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SUBDIRECTLY IRREDUCIBLE GENERALIZED ALMOST DISTRIBUTIVE FUZZY LATTICES

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Abstract. In this paper we introduce the concept of ideals, filters and congruences in a GADFL and we give an equivalent condition for a GADFL to become an ADFL interms of ideals, filters and congruence relations. We also characterize Subdirectly Irreducible GADFLs.

1. Introduction

The concept of Generalized Almost Distributive Lattices(GADLs) was introduced by G. C. Rao, Ravi Kumar Bandaru and N. Rafi [5] as a generalization of an Almost Distributive Lattices(ADLs) [6] which was a common abstraction of almost all the existing ring theoretic generalization of a Boolean algebra on one hand and distributive lattices on the other. On the other hand, L. A. Zadeh [7] in 1965 introduced the notion of fuzzy set. Again in 1971, L. A. Zadeh [8] defined a fuzzy ordering as a generalization of the concept of ordering, that is, a fuzzy ordering is a fuzzy relation that is transitive. In particular, a fuzzy partial ordering is a fuzzy ordering that is reflexive and antisymmetric. In 1994, N. Ajmal and K. V. Thomas [1] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. In 2009, I. Chon [4], considering the notion of fuzzy order of Zadeh , introduced a new notion of fuzzy lattices and studied the level sets of fuzzy lattices. He also introduced the notions of distributive and modular fuzzy lattices and considered some basic properties of fuzzy lattices. In 2017, Berhanu *et al.* [2] introduce the concept of an Almost Distributive Fuzzy Lattices (ADFLs) as a generalization of Distributive Fuzzy Lattices and characterized some properties of an ADL using

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the fuzzy partial order relations and fuzzy lattices defined by I. Chon. Later on Berhanu and Yohannes [3] introduce the concept of Generalized Almost Distributive Fuzzy Lattices (GADFLs) as a generalization of ADFLs. As a continuation in this paper we introduce the concept of Subdirectly Irreducible GADFLs.

2. Preliminaries

First we recall certain definitions and properties of Fuzzy Partial Order Relations, Almost Distributive Fuzzy Lattices and Generalized Almost Distributive Fuzzy Lattices that are required in this paper.

Definition 2.1. ([4]) Let X be a set. A function $A : X \times X \longrightarrow [0,\ 1]$ is called a fuzzy relation in X. The fuzzy relation A in X is reflexive iff $A(x, x) = 1$ for all $x \in X$, A is transitive iff $A(x, z) \geqslant \sup_{y \in X} \min(A(x, y), A(y, z))$, and A is antisymmetric iff $A(x, y) > 0$ and $A(y, x) > 0$ implies $x = y$. A fuzzy relation A is fuzzy partial order relation if A is reflexive, antisymmetric and transitive. A fuzzy partial order relation A is a fuzzy total order relation iff $A(x, y) > 0$ or $A(y, x) > 0$ for all $x, y \in R$. If A is a fuzzy partial order relation in a set X, then (X, A) is called a fuzzy partially ordered set or a fuzzy poset. If B is a fuzzy total order relation in a set X, then (X, B) is called a fuzzy totally ordered set or a fuzzy chain.

Definition 2.2. ([2]) Let $(R, \vee, \wedge, 0)$ be an algebra of type (2, 2, 0) and (R, A) be a fuzzy poset. Then we call (R, A) is an Almost Distributive Fuzzy Lattice (ADFL) if the following axioms are satisfied:

(F1) $A(a, a \vee 0) = A(a \vee 0, a) = 1$

(F2) $A(0, 0 \wedge a) = A(0 \wedge a, 0) = 1$

(F3) $A((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)) = A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c) = 1$

(F4) $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$

(F5) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$

(F6) $A((a \vee b) \wedge b, b) = A(b, (a \vee b) \wedge b) = 1$

for all a,b,c $\in$ R.

From the definitions of ADL and ADFL, The following theorem is direct.

Theorem 2.1. ([2]) *Let (R, A) be a fuzzy poset. Then R is an ADL iff (R, A) is an ADFL.*

Theorem 2.2. ([2]) *Let (R, A) be an ADFL . Then*

$$a = b \Leftrightarrow A(a, b) = A(b, a) = 1.$$

Definition 2.3. ([2]) Let (R, A) be an ADFL. Then for any $a, b \in R$

$$a \leqslant b \text{ if and only if } A(a, b) > 0.$$

Theorem 2.3. ([2]) *If (R, A) is an ADFL then*

$$a \wedge b = a \text{ if and only if } A(a, b) > 0.$$

LEMMA 2.1. ([2]) *Let* (R, A) *be an ADFL. Then for each* a *and* b *in* R

(1) $A(a \wedge b, b) > 0$ *and* $A(b \wedge a, a) > 0$

(2) $A(a, a \vee b) > 0$ *and* $A(b, b \vee a) > 0$.

DEFINITION 2.4. ([3]) Let $(R, \vee, \wedge)$ be an algebra of type (2, 2) and (R, A) be a fuzzy poset. Then we call (R, A) is a Generalized Almost Distributive Fuzzy Lattice if it satisfies the following axioms:

(FG1) $A((a \wedge b) \wedge c, a \wedge (b \wedge c)) = A(a \wedge (b \wedge c), (a \wedge b) \wedge c) = 1$

(FG2) $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$

(FG3) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$

(FA1) $A(a \wedge (a \vee b), a) = A(a, a \wedge (a \vee b)) = 1$

(FA2) $A((a \vee b) \wedge a, a) = A(a, (a \vee b) \wedge a) = 1$

(FA3) $A((a \wedge b) \vee b, b) = A(b, (a \wedge b) \vee b) = 1$

for all $a, b, c \in R$.

EXAMPLE 2.1. ([3]) Let $R = \{a, b, c\}$. Define two binary operations $\vee$ and $\wedge$ on R as follows:

$\vee$	a	b	c
a	a	b	a
b	b	b	b
c	c	c	c

and

$\wedge$	a	b	c
a	a	a	c
b	a	b	c
c	a	a	c

Define a fuzzy relation $A : R \times R \longrightarrow [0, 1]$ as follows:

$$A(a, a) = A(b, b) = A(c, c) = 1,\ A(b, a) = A(b, c) = A(c, a) = A(c, b) = 0,$$
$$A(a, b) = 0.2 \text{ and } A(a, c) = 0.4.$$

Clearly (R, A) is a fuzzy poset. Here (R, A) is a GADFL since it satisfies the above six axioms of a GADFL but it is not an ADFL. Since

$$A((c \vee b) \wedge b, (c \wedge b) \vee (b \wedge b)) = A(c \wedge b, a \vee b) = A(a, b) = 0.2 \neq 1$$

and

$$A((c \wedge b) \vee (b \wedge b), (c \vee b) \wedge b) = A(a \vee b, c \wedge b) = A(b, a) = 0 \neq 1.$$

Hence

$$A((c \vee b) \wedge b, (c \wedge b) \vee (b \wedge b)) \neq A((c \wedge b) \vee (b \wedge b), (c \vee b) \wedge b).$$

DEFINITION 2.5. ([3]) Let (R, A) be a GADFL. Then (R, A) is said to be associative if the operation $\vee$ in R is associative.

THEOREM 2.4. ([3]) *Let (R, A) be a GADFL. Then the following are equivalent.*

(1) (R, A) *is an Almost Distributive Fuzzy Lattice;*

(2) $A((a \vee b) \wedge c,\ (a \wedge c) \vee (b \wedge c)) > 0$ *and* $A((a \wedge c) \vee (b \wedge c),\ (a \vee b) \wedge c) > 0$ *;*

(3) $A((a \vee b) \wedge b,\ b) > 0$ *and* $A(b,\ (a \vee b) \wedge b) > 0$*;*

(4) *$A((a \vee b) \wedge c,\ (b \vee a) \wedge c) > 0$ and $A((b \vee a) \wedge c,\ (a \vee b) \wedge c) > 0$;*
for all $a, b, c \in R$.

3. Ideals and Congruences in a GADFL

In this section, we introduce the concept of an ideal and filter in a GADFL. Also we give the definition of a congruence in a GADFL. We give an equivalent condition for a GADFL to become an ADFL in terms of ideals, filters and congruence relations of (R, A).

DEFINITION 3.1. Let (R, A) be a GADFL. A non empty subset I_A of R is said to be an ideal of (R, A), if it satisfies the following conditions:
(i) If $x \in R$, $y \in I_A$ and $A(x, y) > 0$, then $x \in I_A$
(ii) If $x, y \in I_A$, then $x \vee y \in I_A$.

DEFINITION 3.2. Let (R, A) be a GADFL. A non empty subset F_A of R is said to be a filter of (R, A), if it satisfies the following conditions:
(i) If $x \in R$, $y \in F_A$ and $A(y, x) > 0$, then $x \in F_A$
(ii) If $x, y \in F_A$, then $x \wedge y \in F_A$.

THEOREM 3.1. *Let (R, A) be a GADFL. Then the following are equivalent:*
(1) *For $x \in R$, if $a \in I_A$ with $A(x, a) > 0$ then $x \in I_A$*
(2) *If $a \in I_A$ and $x \in R$ then $a \wedge x \in I_A$.*

PROOF. (1) $\Rightarrow$ (2). Suppose (1). For $a \in I_A$ and $x \in R$ we need to show $a \wedge x \in I_A$. Since $(x \wedge a) \vee a = a$ then $x \wedge a \leqslant a$ and hence $A(x \wedge a, a) > 0$ then by (1) $x \wedge a \in I_A$. Now

$$A(a \wedge x, x \wedge a) = A(a \wedge x \wedge x, x \wedge a) = A(x \wedge x, x \wedge a) = A(x, x \wedge a) = A(x \wedge a, x \wedge a) = 1.$$

Similarly, $A(x \wedge a, a \wedge x) = 1$. Hence $a \wedge x = x \wedge a \in I_A$.
(2) $\Rightarrow$ (1). Suppose (2). For $x \in R$, $a \in I_A$ with $A(x, a) > 0$ implies that $x \wedge a = x$. Hence $x = x \wedge a = x \wedge a \wedge a = a \wedge x \wedge a \in I_A$ by (2). Therefore $x \in I_A$. $\square$

THEOREM 3.2. *Let (R, A) be a GADFL and $a \in R$. Define*

$$(a]_A = \{x \in R \mid A(x, a \wedge x) > 0\}.$$

Then $(a]_A$ is the smallest ideal of (R, A) containing a and is called the principal ideal generated by a.

PROOF. Since $A(a, a \wedge a) = A(a, a) = 1$. then $a \in (a]_A$. Therefore $(a]_A \neq \emptyset$. Now, let $r \in (a]_A$ and $y \in R$ such that $A(y, r) > 0$. Since $r \in (a]_A$ we have $A(r, a \wedge r) > 0$ for $x \in R$ and $A(y, r) > 0$ implies that $y \wedge r = y$. Now, $A(y, a \wedge y) = A(y, a \wedge y \wedge r) = A(y, y \wedge a \wedge r) = A(y, y \wedge r) = A(y, y) > 0$. Therefore $y \in (a]_A$. Again, let $r, s \in (a]_A$. Then $A(r, a \wedge r) > 0$ and $A(s, a \wedge s) > 0$. for $r, s \in R$. Now, $A(r \vee s, a \wedge (r \vee s)) = A(r \vee s, (a \wedge r) \vee (a \wedge s)) = A(r \vee s, r \vee s) > 0$. Then $r \vee s \in (a]_A$. Therefore $(a]_A$ is an ideal of (R, A) containing a. To show $(a]_A$ is the smallest. Let I_A be any ideal of (R, A) containing a. Let $r \in (a]_A$ then

$A(r, a \wedge r) > 0$ for $r \in R$. Now as $a \in I_A$ and $r \in R$ then $a \wedge r = r \in I_A$ as I_A is an ideal. Hence $(a]_A \subseteq I_A$. Therefore $(a]_A$ is the smallest ideal of (R, A) containing a. $\square$

Similarly we can prove that

$$[a)_A = \{x \in R \mid A(a \vee x, x) > 0 \text{ and } A(x, a \vee x) > 0\}$$

is the smallest filter of (R, A) containing a and is called the principal filter generated by a.

Now we discuss some important properties of the principle ideals (filters) of (R, A).

LEMMA 3.1. *Let (R, A) be a GADFL and $a, b \in R$. Then*

(i) $a \in (b]_A$ *if and only if* $A(a, b \wedge a) > 0$*;*

(ii) $a \in [b)_A$ *if and only if* $A(b \vee a, a) > 0$ *and* $A(a, b \vee a) > 0$*;*

(iii) $A(a, b) > 0 \Rightarrow (a]_A \subseteq (b]_A$*;*

(iv) $a \in (b]_A \Rightarrow (a]_A \subseteq (b]_A$.

PROOF. The proofs of (i) and (ii) are trivial.

(iii) Suppose $A(a, b) > 0$ and $x \in (a]_A$ then $A(x, a \wedge x) > 0$. Now

$$A(x, b \wedge x) = A(x, b \wedge a \wedge x) = A(x, a \wedge b \wedge x) = A(x, a \wedge x) > 0$$

. Hence $x \in (b]_A$. Therefore $(a]_A \subseteq (b]_A$.

(iv) Let $a \in (b]_A$ and $x \in (a]_A$. Then $A(a, b \wedge a) > 0$ and $A(x, a \wedge x) > 0$. Now

$$A(x, b \wedge x) = A(x, b \wedge a \wedge x) = A(x, a \wedge x) > 0.$$

Hence $x \in (b]_A$. Therefore $(a]_A \subseteq (b]_A$. $\square$

THEOREM 3.3. *Let (R, A) be a GADFL and $a, b \in R$. Then $(a \wedge b]_A = (b \wedge a]_A$.*

PROOF. Suppose (R, A) is a GADFL and $a, b \in R$.

($\Rightarrow$) Let $x \in (a \wedge b]_A$. Then $A(x, a \wedge b \wedge x) > 0$. Now

$$A(x, b \wedge a \wedge x) = A(x, a \wedge b \wedge x) > 0.$$

Therefore $x \in (b \wedge a]_A$

($\Leftarrow$) Assume that $x \in (b \wedge a]_A$. Then $A(x, b \wedge a \wedge x) > 0$. Now

$$A(x, a \wedge b \wedge x) = A(x, b \wedge a \wedge x) > 0.$$

Therefore $x \in (a \wedge b]_A$. Hence $(a \wedge b]_A = (b \wedge a]_A$. $\square$

If (R, A) is an ADFL and $a, b \in R$, then $(a \vee b]_A = (b \vee a]_A$. But if this condition holds in a GADFL then the GADFL becomes an ADFL. We prove this in the following theorem.

THEOREM 3.4. *Let (R, A) be a GADFL, R with 0. Then the following are equivalent:*

(1) (R, A) *is an Almost Distributive Fuzzy Lattice*

(2) *For any $a, b \in R$, $(a \vee b]_A$ is the supremum of $(a]_A$ and $(b]_A$ in $(I_A(R), \subseteq)$,*

where $I_A(R)$ is the set of all ideals in (R, A)
(3) $(a \vee b]_A = (b \vee a]_A$ *for all* $a, b \in R$.

PROOF. (1) $\Rightarrow$ (2). Assume (1). Let $a, b \in R$. Since

$$A(a,\, a \vee b) > 0 \Rightarrow (a]_A \subseteq (a \vee b]_A \text{ (lemma 3.1(iii))}$$

and

$$A(b,\, b \vee a) > 0 \Rightarrow (b]_A \subseteq (b \vee a]_A = (a \vee b]_A$$

as (R, A) is an ADFL. Therefore $(a \vee b]_A$ is an upper bound of $(a]_A$ and $(b]_A$. Let J_A be any ideal of (R, A) such that $(a]_A \subseteq J$ and $(b]_A \subseteq J_A$. Clearly $a \in J_A$ and $b \in J_A$. Therefore $a \vee b \in J_A$ and hence $(a \vee b]_A \subseteq J_A$. Thus $(a \vee b]_A$ is the supremum of $(a]_A$ and $(b]_A$ in $(I_A(R), \subseteq)$.

(2) $\Rightarrow$ (3). Assume (2). Then $(a \vee b]_A$ and $(b \vee a]_A$ both are supremums of $(a]_A$ and $(b]_A$ in the poset $(I_A(R), \subseteq)$. Therefore $(a \vee b]_A = (b \vee a]_A$ by uniqueness of supremum.

(3) $\Rightarrow$ (1). Assume (3). Let $a, b \in R$. Since from GADFL $A(b, (b \vee a) \wedge b) > 0$ then $b \in (b \vee a]_A = (a \vee b]_A$. Hence $A(b, (a \vee b) \wedge b) > 0$ and since $A((a \vee b) \wedge b, b) > 0$. Therefore (R, A) is an Almost Distributive Fuzzy Lattice. $\square$

DEFINITION 3.3. Let (R, A) be a GADFL. An equivalence relation Θ on (R, A) is called a congruence on (R, A) if, for $a, b, c, d \in R$, holds

$$(a, b),\, (c, d) \in \Theta \Rightarrow (a \vee c, b \vee d),\, (a \wedge c, b \wedge d) \in \Theta.$$

THEOREM 3.5. *Let F_A be a filter of a GADFL (R, A). Then the relation $\varphi^{F_A} = \{(x, y) \in R \times R \mid A(a \wedge x,\, a \wedge y) = A(a \wedge y,\, a \wedge x) = 1$, for some $a \in F_A$ $\}$ is a congruence relation on (R, A).*

PROOF. Clearly φ^{F_A} is an equivalence relation on (R, A).
Now, let $(x, y),\, (u, v) \in \varphi^{F_A}$ then

$$A(a \wedge x, a \wedge y) = A(a \wedge y, a \wedge x) = 1$$

and

$$A(b \wedge u, b \wedge v) = A(b \wedge v, b \wedge u) = 1$$

for some $a, b \in F_A$. Hence $a, b \in F_A \Rightarrow a \wedge b \in F_A$ and

$$A(a \wedge b \wedge x \wedge u, a \wedge b \wedge y \wedge v) = A(a \wedge x \wedge b \wedge u, a \wedge b \wedge y \wedge v) =$$
$$A(a \wedge y \wedge b \wedge u, a \wedge b \wedge y \wedge v) = A(a \wedge y \wedge b \wedge v, a \wedge b \wedge y \wedge v) =$$
$$A(a \wedge b \wedge y \wedge v, a \wedge b \wedge y \wedge v) = 1.$$

Similarly $A(a \wedge b \wedge y \wedge v,\, a \wedge b \wedge x \wedge u) = 1$. Therefore $(x \wedge u,\, y \wedge v) \in \varphi^{F_A}$.
Also,
$A((a \wedge b) \wedge (x \vee u), (a \wedge b) \wedge (y \vee v)) = A(a \wedge b \wedge (x \vee u), (a \wedge b) \wedge (y \vee v))$
$= A((a \wedge b \wedge x) \vee (a \wedge b \wedge u)), (a \wedge b) \wedge (y \vee v)) = A((b \wedge a \wedge x) \vee (a \wedge b \wedge u)), (a \wedge b) \wedge (y \vee v))$
$= A((b \wedge a \wedge y) \vee (a \wedge b \wedge u)), (a \wedge b) \wedge (y \vee v)) = A((b \wedge a \wedge x) \vee (a \wedge b \wedge v)), (a \wedge b) \wedge (y \vee v))$

$= A((a \wedge b \wedge y) \vee (a \wedge b \wedge v)), (a \wedge b) \wedge (y \vee v)) = A((a \wedge b) \wedge (y \vee v), (a \wedge b) \wedge (y \vee v))$
$= 1.$

Similarly

$$A((a \wedge b) \wedge (y \vee v), (a \wedge b) \wedge \wedge (x \vee u)) = 1.$$

Therefore

$$(x \vee u, y \vee v) \in \varphi^{F_A}$$

and hence φ^{F_A} is a congruence relation on (R, A). □

Lemma 3.2. *Let (R, A) be a GADFL. Then for any $a \in R$, $\varphi^{[a)_A} = \varphi^{a_A}$.*

Proof. Clearly $\varphi^{a_A} \subseteq \varphi^{[a)_A}$. Let $(x, y) \in \varphi^{[a)_A}$. Then

$$A(t \wedge x, t \wedge y) = A(t \wedge y, t \wedge x) = 1$$

for some $t \in [a)_A$. Now, $t \in [a)_A \Rightarrow a \vee t = t$ by lemma 3.1(ii) and hence $a \wedge t = a$. Since

$$A(a \wedge x, a \wedge y) = A(a \wedge t \wedge x, a \wedge y) = A(a \wedge t \wedge y, a \wedge y) = A(a \wedge y, a \wedge y) = 1$$
$$\Rightarrow A(a \wedge x,\ a \wedge y) = 1.$$

Similarly $A(a \wedge y,\ a \wedge x) = 1$. Hence $(x, y) \in \varphi^{a_A}$. Therefore $\varphi^{[a)_A} \subseteq \varphi^{a_A}$ and hence $\varphi^{[a)_A} = \varphi^{a_A}$. □

In general, for any $a \in R$,

$$\psi_{a_A} = \{\ (x, y) \in R \times R \mid A(x \wedge a,\ y \wedge a) = A(y \wedge a,\ x \wedge a) = 1\}$$

is an equivalence relation but not a congruence relation on (R, A). For in example 2.1, $\psi_{b_A} = \{(a, a),\ (b, b),\ (c, c), (a, c),\ (c, a)\}$ is not congruence relation on (R, A) because for (a, c), $(b, b) \in \psi_{b_A}$, we have

$$A((a \vee b) \wedge b, (c \vee b) \wedge b) = A(b \wedge b, c \wedge b) = A(b, a) = 0$$

and

$$A((c \vee b) \wedge b, (a \vee b) \wedge b) = A(b \wedge b, c \wedge b) = A(a, b) = 0.2.$$

Hence $A((a \vee b) \wedge b, (c \vee b) \wedge b) \neq A((c \vee b) \wedge b, (a \vee b) \wedge b)$. Then $(a \vee b,\ c \vee b)$ is not in ψ_{b_A}. Therefore ψ_{b_A} is not a congruence relation.

Also for any filter F_A of (R, A),

$\psi_{F_A} = \{(x, y) \in R \times R \mid A(x \wedge a,\ y \wedge a) = A(y \wedge a,\ x \wedge a) = 1$, for some $a \in F_A\}$

is an equivalence relation but not a congruence relation on (R, A).

Lemma 3.3. *Let (R, A) be a GADFL. Then for any $a \in R$,*

$$\psi_{a_A} = \{(x, y) \in R \times R \mid A(x \wedge a, y \wedge a) = A(y \wedge a, x \wedge a) = 1\} = \psi_{[a)_A}.$$

Proof. Clearly $\psi_{a_A} \subseteq \psi_{[a)_A}$. Let $(x, y) \in \psi_{[a)_A}$. Then

$$A(x \wedge t,\ y \wedge t) = A(y \wedge t,\ x \wedge t) = 1 \text{ for some } t \in [a)_A.$$

Now, $t \in [a)_A$ implies $a \vee t = t$ by lemma 3.1(ii) and hence $a \wedge t = a$. Also,

$$A(x \wedge a, y \wedge a) = A(x \wedge a \wedge t, y \wedge a) = A(a \wedge x \wedge t, y \wedge a)$$
$$= A(a \wedge y \wedge t, y \wedge a) = A(y \wedge a \wedge t, y \wedge a) = A(y \wedge a, y \wedge a) = 1.$$

Similarly $A(y \wedge a, x \wedge a) = 1$. Hence $(x, y) \in \psi_{a_A}$. Thus $\psi_{[a)_A} \subseteq \psi_{a_A}$. Therefore $\psi_{a_A} = \psi_{[a)_A}$. $\square$

THEOREM 3.6. *Let (R, A) be an associative GADFL, R with 0. Then for any ideal I_A of (R, A), the relation*

$$\vartheta_{I_A} = \{(x, y) \in R \times R \mid A(a \vee x, a \vee y) = A(a \vee y, a \vee x) = 1, \textit{ for some } a \in I_A\}$$

is the smallest congruence on (R, A) containing $I_A \times I_A$.

PROOF. Clearly ϑ_{I_A} is a congruence relation on (R, A). Also, for any $x, y \in I_A$, we have $x \vee y \in I_A$ and $(x, y) \in I_A \times I_A$. Now

$$A((x \vee y) \vee x, (x \vee y) \vee x) = A(x \vee y, x \vee y) = 1 \text{ and } A((x \vee y) \vee y, (x \vee y) \vee x) = 1.$$

Hence $(x, y) \in \vartheta_{I_A}$. Therefore $I_A \times I_A \subseteq \vartheta_{I_A}$. Now, let ϑ_A be any congruence on (R, A) containing $I_A \times I_A$. Then

$$(x, y) \in \vartheta_{I_A} \Rightarrow A(a \vee x, a \vee y) = A(a \vee y, a \vee x) = 1,$$

for some $a \in I_A$. Since $0 \in I_A$ and $a \in I_A$ then

$$\begin{aligned}(0, a) \in I_A \times I_A &\Rightarrow (0, a) \in \vartheta_A \\ &\Rightarrow (0 \vee x, a \vee x) \in \vartheta_A \text{ (since } (x, x) \in \vartheta_A \text{ and } \vartheta_A \text{ is congruence)} \\ &\Rightarrow (x, a \vee x) \in \vartheta_A\end{aligned}$$

and similarly

$$\begin{aligned}(y, a \vee y) \in \vartheta_A &\Rightarrow (a \vee y, y) \in \vartheta_A \\ &\Rightarrow (x, y) \in \vartheta_A \text{ (since } \vartheta_A \text{ is transitive and } a \vee x = a \vee y).\end{aligned}$$

Therefore $\vartheta_{I_A} \subseteq \vartheta_A$. Thus ϑ_{I_A} is the smallest congruence on (R, A) containing $I_A \times I_A$. $\square$

Now we characterize an ADFL in terms of ψ_{F_A} and ψ_{a_A}.

THEOREM 3.7. *Let (R, A) be a GADFL. Then the following are equivalent.*

(i) *(R, A) is an Almost Distributive Fuzzy Lattice*

(ii) *For any filter F_A of (R, A), ψ_{F_A} is a congruence relation on (R, A)*

(iii) *ψ_{a_A} is a congruence relation on (R, A) for all $a \in R$.*

PROOF. Suppose (R, A) is a GADFL.

(i) $\Rightarrow$ (ii). Assume (i). Let F_A be a filter of (R, A). Clearly ψ_{F_A} is an equivalence relation. Let $(a, b), (c, d) \in \psi_F$. Then $A(a \wedge x, b \wedge x) = A(b \wedge x, a \wedge x) = 1$ and $A(c \wedge y, d \wedge y) = A(d \wedge y, c \wedge y) = 1$ for some $x, y \in F_A$. Since F_A is a filter of (R, A), $x \wedge y \in F_A$. Now

$$A(a \wedge c \wedge x \wedge y, b \wedge d \wedge x \wedge \wedge y) = A(a \wedge x \wedge c \wedge y, b \wedge d \wedge x \wedge \wedge y)$$
$$= A(a \wedge x \wedge d \wedge y, b \wedge d \wedge x \wedge \wedge y) = A(b \wedge x \wedge d \wedge y, b \wedge d \wedge x \wedge \wedge y)$$
$$= A(b \wedge d \wedge x \wedge y, b \wedge d \wedge x \wedge \wedge y) = 1.$$

Similarly

$$A(b \wedge d \wedge x \wedge y, a \wedge c \wedge x \wedge \wedge y) = 1$$

and hence $(a \wedge c,\, b \wedge d) \in \psi_{F_A}$. Also,

$A((a \vee c) \wedge x \wedge y, (b \vee d) \wedge x \wedge y) = A((a \wedge x \wedge y) \vee (c \wedge x \wedge y), (b \vee d) \wedge x \wedge y)$

$= A((b \wedge x \wedge y) \vee (x \wedge c \wedge y), (b \vee d) \wedge x \wedge y) = A([b \wedge (x \wedge y)] \vee [x \wedge d \wedge y], (b \vee d) \wedge x \wedge y)$

$= A([b \wedge (x \wedge y)] \vee [d \wedge (x \wedge y)], (b \vee d) \wedge x \wedge y)$

$= A((b \vee d) \wedge x \wedge y, (b \vee d) \wedge x \wedge y) = 1.$

Similarly $A((b \vee d) \wedge x \wedge y,\, (a \vee c) \wedge x \wedge y) = 1$. Therefore $(a \vee c,\, b \vee d) \in \psi_{F_A}$. Thus ψ_{F_A} is a congruence relation on (R, A).

(ii) $\Rightarrow$ (iii) Follows from lemma 3.3.

(iii) $\Rightarrow$ (i). Assume (iii). Let $a, b \in R$. Since

$$A(a \wedge b, (a \wedge b) \wedge b) = A((a \wedge b) \wedge b, a \wedge b) = 1.$$

then $(a,\, a \wedge b \in)\ \psi_{b_A}$. Also, $A(b \wedge b,\, b \wedge b) = 1$. Hence $(b, b) \in \psi_{b_A}$. Since ψ_{b_A} is a congruence relation on (R, A), $(a \vee b,\, (a \wedge b) \vee b) \in \psi_{b_A}$. Hence

$A((a \vee b) \wedge b,\, [(a \wedge b) \vee b] \wedge b) = A([(a \wedge b) \vee b] \wedge b,\, (a \vee b) \wedge b) = 1 \Rightarrow$

$A((a \vee b) \wedge b,\, b \wedge b) = A(b \wedge b,\, (a \vee b) \wedge b) = 1 \Rightarrow$

$A((a \vee b) \wedge b,\, b) > 0$ and $A(b,\, (a \vee b) \wedge b) > 0$.

Therefore (R, A) is an Almost Distributive Fuzzy Lattice. □

4. Subdirectly Irreducible GADFLs

In this section we characterize Subdirectly Irreducible associative GADFLs.

Definition 4.1. Let (R, A) be a GADFL,

$$\triangle_{(R,A)} = \{(x, y) \in R \times R | A(x, y) = A(y, x) = 1\}$$

is called zero congruence in (R, A).

Definition 4.2. A GADFL (R, A) is said to be subdirectly irreducible if the intersection of any family of nonzero congruences in (R, A) is again a nonzero congruence.

Lemma 4.1. *Let (R, A) be a GADFL. For any $a \in R$,*

$$\varphi^{a_A} = \{(x, y) \in R \times R \mid A(a \wedge x, a \wedge y) = A(a \wedge y, a \wedge x) = 1\}$$

is a congruence relation on (R, A). Further, $\varphi^{a_A} = \Delta_{(R,A)}$ if and only if a is a left identity element of R and $\varphi^{a_A} = R \times R$ if and only if $A(a, 0) = A(0, a) = 1$.

Proof. Clearly φ^{a_A} is an equivalence relation on (R, A). Let $(u, v), (c, d) \in \varphi^{a_A}$. Then $A(a \wedge u,\, a \wedge v) = A(a \wedge v,\, a \wedge u) = 1$ and $A(a \wedge c,\, a \wedge d) = A(a \wedge d,\, a \wedge c) = 1$. Now

$A(a \wedge (u \wedge c), a \wedge (v \wedge d)) = A((a \wedge u) \wedge c, a \wedge (v \wedge d)) = A((a \wedge v) \wedge c, a \wedge (v \wedge d))$

$= A(a \wedge v \wedge c, a \wedge (v \wedge d)) = A(v \wedge a \wedge c, a \wedge (v \wedge d)) = A(v \wedge a \wedge d, a \wedge (v \wedge d))$

$= A(a \wedge (v \wedge d), a \wedge (v \wedge d)) = 1.$

Similarly $A(a \wedge (v \wedge d),\, a \wedge (u \wedge c)) = 1$. Hence $(u \wedge c,\, v \wedge d) \in \varphi^{a_A}$. Also,

$A(a \wedge (u \vee c), a \wedge (v \vee d)) = A((a \wedge u) \vee (a \wedge c), a \wedge (v \vee d))$

$= A((a \wedge u) \vee (a \wedge d), a \wedge (v \vee d)) = A(a \wedge (v \vee d), a \wedge (v \vee d)) = 1.$

Similarly $A(a \wedge (v \vee d),\, a \wedge (u \vee c)) = 1$. Hence $(u \vee c,\, v \vee d) \in \varphi^{a_A}$. Therefore φ^{a_A} is a congruence relation on (R, A).

Suppose $\varphi^{a_A} = \Delta_{(R,A)}$. Let $x \in$ R. Then

$$A(a \wedge (a \wedge x), a \wedge x) = A(a \wedge a \wedge x, a \wedge x) = A(a \wedge x, a \wedge x) = 1.$$

Similarly $A(a \wedge x,\, a \wedge (a \wedge x)) = 1$. Hence $(a \wedge x,\, x) \in \varphi^{a_A} = \Delta_{(R,A)}$ and then $a \wedge x = x$. Thus a is left identity element.

Conversely suppose a is left identity element and $(x, y) \in \varphi^{a_A}$. Then

$$A(a \wedge x, a \wedge y) = A(a \wedge y, a \wedge x) = 1.$$

That is $A(x, y) = A(y, x) = 1$ and hence $\varphi^{a_A} = \Delta_{(R,A)}$. Also suppose $\varphi^{a_A} = R \times R$, since $(a, 0) \in R \times R$ then $(a, 0) \in \varphi^{a_A}$. Hence

$$A(a \wedge a, a \wedge 0) = A(a \wedge 0, a \wedge a) = 1 \Rightarrow A(a, 0) = A(0, a) = 1.$$

Conversely suppose $A(a, 0) = A(0, a) = 1$ then $a = 0$. Clearly $\varphi^{a_A} \subseteq R \times R$. Now let $(x, y) \in R \times R$. Since $A(a \wedge x, a \wedge y) = A(0 \wedge x, 0 \wedge y) = A(0, 0) = 1$. Similarly $A(a \wedge y, a \wedge x) = 1$. Then $(x, y) \in \varphi^{a_A}$. Therefore $\varphi^{a_A} = R \times R$. □

The following result can be easily verified.

LEMMA 4.2. *For any $a \in R$,*

$$\vartheta_{a_A} = \{(x, y) \in R \times R \mid A(a \vee x, a \vee y) = A(a \vee y, a \vee x) = 1\}$$

is an equivalent relation on (R, A).

In general, ϑ_{a_A} is not a congruence relation on (R, A). In example 2.1,

$$\vartheta_{a_A} = \Delta_{(R,A)} \cup \{(a, c), (c, a)\}$$

is not a congruence relation on (R, A) because for $(a, c), (b, b) \in \vartheta_{a_A}$, we have that $(a \vee b,\, c \vee b) = (b, c)$ is not in ϑ_{a_A}. But if $\vee$ is associative in R, then ϑ_{a_A} is a congruence relation on (R, A). In fact we prove the following.

THEOREM 4.1. *Let (R, A) be a GADFL. Then ϑ_{a_A} is a congruence on (R, A) if and only if $\vee$ is associative. Further, $\vartheta_{a_A} = \Delta_{(R,A)}$ if and only if a is the zero (least) element of R.*

PROOF. Suppose (R, A) is a GADFL.
Claim: (i) ϑ_{a_A} is a congruence on (R, A) if and only if $\vee$ is associative.
(ii) $\vartheta_{a_A} = \Delta_{(R,A)}$ if and only if a is the zero(least) element of R.

(i) Suppose ϑ_{a_A} is a congruence on (R, A), for all $a \in R$. Let $a, b, c \in R$, Since $A(a \vee 0,\, a \vee a) = A(a \vee a,\, a \vee 0) = 1$. Then $(0, a) \in \vartheta_{a_A}$. Now, $(0, a) \in \vartheta_{a_A}$ and $(b, b) \in \vartheta_{a_A} \Rightarrow (0 \vee b,\, a \vee b) \in \vartheta_{a_A} \Rightarrow (b,\, a \vee b) \in \vartheta_{a_A}$. Since $(c, c) \in \vartheta_{a_A}$ then $(b \vee c,\, (a \vee b) \wedge c) \in \vartheta_{a_A}$. Therefore A$(a \vee (b \vee c),\, a \vee [(a \vee b) \vee c]) = A(a \vee [(a \vee b) \vee c],\, a \vee (b \vee c)) = 1$. Then $a \vee (b \vee c) = a \vee [(a \vee b) \vee c] = (a \vee b) \vee c$ (since $a \wedge [(a \vee b) \vee c] = a$ and $a \vee b = b \Leftrightarrow a \wedge b = a$). Thus $\vee$ is associative.

Conversely suppose $\vee$ is associative. By lemma 4.4, ϑ_{a_A} is an equivalence relation on (R, A). Let (u, v) and $(c, d) \in \vartheta_{a_A}$. Then

$$A(a \vee u, a \vee v) = A(a \vee v, a \vee u) = 1$$

and

$$A(a \vee c, a \vee d) = A(a \vee d, a \vee c) = 1.$$

Now

$A(a \vee (u \wedge c), a \vee (v \wedge d)) = A((a \vee u) \wedge (a \vee c), a \vee (v \wedge d))$
$= A((a \vee v) \wedge (a \vee c), a \vee (v \wedge d)) = A((a \vee v) \wedge (a \vee d), a \vee (v \wedge d))$
$= A(a \vee (v \wedge d), a \vee (v \wedge d)) = 1.$

Similarly $A(a \vee (v \wedge d),\ a \vee (u \wedge c)) = 1$. Hence $(u \wedge c,\ v \wedge d) \in \vartheta_{a_A}$. Also,
$A(a \vee (u \vee c), a \vee (v \vee d)) = A((a \vee u) \vee c, a \vee (v \vee d)) = A((a \vee v) \vee c, a \vee (v \vee d))$
$= A([(a \vee v) \vee a] \vee c, a \vee (v \vee d)) = A((a \vee v) \vee (a \vee c), a \vee (v \vee d))$
$= A((a \vee v) \vee (a \vee d), a \vee (v \vee d)) = A([(a \vee v) \vee a] \vee d), a \vee (v \vee d))$
$= A((a \vee v) \vee d, a \vee (v \vee d)) = A(a \vee (v \vee d), a \vee (v \vee d)) = 1.$

Similarly $A(a \vee (v \vee d),\ a \vee (u \vee c)) = 1$. Hence $(u \vee c,\ v \vee d) \in \vartheta_{a_A}$. Therefore ϑ_{a_A} is a congruence relation on (R, A).

(ii) Suppose $\vartheta_{a_A} = \Delta_{(R,A)}$. Then for any $x \in R$, we have

$$A(a \vee a, a \vee (a \wedge x)) = A(a \vee (a \wedge x), a \vee a) = A(a, a) = 1.$$

So that $(a, a \wedge x) \in \vartheta_{a_A} \Rightarrow (a, a \wedge x) \in \Delta_{(R,A)}$ and hence $a = a \wedge x$. Thus $a \leqslant x$ for all $x \in R$. Hence a is the zero element of R.

Conversely suppose a is the zero element of R. Let $(x, y) \in \vartheta_{a_A}$. Then

$$A(a \vee x, a \vee y) = A(a \vee y, a \vee x) = 1 \Rightarrow A(x, y) = A(y, x) = 1.$$

Hence $\vartheta_{a_A} = \Delta_{(R,A)}$. □

In the following theorem we characterize a subdirectly irreducible associative GADFL.

Theorem 4.2. *Let (R, A) be an associative GADFL. Then (R, A) is subdirectly irreducible if and only if every nonzero element of R is left identity and R contains at most two nonzero elements.*

Proof. Let (R, A) be an associative GADFL. Suppose (R, A) is subdirectly irreducible.

Claim: (i) Every nonzero element of R is left identity

(ii) R contains at most two nonzero elements.

(i) Let ϑ_A be the smallest nonzero congruence relation on (R, A). Choose $x, y \in R$ with $x \neq y$ such that $(x, y \in)\ \vartheta_A$. Assume that x and y both are not left identity elements of R. Then $\varphi^{x_A} \neq \Delta_{(R,A)} \neq \varphi^{x_A}$, so that $(x, y) \in \varphi^{x_A} \cap \varphi^{y_A}$ (since ϑ_A is the smallest nonzero congruence on a subdirect irreducible (R, A)). Hence

$$A(x \wedge x, x \wedge y) = A(x \wedge y, x \wedge x) = 1$$

and

$$A(y \wedge y, y \wedge x) = A(y \wedge x, y \wedge y) = 1 \Rightarrow x = x \wedge y$$

and $y = y \wedge x$. Thus $A(x, y) > 0$ and $A(y, x) > 0$ and hence $x = y$, which is a contradiction. Thus at least one of x, y is a left identity element. Without loss of generality, assume that x is a left identity element. Let a be a nonzero element of R. Suppose a is not left identity element. Now x is left identity element implies that $\mathrm{A}(x \wedge a,\, a) = \mathrm{A}(a,\, x \wedge a) = 1$. As $a \wedge x \wedge a = x \wedge a \wedge a = x \wedge a = a$ so that $a \wedge x$ is a nonzero element of R. Therefore $\vartheta_{(a \wedge x)_A} \neq \Delta_{(R,A)}$ (by theorem 4.1). Hence $(x, y) \in \vartheta_{(a \wedge x)_A}$. Also, since a is not left identity element, we get φ^{a_A} is a nonzero congruence(by lemma 4.1) and hence $(x, y) \in \varphi^{a_A}$. Now

$$\begin{aligned} A(x, y) &= A((a \wedge x) \vee x, y) = A((a \wedge x) \vee x, y) \\ &= A((a \wedge x) \vee y, y) = A((a \wedge y) \vee y, y) = 1. \end{aligned}$$

Similarly $\mathrm{A}(y, x) = 1$. Hence $\mathrm{A}(x, y) = \mathrm{A}(y, x) = 1 \Rightarrow x = y$. Which is a contradiction. Thus a is left identity element.

(ii) Suppose $a, b, c \in R$ be three distinct nonzero elements of R. Then a, b, c are left identity elements. Hence we get $\varphi_A = \Delta_{(R,A)} \cup \{(a, b),\, (b, a)\}$ and $\psi_A = \Delta_{(R,A)} \cup \{(b, c),\, (c, b)\}$ are two nonzero congruences on (R, A) such that $\varphi_A \cap \psi_A = \Delta_{(R,A)}$. This contradicts the fact that (R, A) is subdirectly irreducible. Hence R has at most two nonzero elements. Converse is trivial. □

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