THE k-DISTANCE NEIGHBORHOOD POLYNOMIAL OF SPLITTING GRAPH OF A GRAPH

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ABSTRACT. The k-distance neighborhood polynomial (N_k-polynomial) of a graph G is the ordinary generating function for the number of k-distance neighborhood of the vertices of G and defined as

$$N_k(G,x) = \sum_{k=0}^{diam(G)} \left(\sum_{i=1}^{n} d_k(v_i) \right) x^k,$$

where $diam(G)$ is the diameter of G and $d_k(v) = |\{u \in V(G) : d(v,u) = k\}|$. In this paper, we establish the exact formulas of the N_k-polynomial for a splitting graph $S(G)$ of a graph G. The N_k-polynomials of some well-known graphs as complete K_n, path P_n, cycle C_n, complete bipartite $K_{r,s}$ and star $K_{1,n}$ are presented.

1. Introduction

All the graphs $G = (V,E)$ considered here are finite undirected with no loops and multiple edges. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively, unless we refer otherwise. The distance $d(u,v)$ between any two vertices u and v of G is the length of a minimum path connecting them. For a vertex $v \in V(G)$ and a positive integer k, the open k-distance neighborhood of v in a graph G, denote $N_k(v/G)$, ($N_k(v)$, if no confuse), is $N_k(v/G) = \{u \in V(G) : d(u,v) = k\}$ and the k-distance degree of a vertex v, denote $d_k(v/G)$, ($d_k(v)$, if no confuse), is $d_k(v) = |N_k(v)|$. It is clearly that $d_1(v) = d(v)$. A graph G is called non-trivial if it has at least one edge. The complement of a graph G, denoted $\overline{G}$, is a graph with vertex set V and edge set

$\overline{E}$, such that $e \in \overline{E}$, if and only if $e \notin E$. A graph G is self-complementary if G is isomorphic to its complement. $\overline{K_n}$ is the empty or total disconnected graph with n vertices, i.e., the graph with n vertices no two of which are adjacent. A graph G is called d-regular graph if the degree $d_1(v)$ of each vertex v in G is equal to d. For a vertex v of G, the eccentricity $e(v) = \max\{d(v,u) : u \in V(G)\}$. The radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$ and the diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$. For any terminology or notation not mention here, we refer the reader to the books [**3, 4**].

A graph can be characterized by a number (index), a matrix or a polynomial. The characterization of graphs by a single topological index is usually impossible. For example, it is possible to find infinite pairs of graphs with the same Wiener index [**13**]. On the other hand, it is possible to characterize graphs by matrices. A well-known example of such matrices is an adjacency matrix [**4**]. But the characterization of graphs by polynomials is a new branch of research in modern graph theory. For more details in the topological indices and polynomials of a graph, we refer the reader to [**1, 2, 5, 6, 9, 11, 13, 14, 15, 16, 17**], and the references therein.

1.1. Splitting Graph of a graph. A Splitting graph of a graph G, denoted by $S(G)$, were first introduced by Sampathkumar and Walikar [**10**], and were further developed by Patil and Thangamani [**8**].

DEFINITION 1.1. For a graph G of order n with vertex set $v_1, v_2, ..., v_n$ the splitting graph of G, denoted by $S(G)$, is obtained by adding a new vertex u_i corresponding to each vertex v_i of G, $i = 1, 2, ..., n$ and then join u_i to all vertex of G adjacent to v_i.

The following results are some properties of splitting graph $S(G)$ of a graph G find in [**10**], which require to prove our main results.

PROPOSITION 1.1. *Let G be a graph of order n with vertex set $v_1, v_2, ..., v_n$, size m and diameter $diam(G) \geqslant 1$. Then*

(1) $|V(S(G))| = 2n$ and $|E(S(G))| = 3m$.
(2) $d_0(v_i/S(G)) = d_0(u_i/S(G)) = d_0(v_i/G) = 1$, $d_1(v_i/S(G)) = 2d_1(v_i/G)$ and $d_1(u_i/S(G)) = d_1(v_i/G)$, where u_i, for every $i = 1, 2, ..., n$ is a new vertex added a corresponding to each vertex v_i of G to instruction $S(G)$.
(3) $diam(S(G)) = \begin{cases} diam(G) + 1, & \text{if } diam(G) \in \{1, 2\} \text{ and } G \neq K_2; \\ diam(G) + 2, & \text{if } G = K_2; \\ diam(G), & \text{if } diam(G) \geqslant 3. \end{cases}$

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1.2. The N_k-polynomial of a graph. Soner and Naji [**7, 12**], in (2016), have been introduced a new type of graph topological polynomial, based on distance and degree, called k-distance neighborhood polynomial of a graph. Which, for simplicity of notion, referred as N_k-polynomial and defined by

$$N_k(G, x) = \sum_{k=0}^{e(v_i)} \left(\sum_{i=1}^{n} |N_k(v_i)| \right) x^k$$

where $N_k(v) = \{u \in V(G) : d(v,u) = k\}$ and $e(v) = \max\{d(v,u) : u \in V(G)\}$. They have been obtained some basic properties of N_k-polynomial of graphs and they presented the exact formulas for the N_k-polynomial of some well-known graphs (namely, a path P_n, a cycle C_n, a complete graph K_n, a star graph K_n, a wheel W_n, a complete bipartite $K_{r,s}$ and a complete multipartite $K_{n_1,...,n_r}$). They also established the N_k-polynomial for some graph operations namely cartesian product, join, union, corona product of graphs and complement graph of some graph.

The following are some fundamental results which will be required for many of our arguments in this paper and which are finding in [**7, 12**].

PROPOSITION 1.2. *Let G be a graph of order n, size m and diameter $diam(G) \geqslant 1$ and let $N_k(G,x) = a_0 + a_1 x + a_2 x^2 + ... + a_d x^d$ be an N_k-polynomial of G. then*

(1) $a_0 = n$.
(2) $a_1 = 2m$.
(3) a_i, *for every $i = 1, 2, ..., d$, is an even integer number.*
(4) *The degree d of $N_k(G,x)$ is equal to $diam(G)$.*

PROPOSITION 1.3. *For $n \geqslant 1$*

(1) $N_k(K_n, x) = n + n(n-1)x$.

(2) $N_k(P_n, x) = n + 2 \sum_{i=1}^{n} (n-i)x^i$.

(3) $N_k(C_n, x) = \begin{cases} n + 2n \sum_{i=1}^{\frac{n-1}{2}} x^i, & \text{if } n \text{ is odd;} \\ n + (2n \sum_{i=1}^{\frac{n}{2}} x^i) - nx^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$

(4) $N_k(K_{r,s}, x) = (r+s) + 2rsx + [r(r-1) + s(s-1)]x^2$.

PROPOSITION 1.4. *For any graph G with n vertices, m edges and $diam(G) = 2$, the N_k-polynomial of G is of the form*

$$N_k(G,x) = n + 2m + (n^2 - n - 2m)x^2.$$

In this paper, We established the exact formulas of the N_k-polynomial for a splitting graph $S(G)$ of a graph G. The N_k-polynomials of splitting graph of some well-known graphs as complete K_n, path P_n, cycle C_n, complete bipartite $K_{r,s}$ and star $K_{1,n}$ are presented.

2. Main Results

THEOREM 2.1. *For the complete graph K_n with $n \geqslant 2$, the N_k-polynomial of the splitting graph $S(K_n)$ of K_n is of the form*

$$N_k(S(K_n), x) = \begin{cases} 4 + 6x + 4x^2 + 2x^3, & \text{if } n = 2; \\ 2n + 3n(n-1)x + (n^2 + n)x^2, & \text{if } n \geqslant 3. \end{cases}$$

PROOF. Let K_n be a complete graph with $n \geqslant 2$ vertices. Then by Proposition 1.1,
we have
$$|V(S(K_n))| = 2n, |E(S(K_n))| = 3\frac{n(n-1)}{2} \text{ and } diam(S(K_n)) = \begin{cases} 3, & \text{if } n = 2; \\ 2, & \text{if } n \geqslant 3. \end{cases}$$
Thus, we consider the following cases:

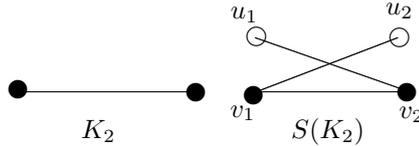**Case:1:** If $n = 2$, then $S(K_2) = P_4$

Figure 1: A graph K_2 and its splitting graph $S(K_2)$.

Hence by Proposition 1.3, we obtain
$$N_k(S(K_2), x) = 4 + 6x + 4x^2 + 2x^3.$$

Case:2: If $n \geqslant 3$, then by applying Proposition 1.2 and Proposition 1.4, we get

$$N_k(S(K_n), x) = |V(S(K_n))| + 2|E(S(K_n))|x + [|V(S(K_n))|^2 - |V(S(K_n))|$$
$$- 2|E(S(K_n))|]x^2$$
$$= 2n + 2[3\frac{n(n-1)}{2}]x + [(2n)^2 - 2n - 2(3\frac{n(n-1)}{2})]x^2$$
$$= 2n + 3n(n-1)x + (n^2 + n)x^2.$$

$\square$

The following Corollary is showing the relationship between the N_k-polynomial of the complete graph and its splitting graph.

COROLLARY 2.1. *For $n \geqslant 3$*
$$N_k(S(K_n), x) = 4N_k(K_n, x) + (n^2 + n)x^2 - n(n-1)x - 2n.$$

THEOREM 2.2. *For the path graph P_n with $n \geqslant 3$, the N_k-polynomial of the splitting graph $S(P_n)$ of P_n is of the form*
$$N_k(S(P_n), x) = 4N_k(P_n, x) + 2\left(x^2 - 1\right)\left((n-1)x + n\right).$$

PROOF. Let P_n be a path graph with $n \geqslant 3$ vertices and let $v_1, v_2, ..., v_n$ be the set vertex of P_n and $u_1, u_2, ..., u_n$ be the set of new vertices introduced in construction of $S(P_n)$. Figure 2, showing the splitting graph of P_n.
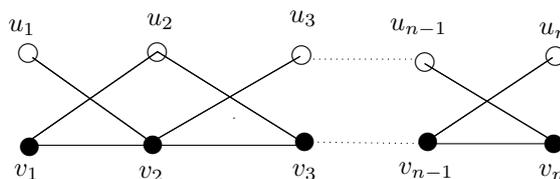
Figure 2: Splitting graph of the path P_n.

For $n \geqslant 3$, we have by Proposition 1.1,

$$diam(S(P_3)) = \begin{cases} 3, & \text{if } n = 3; \\ diam(P_n), & \text{otherwise.} \end{cases}$$

Hence we consider the following cases:

Case:1: If $n = 3$, then by Propositions 1.1 and 1.2, we obtain

$$N_k(S(P_3), x) = \sum_{k=0}^{3} \Big(\sum_{i=1}^{6} d_k(w_i/S(P_3)) \Big) x^k, where \ w_i = v_i \ or \ u_i, \ for \ i = 1, 2, ..., n$$

$$= \Big(\sum_{i=1}^{6} d_0(w_i/S(P_3)) \Big) x^0 + \Big(\sum_{i=1}^{6} d_1(w_i/S(P_3)) \Big) x$$

$$+ \Big(\sum_{i=1}^{6} d_2(w_i/S(P_3)) \Big) x^2 + \Big(\sum_{i=1}^{6} d_3(w_i/S(P_3)) \Big) x^3$$

$$= |V(S(P_3))| + 2|E(S(P_3))|x$$

$$+ \Big(\sum_{i=1}^{3} d_2(v_i/S(P_3)) + \sum_{i=1}^{3} d_2(u_i/S(P_3)) \Big) x^2$$

$$+ \Big(\sum_{i=1}^{3} d_3(v_i/S(P_3)) + \sum_{i=1}^{3} d_3(u_i/S(P_3)) \Big) x^3$$

$$= 6 + 12x + (3 + 1 + 3 + 3 + 1 + 3)x^2 + (0 + 0 + 0 + 1 + 2 + 1)x^3$$

$$= 6 + 12x + 14x^2 + 4x^3.$$

Since, $N_k(P_3, x) = 3 + 4x + 2x^2$, it follows that

$$N_k(S(P_3), x) = 4N_k(P_3, x) + 2(x^2 - 1)(2x + 3).$$

Case:2: If $n \geqslant 4$, we obtain for every $i \in \{1, 2, ..., n\}$ that

$d_0(v_i/S(P_n)) = d_0(v_i/P_n)$ and $d_0(u_i/S(P_n)) = d_0(v_i/P_n)$

$d_1(v_i/S(P_n)) = 2d_1(v_i/P_n)$ and $d_1(u_i/S(P_n)) = d_1(v_i/P_n)$

$d_2(v_i/S(P_n)) = 2d_2(v_i/P_n) + 1$ and $d_2(u_i/S(P_n)) = 2d_2(v_i/P_n) + 1$

$d_3(v_i/S(P_n)) = 2d_3(v_i/P_n)$ and $d_3(u_i/S(P_n)) = 2d_3(v_i/P_n) + d_1(v_i/P_n)$

$d_4(v_i/S(P_n)) = 2d_4(v_i/P_n)$ and $d_4(u_i/S(P_n)) = 2d_4(v_i/P_n)$

....

$d_k(v_i/S(P_n)) = 2d_k(v_i/P_n)$ and $d_k(u_i/S(P_n)) = 2d_k(v_i/P_n)$,

for every $5 \leqslant k \leqslant n-1$. Hence,

$$
\begin{aligned}
N_k(S(P_n), x) &= \sum_{k=0}^{n-1} \Big(\sum_{w \in V(S(P_n))} d_k(w/S(P_n)) \Big) x^k \\
&= \sum_{k=0}^{n-1} \Big(\sum_{i=1}^{n} d_k(v_i/S(P_n)) \Big) x^k + \sum_{k=0}^{n-1} \Big(\sum_{i=1}^{n} d_k(u_i/S(P_n)) \Big) x^k \\
&= \Big(\sum_{i=1}^{n} d_0(v_i/S(P_n)) \Big) x^0 + \Big(\sum_{i=1}^{n} d_1(v_i/S(P_n)) \Big) x + ... \\
&+ \Big(\sum_{i=1}^{n} d_{n-1}(v_i/S(P_n)) \Big) x^{n-1} \\
&+ \Big(\sum_{i=1}^{n} d_0(u_i/S(P_n)) \Big) x^0 + \Big(\sum_{i=1}^{n} d_1(u_i/S(P_n)) \Big) x + ... \\
&+ \Big(\sum_{i=1}^{n} d_{n-1}(u_i/S(P_n)) \Big) x^{n-1} \\
&= \sum_{i=1}^{n} d_0(v_i/P_n) + \Big(\sum_{i=1}^{n} 2d_1(v_i/P_n) \Big) x + \Big(\sum_{i=1}^{n} (2d_2(v_i/P_n) + 1) \Big) x^2 \\
&+ \Big(\sum_{i=1}^{n} 2d_3(v_i/P_n) \Big) x^3 + \Big(\sum_{i=1}^{n} 2d_4(v_i/P_n) \Big) x^4 + ... \\
&+ \Big(\sum_{i=1}^{n} 2d_{n-1}(v_i/P_n) \Big) x^{n-1} \\
&+ \sum_{i=1}^{n} d_0(v_i/P_n) + \Big(\sum_{i=1}^{n} d_1(v_i/P_n) \Big) x + \Big(\sum_{i=1}^{n} (2d_2(v_i/P_n) + 1) \Big) x^2 \\
&+ \Big(\sum_{i=1}^{n} (2d_3(v_i/P_n) + d_1(v_i/P_n)) \Big) x^3 + \Big(\sum_{i=1}^{n} 2d_4(v_i/P_n) \Big) x^4 + ... \\
&+ \Big(\sum_{i=1}^{n} 2d_{n-1}(v_i/P_n) \Big) x^{n-1} \\
&= 4 \bigg[\sum_{i=1}^{n} d_0(v_i/P_n) + \Big(\sum_{i=1}^{n} d_1(v_i/P_n) \Big) x + \Big(\sum_{i=1}^{n} d_2(v_i/P_n) \Big) x^2 + ... \\
&+ \Big(\sum_{i=1}^{n} d_{n-1}(v_i/P_n) \Big) x^{n-1} \bigg] - 2 \sum_{i=1}^{n} d_0(v_i/P_n) - \Big(\sum_{i=1}^{n} d_1(v_i/P_n) \Big) x \\
&+ \Big(2 \sum_{i=1}^{n} 1 \Big) x^2 + \Big(\sum_{i=1}^{n} d_1(v_i/P_n) \Big) x^3 \\
&= 4N_k(P_n, x) - 2n - 2mx + 2nx^2 + 2mx^3 \\
&= 4N_k(P_n, x) - 2n - 2(n-1)x + 2nx^2 + 2(n-1)x^3 \\
&= 4N_k(P_n.x) + 2 \Big(x^2 - 1 \Big) \Big((n-1)x + n \Big).
\end{aligned}
$$

$\square$

THEOREM 2.3. *For a cycle graph C_n with $n \geqslant 4$ vertices, the N_k-polynomial of $S(C_n)$ is of the form*

$$N_k(S(C_n), x) = 4N_k(C_n, x) + 2n(x^3 + x^2 - x - 1).$$

PROOF. Let C_n be a cycle with $n \geqslant 4$ vertices and let $v_1, v_2, ..., v_n$ be the set vertex of C_n and $u_1, u_2, ..., u_n$ be the set of new vertices introduced in construction of $S(C_n)$. For $n \geqslant 3$, we have by Proposition 1.1,

$$diam(S(C_3)) = \begin{cases} 3, & \text{if } n \in \{4,5\}; \\ diam(C_n), & \text{otherwise.} \end{cases}$$

Hence we consider the following cases:

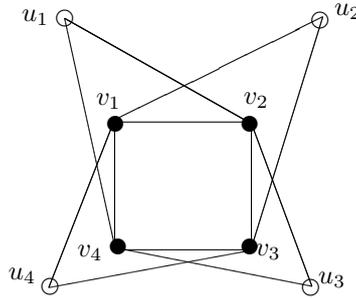**Case1::** If $n = 4$, then $S(C_4))$ is shown in figure 3,

Figure 3: Splitting graph of C_4.

$$N_k(S(C_4), x) = \sum_{k=0}^{3} \Big(\sum_{i=1}^{8} d_k(w_i/S(C_4)) \Big) x^k, where\ w_i = v_i\ or\ u_i,\ for\ i = 1, 2, ..., n$$

$$= \Big(\sum_{i=1}^{8} d_0(w_i/S(C_4)) \Big) x^0 + \Big(\sum_{i=1}^{8} d_1(w_i/S(C_4)) \Big) x$$

$$+ \Big(\sum_{i=1}^{8} d_2(w_i/S(C_4)) \Big) x^2 + \Big(\sum_{i=1}^{8} d_3(w_i/S(C_4)) \Big) x^3$$

$$= |V(S(C_4))| + 2|E(S(C_4))|x$$

$$+ \Big(\sum_{i=1}^{4} d_2(v_i/S(C_4)) + \sum_{i=1}^{4} d_2(u_i/S(C_4)) \Big) x^2$$

$$+ \Big(\sum_{i=1}^{4} d_3(v_i/S(C_4)) + \sum_{i=1}^{4} d_3(u_i/S(C_4)) \Big) x^3$$

$$= 8 + 24x + (3 + 3 + 3 + 3 + 3 + 3 + 3 + 3)x^2$$

$$+ (0 + 0 + 0 + 0 + 2 + 2 + 2 + 2)x^3$$

$$= 6 + 24x + 24x^2 + 8x^3.$$

Since, $N_k(C_4, x) = 4 + 8x + 4x^2$, (see Proposition 1.3), it follows that

$$N_k(S(C_4), x) = 4(4 + 8x + 4x^2) + 8(x^3 + x^2 - x - 1)$$
$$= 4N_k(C_4, x) + 2(4)(x^3 + x^2 - x - 1).$$

Case2:: If $n = 5$, then similarly as in Case 1, we get

$N_k(S(C_5), x) = 10 + 30x + 50x^2 + 10x^3$.

Since, $N_k(C_5, x) = 5 + 10x + 10x^2$, then

$N_k(S(C_5), x) = 4N_k(C_5, x) + 2(5)(x^3 + x^2 - x - 1)$.

Case3:: If $n \geqslant 6$, we obtain for every $i \in \{1, 2, ..., n\}$ that

$d_0(v_i/S(C_n)) = d_0(v_i/C_n)$ and $d_0(u_i/S(C_n)) = d_0(v_i/C_n)$

$d_1(v_i/S(C_n)) = 2d_1(v_i/C_n)$ and $d_1(u_i/S(C_n)) = d_1(v_i/C_n)$

$d_2(v_i/S(C_n)) = 2d_2(v_i/C_n) + 1$ and $d_2(u_i/S(C_n)) = 2d_2(v_i/C_n) + 1$

$d_3(v_i/S(C_n)) = 2d_3(v_i/C_n)$ and $d_3(u_i/S(C_n)) = 2d_3(v_i/C_n) + 2$

$d_4(v_i/S(C_n)) = 2d_4(v_i/C_n)$ and $d_4(u_i/S(C_n)) = 2d_4(v_i/C_n)$

....

$d_k(v_i/S(C_n)) = 2d_k(v_i/C_n)$ and $d_k(u_i/S(C_n)) = 2d_k(v_i/C_n)$,

for every $6 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$. Hence,

$$N_k(S(C_n), x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Big(\sum_{w \in V(S(P_n))} d_k(w/S(C_n)) \Big) x^k$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Big(\sum_{i=1}^{n} d_k(v_i/S(C_n)) \Big) x^k + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Big(\sum_{i=1}^{n} d_k(u_i/S(C_n)) \Big) x^k$$

$$= \Big(\sum_{i=1}^{n} d_0(v_i/S(C_n)) \Big) x^0 + \Big(\sum_{i=1}^{n} d_1(v_i/S(C_n)) \Big) x + ...$$

$$+ \Big(\sum_{i=1}^{n} d_{\lfloor \frac{n}{2} \rfloor}(v_i/S(C_n)) \Big) x^{\lfloor \frac{n}{2} \rfloor}$$

$$+ \Big(\sum_{i=1}^{n} d_0(u_i/S(C_n)) \Big) x^0 + \Big(\sum_{i=1}^{n} d_1(u_i/S(C_n)) \Big) x + ...$$

$$+ \Big(\sum_{i=1}^{n} d_{\lfloor \frac{n}{2} \rfloor}(u_i/S(P_n)) \Big) x^{\lfloor \frac{n}{2} \rfloor}$$

$$= \sum_{i=1}^{n} d_0(v_i/C_n) + \Big(\sum_{i=1}^{n} 2d_1(v_i/C_n) \Big) x + \Big(\sum_{i=1}^{n} (2d_2(v_i/C_n + 1) \Big) x^2$$

$$+ \Big(\sum_{i=1}^{n} 2d_3(v_i/C_n) \Big) x^3 + \Big(\sum_{i=1}^{n} 2d_4(v_i/C_n) \Big) x^4 + ...$$

$$+ \Big(\sum_{i=1}^{n} 2d_{\lfloor \frac{n}{2} \rfloor}(v_i/C_n) \Big) x^{\lfloor \frac{n}{2} \rfloor}$$

$$+ \sum_{i=1}^{n} d_0(v_i/C_n) + \Big(\sum_{i=1}^{n} d_1(v_i/C_n) \Big)x + \Big(\sum_{i=1}^{n} (2d_2(v_i/C_n + 1) \Big)x^2$$

$$+ \Big(\sum_{i=1}^{n} (2d_3(v_i/C_n) + 2) \Big)x^3 + \Big(\sum_{i=1}^{n} 2d_4(v_i/C_n) \Big)x^4 + \dots$$

$$+ \Big(\sum_{i=1}^{n} 2d_{\lfloor \frac{n}{2} \rfloor}(v_i/C_n) \Big)x^{\lfloor \frac{n}{2} \rfloor}$$

$$= 4 \left[\sum_{i=1}^{n} d_0(v_i/C_n) + \Big(\sum_{i=1}^{n} d_1(v_i/C_n) \Big)x + \dots + \Big(\sum_{i=1}^{n} d_{\lfloor \frac{n}{2} \rfloor}(v_i/C_n) \Big)x^{\lfloor \frac{n}{2} \rfloor} \right]$$

$$- 2 \sum_{i=1}^{n} d_0(v_i/C_n) - \Big(\sum_{i=1}^{n} d_1(v_i/C_n) \Big)x + \Big(2 \sum_{i=1}^{n} 1 \Big)x^2 + \Big(\sum_{i=1}^{n} 2 \Big)x^3.$$

$$= 4N_k(C_n, x) - 2n - 2mx + 2nx^2 + 2mx^3$$

$$= 4N_k(C_n, x) - 2n - 2nx + 2nx^2 + 2nx^3.$$

Therefore, $N_k(S(C_n), x) = 4N_k(C_n.x) + 2n\Big(x^3 + x^2 - x - 1\Big)$. $\square$

THEOREM 2.4. *For a complete bipartite graph $K_{r,s}$ with partition vertex sets V and U with $r \geqslant 1$ and $s \geqslant 1$ vertices, respectively, the N_k-polynomial of splitting graph $S(K_{r,s})$ of $K_{r,s}$ is of form*

$$N_k\big(S(K_{r,s}, x) = 2(r+s) + 6rsx + \Big[2r(r-1) + 2s(s-1) \Big]x^2 + rsx^3.$$

PROOF. Let $K_{r,s}$ be a complete bipartite graph with partition vertex sets $V = \{v_1, v_2, ..., v_r\}$ and $U = \{u_1, u_2, ..., u_s\}$ and let the sets $X = \{x_1, x_2, ..., x_r\}$ and $Y = \{y_1, y_2, ..., y_s\}$ be the new vertex sets introduced in construction a splitting graph $S(K_{r,s})$ of $K_{r,s}$ corresponding to partition sets V and U, respectively.
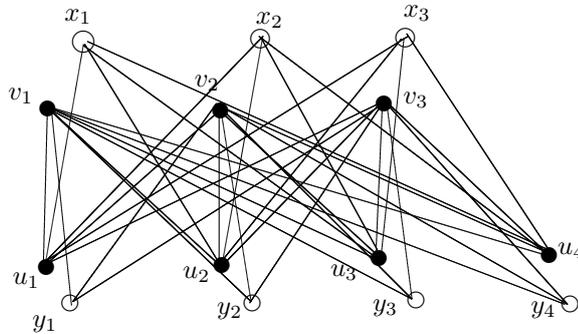

Figure 4: Splitting graph of $K_{3,4}$

Thus, by Proposition 1.1, $|V(S(K_{r,s}))| = 2(r+s)$, $E(S(K_{r,s}))| = 3rs$ and $diam(S(K_{r,s})) = 3$. Hence, by Proposition 1.2, we get

$$\sum_{w \in V(S(K_{r,s}))} d_0(w/S(K_{r,s})) = |V(S(K_{r,s}))| = 2(r+s) \text{ and}$$

$$\sum_{w \in V(S(K_{r,s}))} d_1(w/S(K_{r,s})) = 2|E(S(K_{r,s}))| = 2(3rs) = 6rs.$$

Since,

$d_2(v_i/S(K_{r,s})) = 2r - 1$ and $d_3(v_i/S(K_{r,s})) = 0$

$d_2(u_i/S(K_{r,s})) = 2s - 1$ and $d_3(u_i/S(K_{r,s})) = 0$

$d_2(x_i/S(K_{r,s})) = 2r - 1$ and $d_3(x_i/S(K_{r,s})) = s$

$d_2(y_i/S(K_{r,s})) = 2s - 1$ and $d_3(y_i/S(K_{r,s})) = r$

Then

$$\begin{aligned}
N_k(S(K_{r,s}), x) &= \sum_{k=0}^{3} \Big(\sum_{w \in V(S(K_{r,s}))} d_k(w/S(K_{r,s})) \Big) x^k \\
&= \Big(\sum_{w \in V(S(K_{r,s}))} d_0(w/S(K_{r,s})) \Big) x^0 \\
&\quad + \Big(\sum_{w \in V(S(K_{r,s}))} d_1(w/S(K_{r,s})) \Big) x \\
&\quad + \Big(\sum_{w \in V(S(K_{r,s}))} d_2(w/S(K_{r,s})) \Big) x^2 \\
&\quad + \Big(\sum_{w \in V(S(K_{r,s}))} d_3(w/S(K_{r,s})) \Big) x^3.
\end{aligned}$$

$$N_k(S(K_{r,s}), x) = |V(S(K_{r,s}))| + 2|E(S(K_{r,s}))|x$$

$$+ \Big[\sum_{i=1}^{r} \Big(d_2(v_i/S(K_{r,s})) + d_2(x_i/S(K_{r,s})) \Big) +$$

$$\sum_{i=1}^{s} \Big(d_2(u_i/S(K_{r,s})) + d_2(y_i/S(K_{r,s})) \Big) \Big] x^2 + \Big[\sum_{i=1}^{r} \Big(d_3(v_i/S(K_{r,s}))$$

$$+ d_3(x_i/S(K_{r,s})) \Big) + \sum_{i=1}^{s} \Big(d_3(u_i/S(K_{r,s})) + d_3(y_i/S(K_{r,s})) \Big) \Big] x^3$$

$$= 2(r+s) + 6rsx + \Big[\sum_{i=1}^{r} \Big((2r-1) + (2r-1) \Big)$$

$$+ \sum_{i=1}^{s} \Big((2s-1) + (2s-1) \Big) \Big] x^2$$

$$+ \Big[\sum_{i=1}^{r} \Big(0+s \Big) + \sum_{i=1}^{s} \Big(0+r) \Big) \Big] x^3$$

$$= 2(r+s) + 6rsx + \Big[2r(r-1) + 2s(s-1) \Big] x^2 + 2rsx^3.$$

$\square$

COROLLARY 2.2. *For a star graph $K_{1,n}$ with $1+n$ vertices, the N_k-polynomial of a splitting $S(K_{1,n})$ of a star $K_{1,n}$ is of form*

$$N_k(S(K_{1,n}), x) = 2(n+1) + 6nx + (4n^2 - 2n + 2)x^2 + nx^3.$$

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