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Some Geometric Properties of Multivalent Functions Defined on Hilbert Space

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Abstract

The main goal of this paper is to study applications of the fractional calculus techniques for a certain subclass of multivalent analytic functions on Hilbert Space. Also, we obtain the coefficient estimates, extreme points, convex combination and hadamard product.

Keywords: Multivalent functions, Fractional calculus, Extreme points, Convex combination, Hilbert Space, Hadamard product.

بعض الخصائص الهندسية للدالة المتعددة التكافؤ المعرفة حول فضاء هلبرت

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الخلاصة

الهدف الرئيسي من هذا البحث هو دراسة تطبيقات التفاضل الكسري لصنف جزئي من الدوال التحليلية المتعددة التكافؤ حول فضاء هلبرت. كذلك، نحن حصلنا على تقديرات المعاملات، النقاط الحرجة، التركيب المحدب وضرب هادمرد (ضرب الالتواء).

1- Introduction:

Let $\text{Ł}(\mathrm{p},\mathrm{m})$ represents a class of functions as below:

$$\mathfrak{f}(z) = z^{p} + \sum_{n=p+m}^{\infty} a_n z^n, \qquad (p, m \in \mathbb{N} = \{1,2,\dots\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disk $\grave{U} = \{z \in \mathbb{C}: |z| < 1\}$.

Let $\mathcal{K}(p,m)$ represents a subclass 0f $\text{Ł}(p,m)$ contains functions of the form:

$$\mathfrak{f}(z) = z^{p} - \sum_{n=p+m}^{\infty} a_n z^n, \qquad (a_n \geq 0\ p, m \in \mathbb{N} = \{1,2,\dots\}), \tag{1.2}$$

A function $\mathfrak{f} \in \text{Ł}(p,m)$ is said to be starlike of order $\delta(0 \leq \delta < p)$ if it satisfies the condition:

$$Re\left\{\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}\right\} > \delta, \qquad (z \in \grave{U}),$$

and is said to be convex of order $\delta(0 \leq \delta < p)$ if it satisfies the condition:

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$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\delta, \qquad (z\in \grave{U}).$$

Denote by $S_m^*(p,\delta)$ and $C_m(p,\delta)$, the classes of Multivalent starlike and convex functions of order δ, respectively, which were introduced and studied by Owa [1]. It is known that (see [2] and [1])

$$f\in C_m(p,\delta) \text{ if and only if } \frac{zf'(z)}{p}\in S_m^*(p,\delta).$$

The classes $S_m^*=S^*(p,\delta)$ and $C_1(p,\delta)=C(p,\delta)$ were studied by Owa [3].

Let Ή be a complex Hilbert Space. Using Ţ as a linear operator on Ή. For a complex analytic f on the unit disk Ù, f(Ţ) is represented as operator know by the usual Riesz-Dunford integral [4]

$$f(T)=\frac{1}{2\pi i}\int_c f(z)(zI-T)^{-1}dz,$$

where I is the identity operator on Ή, c is a positively oriented simple closed rectifiable contour lying in Ù and containing the spectrum α(Ţ) of Ţ in its interior domain [5]. Also f(Ţ) can be defined by the series

$$f(T)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}T^n,$$

which converges in the norm topology [6].

Definition (1.1) [7]**:**

The fractional integral operator 0f order ζ (ζ > 0) is known by

$$Đ_T^{-\zeta}f(T)=\frac{1}{\Gamma(\zeta)}\int_0^1\frac{T^{\zeta}f(tT)}{(1+t)^{1-\zeta}}dt,$$

where f is analytic function in a simple connected region of z - plane containing the origin.

Definition (1.2) [7]**:**

The fractional derivative for operator of order ζ (0 ≤ ζ < 1) is defined by

$$Đ_T^{\zeta}f(T)=\frac{1}{\Gamma(1-\zeta)}\frac{d}{dT}\int_0^1\frac{T^{1-\zeta}f(tT)}{(1-t)^{\zeta}}dt,$$

where f is analytic in a simply connected region of the z - plane containing the origin.

For $f\in K(p,m)$, from Definitions (1.1) and (1.2) by applying a simple calculation, we get

$$Đ_T^{-\zeta}f(T)=\frac{\Gamma(p+1)}{\Gamma(p+\zeta+1)}T^{p+\zeta}-\sum_{n=p+m}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\zeta+1)}a_nT^{n+\zeta} \tag{1.3}$$

and

$$Đ_T^{\zeta}f(T)=\frac{\Gamma(p+1)}{\Gamma(p-\zeta+1)}T^{p-\zeta}-\sum_{n=p+m}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-\zeta+1)}a_nT^{n-\zeta} \tag{1.4}$$

Definition (1.3):

A function $f\in K(p,m)$ is defined in the class $ŁK(p,m,\beta,\sigma,\vartheta,T)$ iff satisfies the inequality:

$$\|T^2f''(T)-p(Tf'(T)-f(T))\|<\vartheta\|(p-\beta\sigma)(Tf'(T)-f(T))+(\beta-1)T^2f''(T)\|, \tag{1.5}$$

where $0<\beta\le 1, 0\le\sigma<\frac{1}{2}, 0<\vartheta\le 1$ and for all operator Ţ with $\|T\|<1$ and $T\neq\emptyset$ (∅ denote the zero operator on Ή).

The operators on Hilbert Space were considered by Xiaopei [8], Joshi [9], Chrakim et al. [10], Ghanim and Darus [11], Selvaraj et al. [7] and Wanas [12].

2- Coefficient Estimates:

In this section, we obtain coefficient estimates for the function f to be in the class $ŁK(p,m,\beta,\sigma,\vartheta,T)$.

Theorem(2.1): Let $f\in K(p,m)$ be defined by (1.2). Then $f\in ŁK(p,m,\beta,\sigma,\vartheta,T)$ for all $T\neq\emptyset$ iff

$$\sum_{n=p+m}^{\infty}(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]a_n\le\beta\vartheta(p-1)(p-\sigma), \tag{2.1}$$

where $0<\beta\le 1, 0\le\sigma<\frac{1}{2}, 0<\vartheta\le 1$.

The result is sharp for the function f given by

$$f(z)=z^p-\frac{\beta\vartheta(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}z^n, n\ge 2, \tag{2.2}$$

Proof: Assume that the inequality (2.1) considered. Then, we have

$$\|Ʈ^2 f''(Ʈ) - ꝑ(Ʈf'(Ʈ) - f(Ʈ))\| - \vartheta\|(ꝑ - \beta\sigma)(Ʈf'(Ʈ) - f(Ʈ)) + (\beta - 1)Ʈ^2 f''(Ʈ)\|$$
$$= \left\| \sum_{n=ꝑ+m}^{\infty} (n-1)(ꝑ-n) a_n Ʈ^n \right\|$$
$$-\vartheta \left\| \beta(ꝑ-1)(ꝑ-\sigma)Ʈ^ꝑ - \sum_{n=ꝑ+m}^{\infty} (n-1)[\beta(n-\sigma) + (ꝑ-n)] a_n Ʈ^n \right\|$$
$$\leq \sum_{n=ꝑ+m}^{\infty} (n-1)(ꝑ-n) a_n \|Ʈ\|^n - \vartheta\beta(ꝑ-1)(ꝑ-\sigma)\|Ʈ\|^ꝑ$$
$$+ \sum_{n=ꝑ+m}^{\infty} \vartheta(n-1)[\beta(n-\sigma) + (ꝑ-n)] a_n \|Ʈ\|^n$$
$$\leq \sum_{n=ꝑ+m}^{\infty} (n-1)[\beta\vartheta(n-\sigma) + (ꝑ-n)(1+\vartheta)] a_n - \vartheta\beta(ꝑ-1)(ꝑ-\sigma) \leq 0.$$

Hence $f \in ŁƘ(ꝑ, m, \beta, \sigma, \vartheta, Ʈ)$.

To show the converse, assume that $f \in ŁƘ(ꝑ, m, \beta, \sigma, \vartheta, Ʈ)$. Therefore

$$\|Ʈ^2 f''(Ʈ) - ꝑ(Ʈf'(Ʈ) - f(Ʈ))\| < \vartheta\|(ꝑ - \beta\sigma)(Ʈf'(Ʈ) - f(Ʈ)) + (\beta - 1)Ʈ^2 f''(Ʈ)\|,$$

gives

$$\left\| \sum_{n=ꝑ+m}^{\infty} (n-1)(ꝑ-n) a_n Ʈ^n \right\|$$
$$< \vartheta\left\|\beta(ꝑ-1)(ꝑ-\sigma)Ʈ^ꝑ - \sum_{n=ꝑ+m}^{\infty}(n-1)[\beta(n-\sigma) + (ꝑ-n)] a_n Ʈ^n\right\|.$$

Setting $Ʈ = r$ $(0 < r < 1)$ in the a above inequality, we get

$$\frac{\sum_{n=ꝑ+m}^{\infty}(n-1)(ꝑ-n) a_n r^n}{\beta(ꝑ-1)(ꝑ-\sigma) r^ꝑ - \sum_{n=ꝑ+m}^{\infty}(n-1)[\beta(n-\sigma)+(ꝑ-n)] a_n r^n} < \vartheta. \tag{2.3}$$

By taking (2.3) with $r \longrightarrow 1^-$, we obtain

$$\sum_{n=ꝑ+m}^{\infty} (n-1)(ꝑ-n) a_n$$
$$< \vartheta\beta(ꝑ-1)(ꝑ-\sigma) - \sum_{n=ꝑ+m}^{\infty} \vartheta(n-1)[\beta(n-\sigma) + (ꝑ-n)] a_n$$

or

$$\sum_{n=ꝑ+m}^{\infty} (n-1)[\beta\vartheta(n-\sigma) + (ꝑ-n)(1+\vartheta)] a_n \leq \beta\vartheta(ꝑ-1)(ꝑ-\sigma),$$

which is the property is proved.

Corollary (2.1): If $f \in ŁƘ(ꝑ, m, \beta, \sigma, \vartheta, Ʈ)$, then

$$a_n \leq \frac{\beta\vartheta(ꝑ-1)(ꝑ-\sigma)}{(n-1)[\beta\vartheta(n-\sigma) + (ꝑ-n)(1+\vartheta)]}, \qquad n \geq 2. \tag{2.4}$$

3- Extreme Points:

We obtain here an extreme points of the class $ŁƘ(ꝑ, m, \beta, \sigma, \vartheta, Ʈ)$.

Theorem (3.1): Let $f_ꝑ(ƶ) = ƶ^ꝑ$ and $f_n(ƶ) = ƶ^ꝑ - \frac{\beta\vartheta(ꝑ-1)(ꝑ-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(ꝑ-n)(1+\vartheta)]} ƶ^n, \ n \geq ꝑ + m.$

Then $f \in ŁƘ(ꝑ, m, \beta, \sigma, \vartheta, Ʈ)$ if and only if can be expressed in the form:

$$f(ƶ) = \tau_ꝑ ƶ^ꝑ + \sum_{n=ꝑ+m}^{\infty} \tau_n f_n(ƶ). \tag{3.1}$$

Where $(\tau_ꝑ \geq 0, \tau_n \geq 0, \ n \geq ꝑ + m)$ and $\tau_ꝑ + \sum_{n=ꝑ+m}^{\infty} \tau_n = 1$.

Proof: Suppose that f is expressed in the form (3.1). Then, we have

$$f(z) = \tau_p z^p + \sum_{n=p+m}^{\infty} \tau_n \left[z^p - \frac{\beta\vartheta(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]} z^n \right]$$
$$= z^p - \sum_{n=p+m}^{\infty} \tau_n \frac{\beta\vartheta(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]} z^n.$$

Hence

$$\sum_{n=p+m}^{\infty} \frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)} \times \tau_n \frac{\beta\vartheta(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}$$
$$= \sum_{n=p+m}^{\infty} \tau_n = 1 - \tau_p \leq 1.$$

Then $f \in ŁΚ(p, m, \beta, \sigma, \vartheta, Ţ)$.
Conversely, suppose that $f \in ŁΚ(p, m, \beta, \sigma, \vartheta, Ţ)$, we may set

$$\tau_n = \frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)} a_{n,}$$

where an is given by (2.4). Then

$$f(z) = z^p - \sum_{n=p+m}^{\infty} a_n z^n = z^p - \sum_{n=p+m}^{\infty} \tau_n \frac{\beta\vartheta(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]} z^n$$
$$= z^p - \sum_{n=p+m}^{\infty}\left(z^p - f_n(z)\right)\tau_n = \left(1 - \sum_{n=p+m}^{\infty} \tau_n\right) z^p + \sum_{n=p+m}^{\infty} \tau_n f_n(z)$$
$$= \tau_p z^p + \sum_{n=p+m}^{\infty} \tau_n f_n(z).$$

This completes the proof of the theorem.

4- Convex Combination:

Theorem (4.1): The class $ŁΚ(p, m, \beta, \sigma, \vartheta, Ţ)$ is closed under convex combinations.

Proof: For $i = 1,2, ...$, let $f_i \in ŁΚ(p, m, \beta, \sigma, \vartheta, Ţ)$, where f_i is given by

$$f_i(z) = z^p - \sum_{n=p+m}^{\infty} a_{n,i} z^n.$$

Then by (2.1), we have

$$\sum_{n=p+m}^{\infty} (n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)] a_{n,i} \leq \beta\vartheta(p-1)(p-\sigma), \quad (4.1)$$

For $\sum_{i=1}^{\infty} \nu_i = 1, 0 \leq \nu_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} \nu_i f_i(z) = z^p - \sum_{n=p+m}^{\infty} \left(\sum_{i=1}^{\infty} \nu_i\, a_{n,i} \right) z^n.$$

Thus, by (4.1), we get

$$\sum_{n=p+m}^{\infty} (n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)] \left(\sum_{i=1}^{\infty} \nu_i\, a_{n,i} \right)$$
$$= \sum_{i=1}^{\infty} \nu_i \left(\sum_{n=p+m}^{\infty} (n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)] a_{n,i} \right)$$
$$\leq \sum_{i=1}^{\infty} \nu_i \left(\beta\vartheta(p-1)(p-\sigma) \right) = \beta\vartheta(p-1)(p-\sigma).$$

Therefore

$$\sum_{i=1}^{\infty} \nu_i f_i(z) \in ŁΚ(p, m, \beta, \sigma, \vartheta, Ţ).$$

Corollary (4.1): The class $ŁΚ(p, m, \beta, \sigma, \vartheta, Ţ)$ is a convex set.

5- Applications of the Fractional Calculus:

Theorem (5.1): If $f \in ŁƘ(p, m, \beta, \sigma, \vartheta, Ţ)$, then

$$\left\| Đ_{Ţ}^{-\zeta} f(Ţ) \right\| \leq \frac{\Gamma(p+1)}{\Gamma(p+\zeta+1)} \|Ţ\|^{p+\zeta} \left[1 + \frac{\Gamma(p+m+1)\Gamma(p+\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m+\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]} \|Ţ\|^{m}\right] \quad (5.1)$$

and

$$\left\| Đ_{Ţ}^{-\zeta} f(Ţ) \right\| \geq \frac{\Gamma(p+1)}{\Gamma(p+\zeta+1)} \|Ţ\|^{p+\zeta} \left[1 - \frac{\Gamma(p+m+1)\Gamma(p+\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m+\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]} \|Ţ\|^{m}\right]. \quad (5.2)$$

For the function f, the result is sharp as follows

$$f(z) = z^{p} - \frac{\beta\vartheta(p-1)(p-\sigma)}{(p+m-1)[\beta\vartheta(p+m-\sigma) - m(1+\vartheta)]} z^{p+m}, (p, m \in \mathbb{N}). \quad (5.3)$$

Proof: Let $f \in ŁƘ(p, m, \beta, \sigma, \vartheta, Ţ)$. By (1.3), we have

$$\frac{\Gamma(p+\zeta+1)}{\Gamma(p+1)} Ţ^{-\zeta} Đ_{Ţ}^{-\zeta} f(Ţ) = Ţ^{p} - \sum_{n=p+m}^{\infty} \frac{\Gamma(n+1)\Gamma(p+\zeta+1)}{\Gamma(n+\zeta+1)\Gamma(p+1)} a_n Ţ^{n}.$$

Setting

$$\Psi(n,\zeta) = \frac{\Gamma(n+1)\Gamma(p+\zeta+1)}{\Gamma(n+\zeta+1)\Gamma(p+1)}, \qquad (n \geq p+m, p, m \in \mathbb{N}),$$

we get

$$\frac{\Gamma(p+\zeta+1)}{\Gamma(p+1)} Ţ^{-\zeta} Đ_{Ţ}^{-\zeta} f(Ţ) = Ţ^{p} - \sum_{n=p+m}^{\infty} \Psi(n,\zeta) a_n Ţ^{n}.$$

Since for $\Psi(n, \zeta)$, Ψ is a decreasing function, then we have

$$0 < \Psi(n,\zeta) \leq \Psi(p+m,\zeta) = \frac{\Gamma(p+m+1)\Gamma(p+\zeta+1)}{\Gamma(p+m+\zeta+1)\Gamma(p+1)}. \quad (5.4)$$

Now, by the application of **Theorem (2.1)** and (5.4), we obtain

$$\left\| \frac{\Gamma(p+\zeta+1)}{\Gamma(p+1)} Ţ^{-\zeta} Đ_{Ţ}^{-\zeta} f(Ţ) \right\| \leq \|Ţ\|^{p} + \sum_{n=p+m}^{\infty} \Psi(n,\zeta) a_n \|Ţ\|^{n}$$

$$\leq \|Ţ\|^{p} + \Psi(p+m,\zeta)\|Ţ\|^{p+m} \sum_{n=p+m}^{\infty} a_n$$

$$\leq \|Ţ\|^{p} + \frac{\Gamma(p+m+1)\Gamma(p+\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m+\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma) - m(1+\vartheta)]} \|Ţ\|^{p+m},$$

which gives (5.1) Similarly, we also have also have

$$\left\| \frac{\Gamma(p+\zeta+1)}{\Gamma(p+1)} Ţ^{-\zeta} Đ_{Ţ}^{-\zeta} f(Ţ) \right\| \geq \|Ţ\|^{p} - \sum_{n=p+m}^{\infty} \Psi(n,\zeta) a_n \|Ţ\|^{n}$$

$$\geq \|Ţ\|^{p} - \Psi(p+m,\zeta)\|Ţ\|^{p+m} \sum_{n=p+m}^{\infty} a_n$$

$$\geq \|Ţ\|^{p} - \frac{\Gamma(p+m+1)\Gamma(p+\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m+\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]} \|Ţ\|^{p+m},$$

which gives (5.2).

By taking $\zeta = 1$ in **Theorem (5.1)**, we obtain the following corollary:

Corollary (5.1): If $f \in ŁƘ(p, m, \beta, \sigma, \vartheta, Ţ)$, then

$$\left\| \int_0^1 Ţf(tŢ)dt \right\| \leq \frac{\|Ţ\|^{p+1}}{p+1} \left[1 + \frac{\beta\vartheta(p^2-1)(p-\sigma)}{((p+m)^2-1)[\beta\vartheta(p+m-\sigma) - m(1+\vartheta)]} \|Ţ\|^{m}\right]$$

and

$$\left\| \int_0^1 Ţf(tŢ)dt \right\| \geq \frac{\|Ţ\|^{p+1}}{p+1} \left[1 - \frac{\beta\vartheta(p^2-1)(p-\sigma)}{((p+m)^2-1)[\beta\vartheta(p+m-\sigma) - m(1+\vartheta)]} \|Ţ\|^{m}\right].$$

Proof: By **Definition (1.1)** and **Theorem (5.1)** for $\zeta = 1$, we have $Đ_{Ţ}^{-\zeta} f(Ţ) = \int_0^1 Ţf(tŢ)dt$, the result is true.

Theorem (5.2): If $f \in \text{ŁK}(p, m, \beta, \sigma, \vartheta, \text{Ţ})$, then

$$\left\|Đ_{Ţ}^{\zeta} f(Ţ)\right\| \leq \frac{\Gamma(p+1)}{\Gamma(p-\zeta+1)} \|Ţ\|^{p-\zeta} \left[1 + \frac{\Gamma(p+m+1)\Gamma(p-\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m-\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]} \|Ţ\|^{m}\right] \quad (5.5)$$

and

$$\left\|Đ_{Ţ}^{\zeta} f(Ţ)\right\| \geq \frac{\Gamma(p+1)}{\Gamma(p-\zeta+1)} \|Ţ\|^{p-\zeta} \left[1 - \frac{\Gamma(p+m+1)\Gamma(p-\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m-\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]} \|Ţ\|^{m}\right]. \quad (5.6)$$

The result is sharp for the function f *given by (5.3).*

Proof: Let $f \in \text{ŁK}(p, m, \beta, \sigma, \vartheta, Ţ)$. By (1.4), we have

$$\frac{\Gamma(p-\zeta+1)}{\Gamma(p+1)} Ţ^{\zeta} Đ_{Ţ}^{\zeta} f(Ţ) = Ţ^{p} - \sum_{n=p+m}^{\infty} \frac{\Gamma(n+1)\Gamma(p-\zeta+1)}{\Gamma(n-\zeta+1)\Gamma(p+1)} a_n Ţ^{n}$$

$$= Ţ^{p} - \sum_{n=p+m}^{\infty} \mathcal{M}(n,\zeta) a_n Ţ^{n},$$

where

$$\mathcal{M}(n,\zeta) = \frac{\Gamma(n+1)\Gamma(p-\zeta+1)}{\Gamma(n-\zeta+1)\Gamma(p+1)}, \qquad (n \geq p+m, p, m \in \mathbb{N}).$$

Since for $n \geq p+m$, $\mathcal{M}$ is a decreasing function, thus we have

$$0 < \mathcal{M}(n,\zeta) \leq \mathcal{M}(p+m,\zeta) = \frac{\Gamma(p+m+1)\Gamma(p-\zeta+1)}{\Gamma(p+m-\zeta+1)\Gamma(p+1)}.$$

Also, by using **Theorem (2.1)**, we get

$$\sum_{n=p+m}^{\infty} a_n \leq \frac{\beta\vartheta(p-1)(p-\sigma)}{(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]}.$$

Thus

$$\left\|\frac{\Gamma(p-\zeta+1)}{\Gamma(p+1)} Ţ^{\zeta} Đ_{Ţ}^{\zeta} f(Ţ)\right\| \leq \|Ţ\|^{p} - \sum_{n=p+m}^{\infty} \Psi(n,\zeta) a_n \|Ţ\|^{n}$$

$$\geq \|Ţ\|^{p} + \mathcal{M}(p+m,\zeta)\|Ţ\|^{p+m} \sum_{n=p+m}^{\infty} a_n$$

$$\leq \|Ţ\|^{p} - \frac{\Gamma(p+m+1)\Gamma(p+\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m+\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]} \|Ţ\|^{p+m}.$$

Then

$$\left\|Đ_{Ţ}^{\zeta} f(Ţ)\right\| \leq \frac{\Gamma(p+1)}{\Gamma(p-\zeta+1)} \|Ţ\|^{p-\zeta} \left[1 + \frac{\Gamma(p+m+1)\Gamma(p-\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m-\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]} \|Ţ\|^{m}\right].$$

and by the same way, we obtain

$$\left\|Đ_{Ţ}^{\zeta} f(Ţ)\right\| \geq \frac{\Gamma(p+1)}{\Gamma(p-\zeta+1)} \|Ţ\|^{p-\zeta} \left[1 - \frac{\Gamma(p+m+1)\Gamma(p-\zeta+1)\beta\vartheta(p-1)(p-\sigma)}{\Gamma(p+1)\Gamma(p+m-\zeta+1)(p+m-1)[\beta\vartheta(p+m-\sigma)-m(1+\vartheta)]} \|Ţ\|^{m}\right].$$

6- Hadamard product

Let the function $f_j(z) (j = 1,2)$ be defined by

$$f_j(z) = z^{p} - \sum_{n=p+m}^{\infty} a_{n,j} z^{n}, \qquad (a_{n,j} \geq 0). \qquad (6.1)$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^{p} - \sum_{n=p+m}^{\infty} a_{n,1} a_{n,2} z^{n} = (f_2 * f_1)(z).$$

Theorem(6.1): Let the function $f_j(z) (j = 1,2)$ be in the class $\text{ŁK}(p, m, \beta, \sigma, \vartheta, Ţ)$. Then $(f_1 * f_2)(z) \in \text{ŁK}(p, m, \eta, \sigma, \vartheta, Ţ)$, where

$$\eta \leq \frac{\beta^2\vartheta(p-1)(p-\sigma)(p-n)(1+\vartheta)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]^2-\beta^2\vartheta^2(p-1)(p-\sigma)(n-\sigma)}.$$

The result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_j(z)=z^p-\frac{\beta\vartheta(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}z^n,(j=1,2). \quad (6.2)$$

Proof: Employing the technique used earlier by Atshan and Buti [13], we need to find the largest η such that

$$\sum_{n=p+m}^{\infty}\frac{(n-1)[\eta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\eta\vartheta(p-1)(p-\sigma)}a_{n,1}a_{n,2}\leq 1.$$

Since $f_j(z)\in ŁK(p,m,\beta,\sigma,\vartheta,T),(j=1,2)$, we readily see that

$$\sum_{n=p+m}^{\infty}\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}a_{n,1}\leq 1,$$

and

$$\sum_{n=p+m}^{\infty}\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}a_{n,2}\leq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{n=p+m}^{\infty}\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}\sqrt{a_{n,1}a_{n,2}}\leq 1. \quad (6.3)$$

Thus it is sufficient to show that

$$\frac{(n-1)[\eta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\eta\vartheta(p-1)(p-\sigma)}a_{n,1}a_{n,2}\leq\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}\sqrt{a_{n,1}a_{n,2}}$$

or equivalently, that

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{\eta[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta[\eta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}.$$

Hence, in the right of inequality (6.3), it is sufficient to prove that

$$\frac{\beta\vartheta(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}\leq\frac{\eta[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta[\eta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}. \quad (6.4)$$

which implies

$$\eta \leq \frac{\beta^2\vartheta(p-1)(p-\sigma)(p-n)(1+\vartheta)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]^2-\beta^2\vartheta^2(p-1)(p-\sigma)(n-\sigma)}.$$

Theorem (6.2): Let the functions $f_j(z),(j=1,2)$ defined by (6.1) be in the class $ŁK(p,m,\beta,\sigma,\vartheta,T)$. Then the function

$$h(z)=z^p+\sum_{n=p+m}^{\infty}\left(a_{n,1}^2+a_{n,2}^2\right)z^n$$

belong to the class $ŁK(p,m,\delta,\sigma,\vartheta,T)$, where

$$\delta=\frac{2\beta^2\vartheta(p-n)(1+\vartheta)(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]^2-2\beta^2\vartheta^2(p-1)(p-\sigma)(n-\sigma)}.$$

The result is sharp for the function $f_j(z)(j=1,2)$ given by

$$f_j(z)=z^p-\frac{\beta\vartheta(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}z^n, \quad (6.5)$$

Proof: By using **Theorem (2.1)**, we obtain

$$\sum_{n=p+m}^{\infty}\left[\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}\right]^2 a_{n,1}^2$$

$$\leq\left[\sum_{n=p+m}^{\infty}\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}a_{n,1}^2\right]^2\leq 1, \quad (6.6)$$

and

$$\sum_{n=p+m}^{\infty}\left[\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}\right]^2 a_{n,2}^2$$

$$\leq\left[\sum_{n=p+m}^{\infty}\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}a_{n,2}^2\right]^2\leq 1. \quad (6.7)$$

It follows from (6.6) and (6.7) that

$$\sum_{n=p+m}^{\infty}\frac{1}{2}\left[\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}\right]^2\left(a_{n,1}^2+a_{n,2}^2\right)\leq 1.$$

Therefore, we need to find the largest δ such that

$$\frac{(n-1)[\delta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\delta\vartheta(p-1)(p-\sigma)}\leq\frac{1}{2}\left[\frac{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]}{\beta\vartheta(p-1)(p-\sigma)}\right]^2.$$

That is

$$\delta\leq\frac{2\beta^2\vartheta(p-n)(1+\vartheta)(p-1)(p-\sigma)}{(n-1)[\beta\vartheta(n-\sigma)+(p-n)(1+\vartheta)]^2-2\beta^2\vartheta^2(p-1)(p-\sigma)(n-\sigma)}.$$

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