The governing momentum and energy equations, under the well-accepted Boussinesq approximation, are

$$\mu\frac{d^2w}{dy^2} + \rho g\beta(T - T_\alpha) = 0 \tag{9}$$

$$k\frac{d^2T}{dy^2} + \mu\left(\frac{dw}{dy}\right)^2 = 0 \tag{10}$$

where μ is the viscosity, ρ is the density of the fluid [8]. The gravitational acceleration is denoted by g and β is the coefficient of thermal expansion, k is the thermal diffusivity. T and T_α are the temperature and the ambient temperature on the walls respectively. If we normalize the lateral coordinate y which is placed at the symmetry axis by L and the axial velocity w by $k/\rho g\beta L^2$ and we have normalized temperature θ as,

$$\theta = \frac{T - T_\alpha}{\mu k/(\rho g\beta L^2)^2} \tag{11}$$

Therefore, Eqs. (9) and (10) become,

$$\theta + \frac{d^2w}{dy^2} = 0 \tag{12}$$

$$\frac{d^2\theta}{dy^2} + \left(\frac{dw}{dy}\right)^2 = 0 \tag{13}$$

Eliminating θ between Eqs. (12) and (13), we obtain the following nonlinear equation

$$\frac{d^4w}{dy^4} = \left(\frac{dw}{dy}\right)^2 \tag{14}$$

where w *and* θ are zero on the wall, hence

$$w(1) = 0 \ , \qquad \frac{d^2w}{dy^2}(1) = 0 \tag{15}$$

Also, symmetry implies

$$\frac{dw}{dy}(0) = 0 \quad , \quad \frac{d^3w}{dy^3}(0) = 0 \tag{16}$$

Of course, one solution to Eqs. (14)-(16) is the trivial solution $w = 0$ which means no motion [8,11]. However, we will also show that Eqs. (14)-(16) have nontrivial solution. We assume that an approximate solution to Eq. (14) is defined in the form

$$w(y) = \sum_{k=0}^{N} W(k)y^k \tag{17}$$

and $W(k)$'s are obtained from the following recurrence relation by applying the differential transform method to Eqs. (14)-(16) and this relation has to be satisfied for $k \geq 0$ as,

$$(k+4)(k+3)(k+2)(k+1)W(k+4) = \sum_{i=0}^{k}(i+1)(k-i+1)W(i+1)W(k-i+1) \tag{18}$$

Imposing Eq. (16) to Eq. (18) implies that $W(1) = W(3) = 0$. Eq. (18) also implies that if $k > 0$ and $W(k) \neq 0$ then $k = 4n + 2$ for some $n \geq 0$. If $W(2) = 0$ then all $W(k) = 0$ for $k > 0$ which forces $w = 0$. Hence, for nontrivial solution we assume that $W(2) = \frac{s}{2} \neq 0$ and $W(0) = w_0$ then, we evaluate other unknown coefficients as

$$\begin{aligned} W(6) &= \frac{1}{360}s^2 \quad , \quad & W(10) &= \frac{1}{151200}s^3 \\ W(14) &= \frac{31}{1816214400}s^4 \quad , \quad & W(18) &= \frac{29}{793945152000}s^5 \end{aligned} \tag{19}$$

Substituting these coefficients into Eq. (17) for sufficient N which satisfy the condition $\lim_{k\to\infty} W(k) = 0$. Hence, imposing Eq. (15) into Eq. (17) then we solve s and $W(0)$ is uniquely determined as follow,

$$s = -13.15530552\ldots \qquad and \qquad W(0) = 6.111485425\ldots$$

As a result, we have

$$\begin{aligned} w(y) = {} & 6.111485425 - 6.577652760y^2 + 0.4807279536y^6 \\ & -0.01505743596y^{10} + 0.0005112088146y^{14} + 1{,}439170306.10^{-5}y^{18} \end{aligned} \tag{20}$$

Plain DTM starts to diverge after N=14 and still it does not give desired solutios to the problem. Now, in order to obtain better approximation to the problem we apply least square technique to Eq. (20). Hence, assuming that the solution is defined as

$$\begin{aligned} w(y) = {} & 6.111485425 - 6.577652760y^2 + 0.4807279536y^6 - 0.01505743596y^{10} \\ & +0.0005112088146y^{14} + 1{,}439170306.10^{-5}y^{18} + k_1y^{22} + k_2y^{26} + k_3y^{30} \end{aligned} \tag{21}$$

where k_1, k_2 and k_3 are unknown coefficients of which values need to be evaluated from the Least square method. By using the method, as a result, we obtain the following solution which gives much better approximation to the problem (three extra terms is enough for convergency at the moment).

$$w(y) = 6.111485425 - 6.577652760y^2 + 0.4807279536y^6 - 0.01505743596y^{10} \\ +0.0005112088146y^{14} + 1{,}439170306.10^{-5}y^{18} + 4{,}031183281.10^{-7}y^{22} \\ -1{,}078395358.10^{-8}y^{26} + 2{,}577470238.10^{-10}y^{30} \quad (22)$$

Error calculations for different N and particular y, and comparisons of these errors for plain DTM and IDTM are given in Table 1.

Table 1 Maximal errors for different N and for particular $E_N(y)(0 \leq y \leq 1)$, and comparisons of these errors for plain DTM and IDTM

N	**6**	**7**	**8**	**9**	**10**
DTM	0.06725	0.003708	0.00018011	$7.85\ 10^{-6}$	$3.625\ 10^{-7}$
IDTM	0.00217	0.000347	0.00000244	$1.65\ 10^{-8}$	$1.794\ 10^{-8}$

Then the total flow rate per width, normalized by $k/(\rho g \beta L)$ is evaluated as

$$q = \int_{-1}^{1} w(y)dy = 7.972440114 \ldots \quad (23)$$

In [11], the total flow rates have been evaluated as 7.972440121 and there is no difference here for up to 7 digits.

The following figures denote where the solution starts to diverge for DTM and how IDTM behaves for the problem. In Fig.1 DTM starts to diverge after $x = 0.8$ however, IDTM gives desired values for the entire interval.

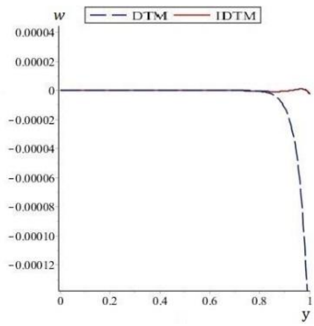


Fig. 1 Parallel plates error estimate for m=6

In Fig. 2 and Fig. 3 DTM starts to diverge after $x = 0.8$ however, IDTM gives desired values for the entire interval.

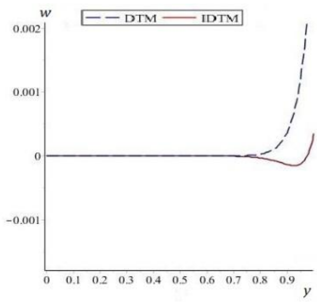


Fig. 2 Parallel plates error estimate for m=8

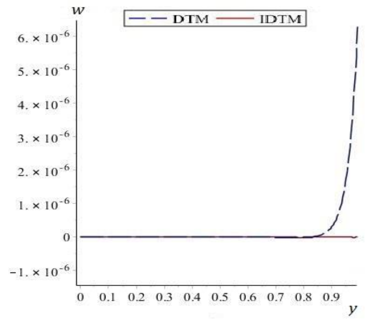


Fig. 3 Parallel plates error estimate for m=9

4. Circular Duct

Now, we consider a circular duct. The governing equation in cylindrical coordinates with similar normalization process can be written for parallel plates case such as,

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)^2 w=\left(\frac{dw}{dr}\right)^2 \tag{24}$$

The corresponding boundary conditions are

$$\frac{dw}{dr}(0)=0 \quad , \quad \frac{d}{dr}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)w(0)=0, \tag{25}$$

$$w(1)=0 \ , \quad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)w(1)=0 \tag{26}$$

Of course one solution to Eqs. (24)-(26) is again the trivial solution $w=0$, it means no motion [11]. We will now also show that, similar to the previous section, Eqs. (24)-(26) have also nontrivial solution. Approximate solution to Eq. (24) is assumed to be in the form:

$$y(r)=\sum_{k=0}^{N} Y(k)r^k \tag{27}$$

If letting $w'=y$ in Eq. (24) and taking the differential transform gives the following recurrence relation:

$$\sum_{i=0}^{k}(k+1)(k+2)(k+3)\delta(k-3)Y(k+3)$$
$$+2\sum_{i=0}^{k}(k+1)(k+2)\delta(k-2)Y(k+2)$$
$$-\sum_{i=0}^{k}(k+1)\delta(k-1)Y(k+1)+Y(k)$$
$$=\sum_{i=0}^{k}\sum_{l=0}^{k-i}\delta(k-3)Y(l)Y(k-i-l) \tag{28}$$

where $\mathrm{Y}(k)$ is the transformed form of y. Eq. (25) implies $Y(0)=Y(2)=0$ and hence, Eq. (28) implies that if $k>0$ and $Y(k)\neq 0$ then $k=4n+2$ for some $n\geq 0$. If $Y(1)=0$ then Eq. (28) implies that all $Y(k)=0$ for $k>0$ which forces $w=0$. Hence, we assume that $Y(1)=\frac{s}{2}\neq 0$. Hence, we evaluate the other unknown coefficients as

$$Y(5)=\frac{1}{384}s^2 \quad , \quad Y(9)=\frac{1}{245760}s^3$$
$$Y(13)=\frac{1}{185794560}s^4 \quad , \quad Y(17)=\frac{79}{13698261319680}s^5 \tag{29}$$

Substituting these coefficients into Eq. (27) for sufficient number of terms (when the required accuracy is satisfied since $\lim_{k\to\infty} Y(k)=0$). From here, integrating Eq. (27) and applying the conditions in Eq. (26) to the integrated form Eq. (27) then, we can solve *s* and $W(0)$ uniquely. Hence, we have

$$s=-77{,}34851423 \quad and \quad W(0) = 16.91496703 \tag{30}$$

As a result, we write

$$w(r)=16{,}91496703-19{,}33712856r^2+2{,}596698201r^6-0{,}1882975760r^{10} +0{,}01376090111r^{14} \tag{31}$$

Eq. (31) still has no quick convergency due to limited number of terms. Now, in order to obtain better approximation to the problem we apply least square technique. Assuming that the solution is defined as

$$w(r)=16{,}91496703-19{,}33712856r^2+2{,}596698201r^6 -0{,}1882975760r^{10}+0{,}01376090111r^{14}\,k_1r^{18}+k_2r^{22}+k_3r^{26} \tag{32}$$

where k_1, k_2 and k_3 are unknown coefficients of which values need to be evaluated from the Least square method. By using the method as it is stated before, as a result, we obtain the following solution which gives the desired solution to the problem.

$$w(r)=16{,}91496703-19{,}33712856r^2+2{,}596698201r^6-0{,}1882975760r^{10} +0{,}01376090111r^{14}-8{,}864471907.10^{-4}r^{18}+5{,}462628689.10^{-5}r^{22} -2{,}647332437.10^{-6}r^{26} \tag{33}$$

Error calculations for particular value of *N* and *w* and comparisons of the error between plain DTM and IDTM have been given in Table 2.

Table 2 Maximal errors for different N and for particular $E_N(r)(0 \leq r \leq 1)$, and comparisons of the error for plain DTM and IDTM

N	**6**	**7**	**8**	**9**	**10**
DTM	9.915605	1.2270191	0.0298474	0.11698318	0.10267140
IDTM	0.895310	0.0159970	0.0003045	$8.422\ 10^{-5}$	$1.307\ 10^{-6}$

Furthermore, the total flow rate per width, normalized by $k/(\rho g \beta L)$ is evaluated as

$$Q = 2\pi \int_0^1 w(r) r dr = 24.79510453 \tag{34}$$

In [11], the total flow rates have been evaluated as 24.79516051 and there is no difference between these two results up to 4 digits.

The following figures denote where the solution starts to diverge for DTM and how IDTM behaves for the problem. After $x = 0.8$ points, the results get far away from the solution when DTM is used. It may be seen in Fig. 4-6 that IDTM solution gives better results for each point on interval.

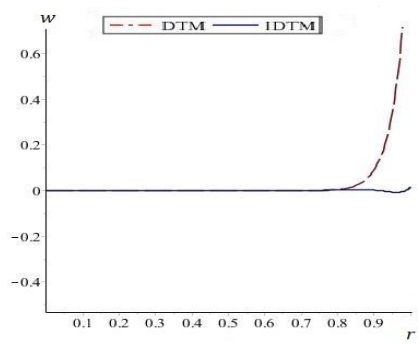


Fig. 4 Circular duct error estimate for m=7

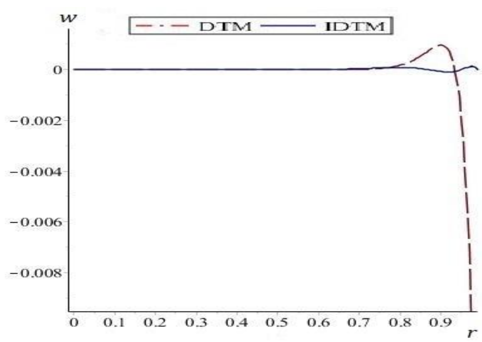


Fig. 5 Circular duct error estimate for m=8

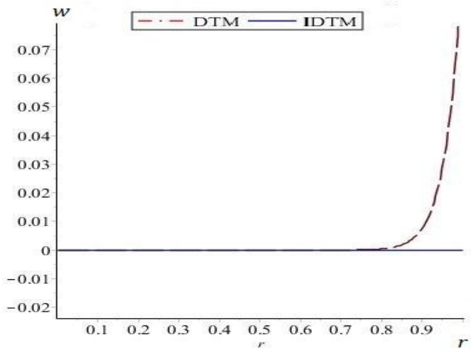


Fig. 6 Circular duct error estimate for m=10

5. Result and Discussion

The main aim of this study is to solve the model problems for completely passive natural convection by a technique which we called is IDTM. The proposed model can be the cooling of a heated body and heating and cooling of rooms and buildings and etc. The model equations of momentum and energy are solved by using improved differential transform method. For the two model problems, we applied our technique to the equations without doing any discretization and linearization and we obtained reliable results for the total flow rate by IDTM which are entirely same results with [11] indeed. Hence, this technique can easily be applied to many complicated case.

References

[1] Ostrach S. Laminar flow with body forces, in: high speed aerodynamics and jet propulsion, Vol.4,edited by F.K.Moore, Princeton University Press, Princeton, 1964.

[2] Jaluria Y. Natural Convection, Pergamon, Oxford, 1964.

[3] Gebhart B. Effects of viscous dissipation in natural convection, Journal of Fluid Mechanics. 1962; 14:225-232. http://dx.doi.org/10.1017/S0022112062001196

[4] Barletta A. Laminar mixed convection with viscous dissipation in a vertical channel, International Journal of Heat and Mass Transfer. 1998; 41(22):3501-3513. http://dx.doi.org/10.1016/S0017-9310(98)00074-X

[5] Johnson CE. Evaluation of correlations for natural convection cooling of electronic equipment, Heat Transfer Engineering, 1986; 7:36-45. http://dx.doi.org/10.1080/01457638608939643

[6] Ellison GN. Methodologies for thermal anlysis of electronic components and systems, Advances in Thermal Modeling of Electronic Components and Systems, 1993; 3:153-156.

[7] Lasance CJM, Joshi Y. Thermal Anlysis of Natural Convection Systems: Status and Challenges, Advances in Thermal Modeling of Electronic Components and Systems, 1998; 2:1-177

[8] White FM. Viscous Fluid Flow, 3rd editon, McGraw-Hill, Boston, 2006.

[9] Ayaz F. On the two-dimensional differential transform method. Applied Mathematics and Computation, 2003; 143:361-374. http://dx.doi.org/10.1016/S0096-3003(02)00368-5

[10] Ayaz F. Solutions of the system of differential equations by differential transform method, Applied Mathematics and Computation, 2004; 147(2):547-567. http://dx.doi.org/10.1016/S0096-3003(02)00794-4

[11] Miklavcic M, Wang CY. Completely passive natural convection, Journal of Applied Mathematics and Mechanics. 2011; 91(7):601-606.