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Comparison between Bayesian and Maximum Likelihood Methods for parameters and the Reliability function of Perks Distribution

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Abstract
In this paper, we have derived Bayesian estimation for the parameters and reliability function of Perks distribution based on two different loss functions, Lindley's approximation has been used to obtain those values. It is assumed that the parameter behaves as a random variable have a Gumbell Type Π prior with non-informative is used. And after the derivation of mathematical formulas of those estimations, the simulation method was used for comparison depending on mean square error (MSE) values and integrated mean absolute percentage error (IMAPE) values respectively. Among of conclusion that have been reached, it is observed that, the LE-NR estimate introduced the best perform for estimating the parameter λ.

Keywords: Perks distribution, Reliability function, Lindley's approximation, Mean square error, Integrated mean absolute percentage error.

مقارنة طريقة بيز مع الامكان الاعظم لمعلمات ودالة المعولية لتوزيع باركس

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الخلاصة
في هذا البحث تم اشتقاق مقدرات بيز لمعلمات ودالة المعولية لتوزيع باركس وبالاعتماد على دالتي خسارة وتم استخدام تقريب ليندلي للحصول على تلك القيم.وتم افتراض ان المعلومات الاولية للمعلمات تخضع لتوزيع كامبل من النوع الثاني. وبعد اشتقاق الصيغ الرياضية لتلك المقدرات تم الاستعانة باسلوب المحاكاة لاجراء المقارنة وبالاعتماد على معيار متوسط مربعات الخطأ ومتوسط النسبة المئوية التكاملية للخطأ المطلق.من بين الاستنتاجات التي تم التوصل اليها لوحظ ان LE-NR قدم افضل اداء لتقدير معلمة λ.

1. Introduction

Perks distribution (PD) has been introduced by perks [1]. The PD plays an important role in actuarial Science include: models for pensioner mortality data [2], parametric mortality projection models [3].

The moments for this distribution do not appear to be available in closed form [4]. The PD has its probability density function as

$$f(x;\beta,\lambda)=\beta\lambda e^{\lambda x}\frac{(1+\beta)}{(1+\beta e^{\lambda x})^2};\beta>0,\lambda>0,x\geq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

where β is the shap parameter and λ is the scale parameter.

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The corresponding cumulative distribution function of the PD is given by
$F(x;\beta,\lambda)=1-\frac{(1+\beta)}{(1+\beta e^{\lambda x})};\beta>0,\lambda>0,x\geq 0$ ……………………………………………………...(2)

The reliability function at time **t** is

$R(t;\beta,\lambda)=\frac{(1+\beta)}{(1+\beta e^{\lambda t})};\beta>0,\lambda>0,t\geq 0$ ……………………………………………………………(3)

and the hazard rate function at time **t** is

$h(t;\beta,\lambda)=\frac{\beta\lambda e^{\lambda t}}{(1+\beta e^{\lambda t})};\beta>0,\lambda>0,t\geq 0$ ……………………………………………(4)

Illustration by drawing, see [5]

2. Maximum Likelihood Estimators

Let $\underline{x}=(x_1,x_2,\dots,x_n)$ be a random sample of size n from PD. The complete –data likelihood function is :

$l(\beta,\lambda|\underline{x})=\prod_{i=1}^{n}f_x(x_i;\ \beta,\lambda)$

$=\prod_{i=1}^{n}\beta\lambda e^{\lambda x_i}\frac{(1+\beta)}{(1+\beta e^{\lambda x_i})^2}$

$=\beta^n\lambda^n e^{\lambda\sum_{i=1}^{n}x_i}\ \frac{(1+\beta)^n}{\prod_{i=1}^{n}(1+\beta e^{\lambda x_i})^2}$ ………………………………………………………………..(5)

$\Rightarrow \ell(\beta,\lambda|\underline{x})=lnl(\beta,\lambda|\underline{x})=n\ln\beta+nln\lambda+\lambda\sum_{i=1}^{n}x_i+nln(1+\beta)-2\sum_{i=1}^{n}ln(1+\beta e^{\lambda x_i})$ .... (6)

Therefore, to obtain the maximum likelihood estimates of β and λ, we find

$\frac{\partial\ell(\beta,\lambda|\underline{x})}{\partial\beta}=\frac{n}{\hat{\beta}}+\frac{n}{1+\hat{\beta}}-2\sum_{i=1}^{n}\frac{e^{\hat{\lambda}x_i}}{1+\beta e^{\hat{\lambda}x_i}}=0$ ………………………………….. (7)

$\frac{\partial\ell(\beta,\lambda|\underline{x})}{\partial\lambda}=\frac{n}{\hat{\lambda}}+\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}\frac{\hat{\beta}x_i e^{\hat{\lambda}x_i}}{1+\hat{\beta}e^{\hat{\lambda}x_i}}=0$ ………………………………….. (8)

mmmmmWe propose here to use Multivariate Newton Raphson (MNR) algorithm to determine the maximum Likelihood estimates of the parameters.

Multivariate Newton- Raphson (MNR) Algorithm

In this algorithm, the solution of the likelihood equation is obtained through an iterative procedure as follows: [6] let ($\beta^{(0)}$,$\lambda^{(0)}$) denote the starting values of β,λ at the (k+1)th step

$$\begin{bmatrix}\hat{\beta}^{(k+1)}\\ \hat{\lambda}^{(k+1)}\end{bmatrix}=\begin{bmatrix}\hat{\beta}^{(k)}\\ \hat{\lambda}^{(k)}\end{bmatrix}-\begin{bmatrix}AA & BB\\ BB & QQ\end{bmatrix}^{-1}_{\substack{\beta=\hat{\beta}^{(k)}\\ \lambda=\hat{\lambda}^{(k)}}}\begin{bmatrix}CC\\ FF\end{bmatrix}_{\substack{\beta=\hat{\beta}^{(k)}\\ \lambda=\hat{\lambda}^{(k)}}} \quad \dots(9)$$

where CC and FF denote the first order derivatives of the natural log- likelihood with respect to the parameter β and λ respectively and $AA, BBand\ QQ$ denote the second –order derivatives of the natural log-likelihood with respect to the parameters are obtained as follows.

$$AA=\frac{\partial^2\ell(\beta,\lambda;\underline{\tilde{x}})}{\partial\beta^2}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}=\frac{-n}{\beta^2}-\frac{n}{(1+\beta)^2}+2\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^2 \quad \dots(10)$$

$$BB=\frac{\partial^2\ell(\beta,\lambda;\underline{\tilde{x}})}{\partial\beta\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}=\frac{\partial^2\ell(\beta,\lambda;\underline{\tilde{x}})}{\partial\lambda\partial\beta}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}=$$

$$-2\sum_{i=1}^{n}\frac{x_i e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}+2\sum_{i=1}^{n}\beta x_i\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^2 \quad \dots(11)$$

$$QQ=\frac{\partial^2\ell(\beta,\lambda;\underline{\tilde{x}})}{\partial\lambda\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}=\frac{-n}{\lambda^2}-2\sum_{i=1}^{n}\frac{\beta x_i^2 e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}+2\sum_{i=1}^{n}\left(\frac{\beta x_i e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^2 \quad \dots(12)$$

The iteration process then continues until $|\beta^{(k+1)}-\beta^{(k)}|<0.0001\ and$
$|\lambda^{(k+1)}-\lambda^{(k)}|<0.0001.$

Now ,depending on the invariant property of MLE, the MLEs of $R(t)$of PD at time t via MNR algorithm can be obtained as,

$\hat{R}_{ML}(t)=\frac{(1+\hat{\beta})}{(1+\hat{\beta}e^{\hat{\lambda}t})};t\geq 0$ ……………………………………...(13)

3. Bayes Estimations

Consider the prior distributions of β and λ of PD are taken to be in dependent Gumbel type II $(1, a)$ and Gumbel type $II(1, b)$ With pdfs.

$$\Pi_1(\beta) = a\left(\frac{1}{\beta}\right)^2 e^{-a/\beta}; \beta > 0, a > 0 \qquad \ldots\ldots(14)$$

and

$$\Pi_2(\lambda) = b\left(\frac{1}{\lambda}\right)^2 e^{-b/\lambda}; \lambda > 0, b > 0 \qquad \ldots\ldots(15)$$

$$\Rightarrow \Pi(\beta,\lambda) = ab\left(\frac{1}{\beta}\right)^2\left(\frac{1}{\lambda}\right)^2 e^{-\left(\frac{a}{\beta}+\frac{b}{\lambda}\right)} \qquad \ldots\ldots(16)$$

The joint posterior density function of β and λ is,

$$h(\beta,\lambda|\underline{x}) = \frac{\Pi(\beta,\lambda|\underline{x})}{\int_\lambda \int_\beta \Pi(\beta,\lambda|\underline{x})\,d\beta\, d\lambda}$$

Where

$$\Pi(\beta,\lambda|\underline{x}) = ab\beta^{n-2}\lambda^{n-2}e^{-\left(\frac{a}{\beta}+\frac{b}{\lambda}\right)}e^{\lambda\sum_{i=1}^n x_i}\frac{(1+\beta)^n}{\prod_{i=1}^n(1+\beta e^{\lambda x_i})^2} \ldots\ldots(17)$$

The squared error loss function (SElF) was proposed by Legendre (1805) and Gauss (1810) in order to develop least square theory. The formula of this loss function for θ is , [7]

$$L(\hat{\theta}_s,\theta) = (\hat{\theta}_s - \theta)^2 \qquad \ldots\ldots(18)$$

where $\hat{\theta}_s$ is an estimate of θ based on SElF.

Now, according to equation (18) Bayes estimator of θ based on SElF is obtained as,

$$\hat{\theta}_{\beta s} = E_h(\theta|\underline{x}) \qquad \ldots\ldots(19)$$

Therefore, Bayes estimator based on SElF represents the mean of the posterior probability density function.

Entropy loss function was first introduced by James and stein for the estimation of dispersion matrix of the multivariate normal distribution. The formula of Entropy loss function (ELF) for θ is, [8]

$$L(\hat{\theta}_e,\theta) = c\left[\frac{\hat{\theta}_e}{\theta} - \ln\left(\frac{\hat{\theta}_e}{\theta}\right) - 1\right], c > 0 \qquad \ldots\ldots(20)$$

Where $\hat{\theta}_e$ is an estimate of θ based an ELF. now, according to equation (20), Bayes estimator of θ based on EIF is obtained as,

$$\hat{\theta}_{Be} = \left[E_h\left(\frac{1}{\theta}|\underline{x}\right)\right]^{-1} \qquad \ldots\ldots(21)$$

where $E_h\left(\frac{1}{\theta}|\underline{x}\right) = \int_0^\infty \frac{1}{\theta}\, h(\theta|\underline{x})\,d\theta$

$h(\theta|\underline{x})$ is the posterior distribution.

According to equation (18) and (20), $\hat{g}_{\beta S}(\beta,\lambda)$ and $\hat{g}_{\beta e}(\beta,\lambda)$ denote Bayes estimation of any function of the parameters, say $g(\beta,\lambda)$, based on SELF and the ELF respectively. where

$$\hat{g}_{\beta s}(\beta,\lambda) = E_h(g(\beta,\lambda)|\underline{x}) = \frac{\int_0^\infty\int_0^\infty g(\beta,\lambda)\Pi(\beta,\lambda|\underline{x})\,d\beta\, d\lambda}{\int_0^\infty\int_0^\infty \Pi(\beta,\lambda|\underline{x})\,d\beta\, d\lambda} \ldots\ldots(22)$$

$$\hat{g}_{\beta e}(\beta,\lambda) = \left(E_h\left(\frac{1}{g(\beta,\lambda)}|\underline{x}\right)\right)^{-1} = \left(\frac{\int_0^\infty\int_0^\infty \frac{1}{g(\beta,\lambda)}\Pi(\beta,\lambda|\underline{x})d\beta d\lambda}{\int_0^\infty\int_0^\infty \Pi(\beta,\lambda|\underline{x})d\beta d\lambda}\right)^{-1} \quad \ldots(23)$$

we can approximate these Bayes estimators by using the Lindley's approximation form to obtain Bayes estimators of the parameters and the reliability function of PD.

Lindley's Approximation

Now, $$E_h(g(\beta,\lambda)|\underline{x}) = \frac{\int_0^\infty\int_0^\infty g(\beta,\lambda)e^{\ell(\beta,\lambda|\underline{x})+q(\beta,\lambda)}d\beta d\lambda}{\int_0^\infty\int_0^\infty e^{\ell(\beta,\lambda|\underline{x})+q(\beta,\lambda)}d\beta d\lambda} \quad \ldots(24)$$

Where

$g\ (\beta,\lambda)$ is a function of β and λ only,

$q\ (\beta,\lambda)$ is natural Log- joint prior density function

Where $q\ (\beta,\lambda) = \ln\Pi(\beta,\lambda)$ ……………………………………………...(25)

Now, $A(\underline{x}) = E(g(\beta,\lambda)|\underline{x})$ let equation (24) can be written as, (see [9])

$$A(\underline{x}) = \hat{g} + \frac{1}{2}[(\hat{g}_{\beta\beta} + 2\hat{g}_\beta\hat{q}_\beta)\hat{\rho}_{\beta\beta} + (\hat{g}_{\beta\lambda} + 2\hat{g}_\beta\hat{q}_\lambda)\hat{\rho}_{\beta\lambda} + (\hat{g}_{\lambda\beta} + 2\hat{g}_\lambda\hat{q}_\beta)\hat{\rho}_{\lambda\beta} + (\hat{g}_{\lambda\lambda} + 2\hat{g}_\lambda\hat{q}_\lambda)\hat{\rho}_{\lambda\lambda}]$$
$$+ \frac{1}{2}[(\hat{g}_\lambda\hat{\rho}_{\lambda\beta} + \hat{g}_\beta\hat{\rho}_{\beta\beta})(\hat{\ell}_{\beta\lambda\lambda}\hat{\rho}_{\lambda\lambda} + \hat{\ell}_{\lambda\beta\beta}\hat{\rho}_{\lambda\beta} + \hat{\ell}_{\beta\lambda\beta}\hat{\rho}_{\beta\lambda} + \hat{\ell}_{\beta\beta\beta}\hat{\rho}_{\beta\beta})$$
$$+ (\hat{g}_\lambda\hat{\rho}_{\lambda\lambda} + \hat{g}_\beta\hat{\rho}_{\lambda\beta})(\hat{\ell}_{\lambda\lambda\lambda}\hat{\rho}_{\lambda\lambda} + \hat{\ell}_{\lambda\beta\lambda}\hat{\rho}_{\lambda\beta} + \hat{\ell}_{\beta\lambda\lambda}\hat{\rho}_{\beta\lambda} + \hat{\ell}_{\beta\beta\lambda}\hat{\rho}_{\beta\beta})] \quad \ldots(26)$$

Where $\hat{\beta}$ and $\hat{\lambda}$ are the MLE's of β and λ respectively.

$$\rho_{\beta\lambda} = \left[\frac{-\partial^2\ell(\beta,\lambda|\underline{x})}{\partial\beta\partial\lambda}\right]^{-1};\ \rho_{\lambda\beta} = \left[\frac{-\partial^2\ell(\beta,\lambda|\underline{x})}{\partial\lambda\partial\beta}\right]^{-1}$$
$$;\ \rho_{\beta\beta} = \left[\frac{-\partial^2\ell(\beta,\lambda|\underline{x})}{\partial\beta^2}\right]^{-1};\ \rho_{\lambda\lambda} = \left[\frac{-\partial^2\ell(\beta,\lambda|\underline{x})}{\partial\lambda^2}\right]^{-1};$$

and, $\hat{g}_\beta = \frac{\partial g}{\partial\beta}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$; $\hat{g}_{\beta\lambda} = \frac{\partial^2 g}{\partial\beta\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$; $\hat{g}_{\beta\beta} = \frac{\partial^2 g}{\partial\beta^2}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$

$\hat{g}_\lambda = \frac{\partial g}{\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$; $\hat{g}_{\lambda\beta} = \frac{\partial^2 g}{\partial\lambda\partial\beta}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$; $\hat{g}_{\beta\lambda} = \frac{\partial^2 g}{\partial\lambda^2}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$;

Differentiating the natural Log- joint prior density function $q\ (\beta,\lambda)$, given by equation (25),partially with respect to $\beta\ and\ \lambda$

$$\hat{q}_\beta = \frac{\partial ln\Pi(\beta,\lambda)}{\partial\beta}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \frac{(-2\hat{\beta}+a)}{\hat{\beta}^2};$$
$$\hat{q}_\lambda = \frac{\partial ln\Pi(\beta,\lambda)}{\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \frac{(-2\hat{\lambda}+b)}{\hat{\lambda}^2}.$$

Now,

$\hat{\ell}_{\beta\beta} = \frac{\partial^2\ell(\beta,\lambda|\underline{x})}{\partial\beta^2}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = AA$ as in equ.(10);

$\hat{\ell}_{\beta\lambda} = \frac{\partial^2\ell(\beta,\lambda|\underline{x})}{\partial\beta\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \frac{\partial^2\ell(\beta,\lambda;\underline{\tilde{x}})}{\partial\lambda\partial\beta}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \hat{\ell}_{\lambda\beta} = BB$ as in equ. (11)

$$\hat{\ell}_{\lambda\lambda} = \frac{\partial^2 \ell(\beta,\lambda|\underline{x})}{\partial \lambda^2}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = QQ \text{ as in equ. (12)}$$

$$\hat{\ell}_{\beta\beta\beta} = \frac{\partial^3 \ell(\beta,\lambda|\underline{x})}{\partial \beta^3}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \frac{2n}{\beta^3} + \frac{2n}{(1+\beta)^3} - 4\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^3$$

$$\hat{\ell}_{\lambda\lambda\lambda} = \frac{\partial^3 \ell(\beta,\lambda|\underline{x})}{\partial \lambda^3}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$$
$$= \frac{2n}{\lambda^3} - 2\sum_{i=1}^{n}\beta {x_i}^3\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right) + 6\sum_{i=1}^{n}\beta^2 {x_i}^3\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^2 - 4\sum_{i=1}^{n}\left(\frac{\beta x_i e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^3$$

$$\hat{\ell}_{\beta\lambda\beta} = \frac{\partial^3 \ell(\beta,\lambda|\underline{x})}{\partial\beta\partial\lambda\partial\beta}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \hat{\ell}_{\lambda\beta\beta} = \frac{\partial^3 \ell(\beta,\lambda|\underline{x})}{\partial\lambda\partial\beta\partial\beta}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \hat{\ell}_{\beta\beta\lambda} = \frac{\partial^3 \ell(\beta,\lambda|\underline{x})}{\partial\beta\partial\beta\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$$
$$= 4\sum_{i=1}^{n} x_i\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^2 - 4\sum_{i=1}^{n}\beta x_i\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^3$$

$$\hat{\ell}_{\lambda\lambda\beta} = \frac{\partial^3 \ell(\beta,\lambda|\underline{x})}{\partial\lambda\partial\lambda\partial\beta}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \hat{\ell}_{\lambda\beta\lambda} = \frac{\partial^3 \ell(\beta,\lambda|\underline{x})}{\partial\lambda\partial\beta\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}} = \hat{\ell}_{\beta\lambda\lambda} = \frac{\partial^3 \ell(\beta,\lambda|\underline{x})}{\partial\beta\partial\lambda\partial\lambda}\Big|_{\substack{\beta=\hat{\beta}\\ \lambda=\hat{\lambda}}}$$
$$= -2\sum_{i=1}^{n} {x_i}^2\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right) + 6\sum_{i=1}^{n}\beta {x_i}^2\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^2 - 4\sum_{i=1}^{n}(\beta x_i)^2\left(\frac{e^{\lambda x_i}}{1+\beta e^{\lambda x_i}}\right)^3$$

3.1Approximate Bayes Estimate of β, λ *and* $R(t)$ based on SELF

Assume that $g\,(\beta,\lambda) = \beta$ in equation (26) and then,

$g\beta = 1, \quad g\beta\beta = g\lambda\lambda = g\lambda = g\beta\lambda = g\lambda\beta = 0.$

$\hat{\beta}_{Bs} = E_h(\beta|\underline{x})$

where $\hat{\beta}_{Bs}$ denote the Bayes estimate of β based on SELF according to Lindley's approximation. Now, assume that $g\,(\beta,\lambda) = \lambda$ in equation (26) and then,

$g\lambda = 1, \quad g\beta\beta = g\lambda\lambda = g\beta = g\lambda\beta = g\beta\lambda = 0.$

$\hat{\lambda}_{Bs} = E_h(\lambda|\underline{x})$

where $\hat{\lambda}_{Bs}$ denote the Bayes estimate of λ based on SELF according to Lindley's approximation and, assume that $g(\beta,\lambda) = R(t) = \left(\frac{1+\beta}{1+\beta e^{\lambda t}}\right)$in equation (26) and then,

$$g_\beta = \frac{1}{1+\beta e^{\lambda t}} - \frac{(1+\beta)e^{\lambda t}}{(1+\beta e^{\lambda t})^2}$$

$$g_{\beta\beta} = \frac{-2e^{\lambda t}}{(1+\beta e^{\lambda t})^2} + \frac{2(1+\beta)e^{2\lambda t}}{(1+\beta e^{\lambda t})^3}$$

$$g_{\beta\lambda} = \frac{-2\beta t e^{\lambda t}}{(1+\beta e^{\lambda t})^2} - \frac{t e^{\lambda t}}{(1+\beta e^{\lambda t})^2} + \frac{2\beta t(1+\beta)e^{2\lambda t}}{(1+\beta e^{\lambda t})^3}$$

$$g_\lambda = -\frac{\beta t(1+\beta)e^{\lambda t}}{(1+\beta e^{\lambda t})^2}$$

$$g_{\lambda\lambda} = -\frac{\beta(1+\beta)t^2 e^{\lambda t}}{(1+\beta e^{\lambda t})^2} + \frac{2(\beta t)^2(1+\beta)e^{2\lambda t}}{(1+\beta e^{\lambda t})^3}$$

$\hat{R}_{Bs}(t) = E_h(R(t)|\underline{x})$

where $\hat{R}_{Bs}(t)$ denote the Bayes estimate of $R(t)$ based on SELF according to Lindley's approximation.

3.2 Approximation Bayes Estimate of β, λ and R(t) based on ELF

Assume that $g(\beta,\lambda)=\frac{1}{\beta}$ in equation (26) and then,

$g_{\beta}=\frac{-1}{\beta^2}, g_{\beta\beta}=\frac{2}{\beta^3}, g_{\lambda}=g_{\lambda\lambda}=g_{\beta\lambda}=g_{\lambda\beta}=0.$

$$\hat{\beta}_{Be}=\left[E_h\left(\frac{1}{\beta}|\underline{x}\right)\right]^{-1}$$

where $\hat{\beta}_{Be}$ denote the Bayes estimate of β based on ELF according to Lindley's approximation.

Now, assume that $g(\beta,\lambda)=\frac{1}{\lambda}$ in equation (26) and then,

$g_{\lambda}=\frac{-1}{\lambda^2}, g_{\beta\beta}=\frac{2}{\lambda^3}, g_{\beta}=g_{\beta\beta}=g_{\beta\lambda}=g_{\lambda\beta}=0.$

$$\hat{\lambda}_{Be}(\beta,\lambda)=\left[E_h\left(\frac{1}{\lambda}|\underline{x}\right)\right]^{-1}$$

where $\hat{\lambda}_{Be}$ denote the Bayes estimate of λ based on ELF according to Lindley's approximation.

and, assume that $g(\beta,\lambda)=\frac{1}{R(t)}=\left(\frac{1+\beta e^{\lambda t}}{1+\beta}\right)$in equation (26)and then,

$$g_{\beta}=\frac{e^{\lambda t}}{1+\beta}-\frac{\left(1+\beta e^{\lambda t}\right)}{(1+\beta)^2}$$

$$g_{\beta\beta}=\frac{-2e^{\lambda t}}{(1+\beta)^2}+\frac{2\left(1+\beta e^{\lambda t}\right)}{(1+\beta)^3}$$

$$g_{\lambda}=\frac{\beta t e^{\lambda t}}{1+\beta}$$

$$g_{\lambda\lambda}=\frac{\beta t^2 e^{\lambda t}}{1+\beta}$$

$$\hat{R}_{Be}(t)=\left[E_h\left(\frac{1}{R(t)}|\underline{x}\right)\right]^{-1}$$

where $\hat{R}_{Be}(t)$denote the Bayes estimate of $R(t)$ based on ELF according to Lindley's approximation.

4. Simulation Study and Results

A Monte Carlo simulation study has been considered to assess the behavior of the obtained estimators for the unknown parameters along with the reliability function of PD. The simulation program has been written by using MATLAB (R2010 b) program.

The general description of the basic four stage of simulation study as follow:

Stage (1): *Choose the sample size (n): n= 15, 60 and 100

*set the true values for the parameters $\beta\, and\ \lambda$ of PD

(β):β=0.5 and 1

(λ) =0.5 and 1.

* Choose the value of hyper- parameters associated with Gumbel typeII prior distributions to be a=b=c=d=0.0001 in order to deal with anon-information.

*Choose four time(t) to assess the estimating reliability function:

t= 1, 2, 3, 4.

*choose the number of sample replicated (L): L= 1000.

Stage (2): Generate a random sample, say $\underline{x}$ of size n distributed as PD through the adoption of inverse transformation method by

$$x=\frac{1}{\lambda}ln\left\{\frac{1}{\beta}\left(\frac{1+\beta}{1-u}-1\right)\right\};0<u<1$$

Where u has the $U(0,1)$ distribution

Stage (3): The initial values required for iterative proceeding algorithm, $\lambda^0=e^1, \beta^0=e^{-3}$.

Stage (4): Repeat the above steps 1000 times and then compare the different estimators for the parameters according to mean squared error and compare different estimators of reliability function with different times according to integrated mean absolute percentage error (IMAPE) as,

$$\mathrm{MSE}(\hat{\beta}) = \frac{\sum_{j=1}^{L}(\hat{\beta}_j - \beta)^2}{L} \qquad \text{.....................(25)}$$

$$\mathrm{MSE}(\hat{\lambda}) = \frac{\sum_{j=1}^{L}(\hat{\lambda}_j - \lambda)^2}{L} \qquad \text{.....................(26)}$$

$$\mathrm{IMAPE}(\hat{R}(t)) = \frac{1}{L}\sum_{j=1}^{L}\left\{\frac{1}{n_t}\sum_{i=1}^{n_t}\left|\frac{[R_j(t_i) - \hat{R}_j(t_i)]}{R_j(t_i)}\right|\right\} \qquad \text{...(27)}$$

Where,

$\hat{\beta}_j, \hat{\lambda}_j$ is the estimate of β *and* λ respectively at the jth run.

L: is the number of sample replicated.

n_t: is the number of times chosen to be (4).

$\hat{R}_j(t_i)$: is the estimates of $R\ (t)$ of the jth run and ith time.

The computational results have been summarized in Tables-(1, 3).

5. Conclusions and Recommendations

From Table-1, approximate Bayes estimate of λ based on ELF via Multivariate Newton- Raphson Algorithm (LE-MNR) introduced the best perform compared with approximate Bayes estimate of λ based on SELF via Multivariate Newton- Raphson Algorithm (LS-NR) for all sample sizes and for all cases.

From Table-2, LS-MNR estimate introduced the best perform compared with LE-MNR according to Lindley´spproximation to estimate β of PD for for all cases and for all sample sizes except for large sample size with $\lambda = \beta = 0.5$.

From Table-3, LE-MNR estimate introduced the best perform compared with LS-MNR according to Lindley´s approximation to estimate $R(t)$ of PD for all cases and for all sample sizes except with $\lambda = \beta = 0.5$ for moderate and large sample sizes.

Based on above conclusions, we recommend

1-choosing LE-MNR to compute Bayes estimates of λ of PD for all sample sizes.

2-choosing LS-MNR to compute Bayes estimates of β of PD especially for small and moderate sample sizes.

3- choosing LE-MNR to compute Bayes estimates of $R(t)$ of PD especially for small sample size.

Table 1-MSE values for Bayes Estimates of λ of PD

N	LS-NR	LE-NR	Best Bayes Estimate
$\boldsymbol{\lambda = 0.5, \beta = 0.5}$			
15	8.7403284	3.5242685	LE-NR
60	3.3423432	2.3470552	LE-NR
100	2.5465965	2.0684431	LE-NR
$\boldsymbol{\lambda = 0.5, \beta = 1}$			
15	5.7705938	3.2152459	LE-NR
60	5.6961760	2.5993199	LE-NR
100	5.3056150	2.4448109	LE-NR
$\boldsymbol{\lambda = 1, \beta = 0.5}$			
15	6.4042988	2.9285690	LE-NR
60	5.4974443	2.8097650	LE-NR
100	5.0957422	2.3290137	LE-NR

Table 2-MSE values for Bayes Estimates of β of PD

n	LS-NR	LE-NR	Best Bayes Estimate
$\boldsymbol{\lambda = 0.5, \beta = 0.5}$			
15	**0.4336745**	**138.9139602**	**LS-NR**
60	**0.4223907**	**1.5252986**	**LS-NR**
100	**0.4221468**	**0.3835587**	**LE-NR**
$\boldsymbol{\lambda = 0.5, \beta = 1}$			
15	**1.7960259**	**412.8577770**	**LS-NR**
60	**1.7534983**	**55.0697165**	**LS-NR**
100	**1.6756107**	**11.4447716**	**LS-NR**
$\boldsymbol{\lambda = 1, \beta = 0.5}$			
15	**0.4665281**	**299.6428458**	**LS-NR**
60	**0.3995523**	**1.9794051**	**LS-NR**
100	**0.3981496**	**0.6421438**	**LS-NR**

Table 3-IMAPE values for Bayes Estimates of $R(t)$ of PD

n	LS-NR	LE-NR	Best Bayes Estimate
$\boldsymbol{\lambda = 0.5, \beta = 0.5}$			
15	**0.6271342**	**0.5056127**	**LE-NR**
60	**0.4492340**	**0.4715720**	**LS-NR**
100	**0.3985375**	**0.4563974**	**LS-NR**
$\boldsymbol{\lambda = 0.5, \beta = 1}$			
15	**0.6025052**	**0.5642312**	**LE-NR**
60	**0.5532955**	**0.4745899**	**LE-NR**
100	**0.5068743**	**0.4718160**	**LE-NR**
$\boldsymbol{\lambda = 1, \beta = 0.5}$			
15	**0.7339687**	**0.5539619**	**LE-NR**
60	**0.6778311**	**0.4868786**	**LE-NR**
100	**0.6776365**	**0.4745774**	**LE-NR**

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