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EXTENDED IDEALS OF ALMOST DISTRIBUTIVE LATTICES

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Abstract. The concept of extended ideals in an Almost distributive Lattice is introduced and studied their properties.

1. Introduction

After Booles axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [6] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set PI(L) of all principal ideals of L forms a distributive lattice. In [**3**], the notion of extended filter of a filter associated to a subset of $Rl-$monoids is defined and derived some properties. In this paper, we introduced the concept of extended ideals in Almost distributive Lattices and studied their properties.

2. Preliminaries

First, we recall certain definitions and properties of ADLs that are required in the paper.

We begin with ADL definition as follows.

Definition 2.1. ([**5**]) An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

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3. $(x \vee y) \wedge y = y$,
4. $(x \vee y) \wedge x = x$,
5. $x \vee (x \wedge y) = x$,
6. $0 \wedge x = 0$ and
7. $x \vee 0 = x$

for all $x, y, z \in L$.

EXAMPLE 2.1. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leqslant b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leqslant$ is a partial ordering on L.

THEOREM 2.1 ([**5**]). *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1). $a \vee b = a \Leftrightarrow a \wedge b = b$
(2). $a \vee b = b \Leftrightarrow a \wedge b = a$
(3). $\wedge$ *is associative in* L
(4). $a \wedge b \wedge c = b \wedge a \wedge c$
(5). $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6). $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(7). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(8). $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
(9). $a \leqslant a \vee b$ *and* $a \wedge b \leqslant b$
(10). $a \wedge a = a$ *and* $a \vee a = a$
(11). $0 \vee a = a$ *and* $a \wedge 0 = 0$
(12). *If* $a \leqslant c$, $b \leqslant c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
(13). $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

THEOREM 2.2 ([**5**]). *Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*

1). $(L, \vee, \wedge, 0)$ *is a distributive lattice*
2). $a \vee b = b \vee a$, *for all* $a, b \in L$
3). $a \wedge b = b \wedge a$, *for all* $a, b \in L$
4). $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a \Rightarrow m = a$.

THEOREM 2.3 ([**5**]). *Let L be an ADL and $m \in L$. Then the following are equivalent:*

1). *m is maximal with respect to $\leqslant$*
2). *$m \vee a = m$, for all $a \in L$*
3). *$m \wedge a = a$, for all $a \in L$*
4). *$a \vee m$ is maximal, for all $a \in L$.*

As in distributive lattices [**1, 2**], a non-empty subset I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

THEOREM 2.4 ([**5**]). *For any x, y in L the following are equivalent:*

1). *$(x] \subseteq (y]$*
2). *$y \wedge x = x$*
3). *$y \vee x = y$*
4). *$[y) \subseteq [x)$.*

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

THEOREM 2.5 ([**4**]). *Let I be an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

3. Extended ideals of ADLs

The concept of extended ideals of an ideal associated to a subset of an ADL is defined and related properties are investigated.

Now, we begin with the following.

DEFINITION 3.1. Let I be an ideal of an ADL L and $A \subseteq L$. We define the extended ideal of I associated with B as follows

$$E_I(A) = \{x \in L \mid x \wedge a \in I, \text{ for all } a \in A\}.$$

We denote $E_I(a)$ instead of $E_I(\{a\})$.

THEOREM 3.1. *Let I be an ADL L and $A \subseteq L$. Then $E_I(A)$ is an ideal of L and $I \subseteq E_I(A)$.*

PROOF. Clearly, $0 \in E_I(A)$. Then $E_I(A)$ is a non-empty set. Let $x, y \in E_I(A)$. Then $x \wedge a \in I$ and $y \wedge a \in I$, for all $a \in A$. Since I is an ideal of L, we get $(x \wedge a) \vee (y \wedge a) \in I$ and hence $(x \vee y) \wedge a \in I$. Therefore $x \vee y \in E_I(A)$. Let $x \in E_I(A)$ and $r \in L$. Then $x \wedge a \in I$, for all $a \in A$. Since I is an ideal of L, we have $x \wedge a \wedge r \in I$ and hence $x \wedge r \wedge a \in I$. Hence $x \wedge r \in E_I(A)$. Therefore $E_I(A)$ is an ideal of L. Let $x \in I$. Since I is an ideal of L, we have $x \wedge x \in I$, for all $a \in A$. Therefore $x \in E_I(A)$. Hence $I \subseteq E_I(A)$. □

DEFINITION 3.2. An ideal I of an ADL L is said to be a stable relative to a subset B of L if $E_I(B) = I$.

THEOREM 3.2. *Let I, J be two ideals of L, and B, $C \subseteq L$. Then*

1. *If $B \subseteq C$ then $E_I(C) \subseteq E_I(B)$*
2. *If $I \subseteq J$ then $E_I(B) \subseteq E_J(B)$*
3. *$E_I(B) = L$ if and only if $B \subseteq I$*
4. *$B \subseteq E_I(E_I(B))$*
5. *If $I \subseteq J$ then $E_I(J) \cap J = I$*
6. *$E_I(E_I(B)) \cap E_I(B) = I$*
7. *If m is any maximal element of L with $m \in B$ then I is stable relative to B*
8. *$E_I(B) = E_I((B])$, where $(B]$ is the ideal generated by B.*

PROOF. 1. Assume that $B \subseteq C$. Let $x \in E_I(C)$. Then $x \wedge c \in I$, for all $c \in C$. That implies $x \wedge b \in I$, since $B \subseteq C$. Therefore $E_I(C) \subseteq E_I(B)$.

2. Assume that $I \subseteq J$. Let $x \in E_I(B)$. Then $x \wedge b \in I$, for all $b \in B$. That implies $x \wedge b \in J$ and hence $x \in E_J(B)$. Therefore $E_I(B) \subseteq E_J(B)$.

3. Assume that $E_I(B) = L$. Let $b \in B$. Then $b \in E_I(B)$. That implies $b \wedge b \in I$ and hence $b \in I$. Therefore $B \subseteq I$. Assume that $B \subseteq I$. Let $x \in L$ and $b \in B$. Then $x \wedge b \in I$. That implies $x \in E_I(B)$ and hence $L \subseteq E_I(B)$. Therefore $E_I(B) = L$.

4.Let $x \in B$ and $y \in E_I(B)$. Then $y \wedge x \in I$. That implies $x \wedge y \in I$. Since $y \in E_I(B)$, we have $x \in E_I(E_I(B))$. Therefore $B \subseteq E_I(E_I(B))$.

5. Assume that $I \subseteq J$. Let $x \in E_I(J) \cap J$. Then $x \in E_I(J)$ and $x \in J$. That implies $x \in I$. Therefore $E_I(J) \cap J = I$.

6. Let $x \in E_I(E_I(B)) \cap E_I(B)$. Then $x \in E_I(E_I(B))$ and $x \in E_I(B)$. that implies $x \in I$. conversely, assume that $x \in I$. Then $x \wedge y \in I$, for all $y \in E_I(B)$. Since $I \subseteq E_I(B)$, we get that $x \in E_I(E_I(B)) \cap E_I(B)$. Therefore $I = E_I(E_I(B)) \cap E_I(B)$.

7. Let m be any maximal element of an ADL L with $m \in B$. we prove that $I = E_I(B)$. Clearly, we have that $I \subseteq E_I(B)$. Let $x \in E_I(B)$ then $x \wedge b \in I$, for all $b \in I$. That implies $x \wedge m \in I$ and hence $m \wedge x \in I$. Therefore $x \in I$. Thus $I = E_I(B)$.

8. Since $B \subseteq (B]$, we have that $E_I((B]) \subseteq E_I(B)$. conversely, let $x \in E_I(B)$. Then $x \wedge b \in I$, for all $b \in B$. Let $z \in (B]$. Then we can write $(\bigvee_{i=1}^{n} c_i) \wedge z = z$, for some $c_i \in B$, and $n \in \mathbb{N}$. Now

$$x \wedge z = x \wedge (c_1 \vee c_2 \vee \cdots \vee c_n) \wedge z$$
$$= (x \wedge z \wedge c_1) \vee (x \wedge z \wedge c_2) \vee (x \wedge z \wedge c_3) \vee \cdots \vee (x \wedge z \wedge c_n) \in I,$$

since $x \wedge z \wedge c_i \in I$, for $1 \leqslant i \leqslant n$. Therefore $x \in E_I((B])$ and hence $E_I(B) \subseteq E_I((B])$. Thus $E_I(B) = E_I((B])$. □

Corollary 3.1. *Let $I,\ J$ be ideals of an ADL L and $B, C \subseteq L$. Then*

1. $E_I(I) = E_I(0) = E_I(E_I(B)) \cap B = L$*;*
2. $E_I(B) = E_I(E_I(E_I(B)))$*;*
3. *I is stable relative to the $\emptyset$;*
4. $E_{E_I(B)}(C) = E_{E_I(C)}(B)$*;*
5. *If $I \subseteq B$ then $(B] \cap E_I((B]) = (B]$.*

Corollary 3.2. *Let I be an ideal of an ADL L and $B \subseteq L$. Then the following conditions are equivalent:*

1. *I is stable relative to B;*
2. *I is stable relative to $(B]$;*
3. $E_I(E_I(B)) = L$.

Theorem 3.3. *Let P be a prime ideal and $E_P(B)$, a proper ideal of an ADL L. Then P is stable relative to B.*

Proof. Let $x \in E_P(B)$. Then $x \wedge b \in P$ for all $b \in B$. Since P is prime, we get $x \in P$ or $b \in P$. If $b \in P$, Then $B \subseteq P$ and hence $E_P(B) = L$, which is a contradiction. Therefore $b \notin P$. So that $x \in P$. Thus $E_P(B) = P$. □

Theorem 3.4. *Let M be a maximal ideal and $E_M(B)$ be a proper ideal of an ADL L. Then M is a stable relative to B*

Theorem 3.5. *Let $B \subseteq C \subseteq L$ and I be an ideal of an ADL L. If I is stable relative to B, then I is a stable relative to C.*

Proof. Assume that I is a stable relative to B. That is $I = E_I(B)$. By theorem 3.2, we get that $I \subseteq E_I(C) \subseteq E_I(B) = I$. Therefore $E_I(C) = I$. □

Proposition 3.1. 1. *If $\{I_i\}_{i\in\Delta}$ is a family of ideals of an ADL L and $B \subseteq L$ then*

$$\bigcap_{i\in\Delta} E_{I_i}(B) = E_{\bigcap\limits_{i\in\Delta} I_i}(B).$$

2. *If $\{I_i\}_{i\in\Delta}$ is a chain of ideals of an ADL L and $B \subseteq L$ then*

$$\bigcup_{i\in\Delta} E_{I_i}(B) = E_{\bigcup\limits_{i\in\Delta} I_i}(B).$$

PROOF. Let $\{I_i\}_{i\in\Delta}$ be a family of ideals of an ADL L and $B \subseteq L$. Now, $x \in \bigcap_{i\in\Delta} E_{I_i}(B) \Leftrightarrow x \in E_{I_i}(B)$, for all $i \in \Delta \Leftrightarrow x \wedge b \in I_i$, for all $b \in B$ and $i \in \Delta \Leftrightarrow x \wedge b \in \bigcap_{i\in\Delta} I_i$, for all $b \in B \Leftrightarrow x \in E_{\bigcap_{i\in\Delta}(I_i)}(B)$. Therefore $\bigcap_{i\in\Delta} E_{I_i}(B) = E_{\bigcap_{i\in\Delta} I_i}(B)$.

Similarly, we get that $\bigcup_{i\in\Delta} E_{I_i}(B) = E_{\bigcup_{i\in\Delta} I_i}(B)$, when $\{I_i\}_{i\in\Delta}$ is a chain of ideals of an ADL L. $\square$

THEOREM 3.6. *Let I, J be ideals of L and $B \subseteq L$. Then*

$$L/\theta_{E_{I\cap J}(B)} = L/\theta_{E_I(B)} \cap L/\theta_{E_J(B)},$$

where

$$L/\theta_{E_I(B)} \cap L/\theta_{E_J(B)} = \{x/\theta_{E_I(B)} \cap y/\theta_{E_J(B)} \mid x/\theta_{E_I(B)} \cap y/\theta_{E_J(B)} \neq \emptyset\}.$$

PROOF. Let $x/\theta_{E_{I\cap J}(B)} \in L/\theta_{E_{I\cap J}(B)}$. Then

$x/\theta_{E_{I\cap J}(B)} = \{y \in L \mid (x, y) \in \theta_{E_{I\cap J}(B)}\}$
$= \{y \in L \mid a \vee x = a \vee y, \text{ for some } a \in E_{I\cap J}(B)\}$
$= \{y \in L \mid a \vee x = a \vee y, \text{ for some } a \in E_I(B) \cap E_J(B)\}$
$= x/\theta_{E_I(B)} \cap x/\theta_{E_J(B)}$ $\square$

THEOREM 3.7. *Let I be ideal of ADL L_1 and J be an ideal of ADL L_2 and $B_1 \subseteq L_1, B_2 \subseteq L_2$. Then $E_{I\times J}(B_1 \times B_2) = E_I(B_1) \times E_J(B_2)$.*

PROOF. Now,

$$E_{I\times J}(B_1 \times B_2) = \{(x,\ y) \in L_1 \times L_2 \mid (x, y) \wedge (b, c) \in I \times J,$$

for all

$$(b, c) \in B_1 \times B_2\} = \{(x, y) \in L_1 \times L_2 \mid (x \wedge b, y \wedge c) \in I \times J$$

for all

$$b \in B_1 \text{ and } c \in B_2\} = \{(x, y) \in L_1 \times L_2 \mid x \wedge b \in I \text{ and } y \wedge c \in J$$

for all

$$b \in B_1,\ c \in B_2\} = \{(x, y) \in L_1 \times L_2 \mid x \in E_I(B_1)$$

and

$$y \in E_J(B_2)\} = E_I(B_1) \times E_J(B_2).$$

$\square$

DEFINITION 3.3. A non-empty subset A of an ADL L, I, an ideal of L and f an endomorphism. We define the $f-$extended ideal of I with respect to A as follows $E_I^f(A) = \{x \in L \mid f(x) \wedge a \in I, \text{ for all } a \in A\}$.

LEMMA 3.1. *$E_I^f(A)$ is an ideal of an ADL L.*

Proof. Since $f(0) = 0$, we have $f(0) \wedge a = 0 \wedge a = 0 \in I$, for all $a \in A$. Therefore $E_I^f(A)$ is a non-empty set. Let $x, y \in E_I^f(A)$. Then $f(x) \wedge a \in I$ and $f(y) \wedge a \in I$, for all $a \in A$. Now $f(x \vee y) \wedge = (f(x) \vee f(y)) \wedge a = (f(x) \wedge a) \vee (f(y) \wedge a) \in I$, since I is an ideal of L. Therefore $x \vee y \in E_I^f(A)$. Let $x \in E_I^f(A)$ and $r \in L$. Then $f(x) \wedge a \in I$, for all $a \in A$. Now, $f(x \wedge x) \wedge a = f(x) \wedge f(r) \wedge a \in I$. Therefore $x \wedge r \in E_I^f(A)$ and hence $E_I^f(A)$ is an ideal of an ADL L. □

Theorem 3.8. *Let I, J be ideals of an ADL L, f be an endomorphism and non-empty subsets A, A' of L. Then we have the following:*

1. *if $I \subseteq J$ then $E_I^f(A) \subseteq E_J^f(A)$;*
2. *if $A \subseteq A'$ then $E_I^f(A') \subseteq E_I^f(A)$;*
3. $E_I^f(\bigcup\limits_{i \in \Delta} A_i) = \bigcap\limits_{i \in \Delta} E_I^f(A_i)$;
4. $E_I^f(A) = \bigcap\limits_{a \in A} E_I^f(a)$;
5. $E^f_{\bigcap\limits_{i \in \Delta} I_i}(A) = \bigcap\limits_{i \in \Delta} E_{I_i}^f(A)$;
6. $E_I^f((A]) = E_I^f(A)$;
7. *$Ker\ f \subseteq E^f_{\{0\}}(A)$ and $E^f_{\{0\}}(L) = Ker\ f$.*

Proof. 1. Assume that $I \subseteq J$. Let $x \in E_I^f(A)$. Then $f(x) \wedge a \in I$, for all $a \in A$. That implies $f(x) \wedge a \in J$, for all $a \in A$. Therefore $x \in E_J^f(A)$ and hence $E_I^f(A) \subseteq E_J^f(A)$. 2. Assume that $A \subseteq A'$. let $x \in E_I^f(A')$. Then $f(x) \wedge a \in I$ for all $a \in A'$. That implies $f(x) \wedge a \in I$, for all $a \in A$ and hence $x \in E_I^f(A)$. Therefore $E_I^f(A') \subseteq E_I^f(A)$.

3. Clearly, we have that $A_i \subseteq \bigcup\limits_{i \in \Delta} A_i$, for all $i \in \Delta$. By (2), we have $E_I^f(\bigcup\limits_{i \in \Delta} A_i) \subseteq \bigcap\limits_{i \in \Delta} E_I^f(A_i)$. Let $x \in \bigcap\limits_{i \in \Delta} E_I^f(A_i)$. Then $x \in E_I^f(A_i)$, for all $i \in \Delta$. That implies $f(x) \wedge a \in I$, for all $a \in A_i$. That implies $f(x) \wedge a \in I$, for all $a \in \bigcup\limits_{i \in \Delta} A_i$ and hence $x \in E_I^f(\bigcup\limits_{i \in \Delta} A_i)$. Therefore $\bigcap\limits_{i \in \Delta} E_I^f(A_i) \subseteq E_I^f(\bigcup\limits_{i \in \Delta} A_i)$. Thus $E_I^f(\bigcup\limits_{i \in \Delta} A_i) = \bigcap\limits_{i \in \Delta} E_I^f(A_i)$.

4. Let $x \in E_I^f(A)$. Then $f(x) \wedge a \in I$, for all $a \in A$. That implies $x \in E_I^f(\{a\})$, since $\{a\} \subseteq A$. Therefore $x \in \bigcap\limits_{a \in A} E_I^f(\{a\})$. Hence $E_I^f(A) \subseteq \bigcap\limits_{a \in A} E_I^f(\{a\})$. Conversely, let $x \in \bigcap\limits_{a \in A} E_I^f(\{a\})$. Then $x \in E_I^f(\{a\})$, for all $a \in A$. That implies $f(x) \wedge a \in I$, for all $a \in A$. Therefore $x \in E_I^f(A)$ and hence $\bigcap\limits_{\in A} E_I^f(\{a\}) \subseteq E_I^f(A)$. Thus $E_I^f(A) = \bigcap\limits_{a \in A} E_I^f(\{a\})$.

5. Now $x \in E^f_{\bigcap\limits_{i\in\Delta} I_i}(A) \Leftrightarrow f(x) \wedge a \in \bigcap\limits_{i\in\Delta} I_i$, for all $a \in A \Leftrightarrow f(x) \wedge a \in I_i$, for all $i \in \Delta \Leftrightarrow x \in E^f_{I_i}(A)$, for all $i \in \Delta \Leftrightarrow x \in \bigcap\limits_{i\in\Delta} E^f_{I_i}(A)$. Therefore $E^f_{\bigcap\limits_{i\in\Delta} I_i}(A) = \bigcap\limits_{i\in\Delta} E^f_{I_i}(A)$.

6. Clearly, we have $A \subseteq (A]$. Then $E^f_I((A]) \subseteq E^f_I(A)$. let $x \in E^f_I(A)$. Then $f(x) \wedge a \in I$, for all $a \in A$. Since $A \subseteq (A]$, we get that $a \in E^f_I((A])$. Therefore $E^f_I(A) \subseteq E^f_I((A])$.

7. Let $x \in Ker\ f$. Then $f(x) = 0$. That implies $f(x) \wedge a = 0 \in \{0\}$, for all $a \in A$. Therefore $x \in E^f_{\{0\}}(A)$. Hence $Ker\ f \subseteq E^f_{\{0\}}(A)$. Let $x \in E^f_{\{0\}}(L)$. Then $f(x) \wedge a \in \{0\}$, for all $a \in L$. That implies $f(x) \wedge a = 0$. In particular, taking $a = f(x)$, we get that $f(x) = 0$ and hence $x \in Ker\ f$. therefore $E^f_{\{0\}}(L) \subseteq Ker\ f$. Conversely, let $x \in Ker\ f$. Then $f(x) = 0$. That implies $f(x) \wedge a = 0 \in \{0\}$, for all $a \in A$. Therefore $x \in E^f_{\{0\}}(L)$ and hence $Ker\ f \subseteq E^f_{\{0\}}(L)$. Thus $Ker\ f = E^f_{\{0\}}(L)$. □

Theorem 3.9. *Let I be an ideal of an ADL L, f be an endomorphism of L. Then for any $a, b \in L$, we have the following:*

1. If $a \leqslant b$ then $E^f_I(b) \subseteq E^f_I(a)$

2. $E^f_I(a \vee b) = E^f_I(a) \cap E^f_I(b)$.

Proof. 1. Assume that $a \leqslant b$. Let $x \in E^f_I(b)$. Then $f(x) \wedge b \in I$. That implies$f(x) \wedge a = f(x) \wedge a \wedge b \in I$ and hence $x \in E^f_I(a)$. Therefore $E^f_I(b) \subseteq E^f_I(a)$.

2. We prove that $E^f_I(a \vee b) \subseteq E^f_I(a) \cap E^f_I(b)$. Conversely, let $x \in E^f_I(a) \cap E^f_I(b)$. Then $f(x) \wedge a \in I$ and $f(x) \wedge b \in I$. That implies $(f(x) \wedge a) \vee (f(x) \wedge b) \in I$ and hence $f(x) \wedge (a \vee b) \in I$. Therefore $x \in E^f_I(a \vee b)$. Thus $E^f_I(a \vee b) = E^f_I(a) \cap E^f_I(b)$. □

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