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Toeplitz Operators on Harmonic Dirichlet spaces

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Abstract In this paper, we completely characterize (semi-)commutativity of Toeplitz operators with harmonic symbols on harmonic Dirichlet space and harmonic Bergman space.

Keywords Toeplitz, Hilbert, Operators, Dirichlet, Bergman, characterize, holomorphic.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and dA the normalized area measure on $\mathbb{D}$. The Sobolev space S is the completion of the space of smooth function f on $\mathbb{D}$ such that

$$\|f\| = \left\{\left|\int_{\mathbb{D}} f\,dA\right|^2 + \int_{\mathbb{D}}\left(\left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2\right) dA\right\}^{\frac{1}{2}} < \infty.$$

Then S is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f\,dA \int_{\mathbb{D}} \bar{g}\,dA + \int_{\mathbb{D}}\left(\frac{\partial f}{\partial z}\overline{\frac{\partial g}{\partial z}} + \frac{\partial f}{\partial \bar{z}}\overline{\frac{\partial g}{\partial \bar{z}}}\right) dA$$

for $f, g \in S$.

It is well known that the classical Drichlet space D is the closed subspace of S consisting of all holomorphic functions in S, and D is a reproducing function space with reproducing kernel

$$K_z(w) = 1 + \log\frac{1}{1-\bar{z}w} = 1 + \sum_{n=1}^{\infty}\frac{(\bar{z}w)^n}{n} \quad , \quad w, z \in \mathbb{D}.$$

The classical Dirichlet space D has been studied extensively, for more information see, for example, survey paper [8] and [10].

In this paper we consider the harmonic Dirichlet space D_h which consists of all harmonic functions in S. As in the harmonic Bergman space (see [6]), it is easy to verify that

$$D_h = D + \overline{D},$$

where $\overline{D} = \{\bar{f} | f \in D\}$, and D_h is also a reproducing function space with reproducing kernel

$$R_z(w) = K_z(w) + \overline{K_z}(w) - 1 \ , \ w, z \in \mathbb{D} \qquad (1)$$

Recall that a nonnegative measure μ on $\mathbb{D}$ is called a D-Carleson measure if

$$\int_{\mathbb{D}} |f|^2\, d\mu \le C\|f\|^2 \ \ , \ \forall f \in D,$$

for some nonnegative constant C. See [9] for the geometric characterization of Carleson measure.

Let $H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$. Denote

$$M = \left\{u \text{ is harmonic on } \mathbb{D} \,\middle|\, \begin{matrix} u = f + \bar{g} & , f, g \in H^\infty(\mathbb{D}) \\ |f'|^2 dA\, , |g'|^2 dA \text{ are } D\text{-Carleson measure} \end{matrix}\right\}.$$

For $\mu \in M$, define Toeplitz operator T_u on D_h as

$$T_u(\emptyset) = Q(u\emptyset) \ \ , \ \forall \emptyset \in D_h,$$

where Q is the orthogonal projection from S onto D_h, and for any $\varphi \in S$,

$$(Q\varphi)(z) = \langle \varphi, R_z \rangle.$$

A direct verification shows that T_u is bounded for $u \in M$.

Let P be the orthogonal projection from S on to D, then for any $\varphi \in S$,

$$(P\varphi)(z) = \langle \varphi, K_z \rangle$$

and by (1), we have

$$T_u(\emptyset) = Q(u\emptyset) = P(u\emptyset) + \overline{P(\overline{u\emptyset})} - P(u\emptyset)(0) \ , \ \forall \emptyset \in D_h.$$

In this paper we will characterize the condition for $u, v \in M$ such that Toepliz operators T_u and T_v on D_h commute.

The study of commutativity of Toeplitz operators traces back to 60s of last century. In [2], commutativity of Toeplitz operators on the Hardy space was characterized. After that, the harmonic symbols of commuting Toeplitz operators on the Bergman space and on the classical Dirichlet space were studied in [1] and [5], respectively. The corresponding problem in harmonic Bergman space was studied in [3], and under certain noncyclicity hypothesis, this problem was solved in [4].

Inspired by the ideal in [3], in this paper, we give a complete characterization for the commutativity of Toeplitz operators on D_h with symbols in M, which is different from the case in classical Dirichlet space D as shown in [5]. Our main result is

Theorem 1.1 Let $u, v \in M$, then $T_u T_v = T_v T_u$ on D_h if and only if a nontrivial linear combination of u and v is constant on $\mathbb{D}$.

Using a similar method, we also characterize semi-commutativity of Toeplitz operators on D_h with symbols in M.

Theorem 1.2 Let $u, v \in M$, then $T_u T_v = T_{uv}$ on D_h if and only if either u or v is constant.

2. The proof of the main results

In this section, $\langle .,. \rangle_2$ denotes the inner product in $L^2(\mathbb{D}, dA)$, the Hilbert space of square integrable Lebesgue measurable functions of the unit disk $\mathbb{D}$.

Note that

$$\left\{1, w^n, w^{-n} \,\middle|\, w \in \mathbb{D}\ ,\ n = 1,2,3, \dots \right\}$$

is an orthogonal basis of D_h, and for $f \in D_h$,

$$\grave{f} = \frac{\partial f}{\partial z} \ ,\ \int_{\mathbb{D}} f dA = f(0).$$

The proof of main results are based on the following lemmas.

Lemma 2.1 If f is holomorphic in M and

$$f(0) = f'(0) = \cdots = f^{(N-1)}(0) = 0$$

for $N \geq 1$, then for $1 \leq m \leq N$,

$$T_f \overline{w}^m = \overline{T_{\bar{f}} w^m} = P(f\overline{w}^m)$$

Proof. By definition, for $1 \leq m \leq N$,

$$T_f \overline{w}^m = Q(f\overline{w}^m) = P(f\overline{w}^m) + \overline{P(\bar{f} w^m)} - P(f\overline{w}^m)(0)$$

and

$$P(\bar{f} w^m)(z) = \langle \bar{f} w^m, K_z \rangle = \int_{\mathbb{D}} \bar{f} w^m dA \int_{\mathbb{D}} \overline{K}_z dA + m \int_{\mathbb{D}} \bar{f} w^{m-1} \overline{K}_z' dA$$

$$= \langle w^m, f \rangle_2 + m \langle w^{m-1}, f K_z' \rangle_2 = \langle w^m, f \rangle_2,$$

since $f(0) = f'(0) = \cdots = f^{(N-1)}(0) = 0$ and $m - 1 < N$, $\langle w^{m-1}, f K_z' \rangle_2 = 0$.

But

$$P(f\overline{w}^m)(0) = \langle f\overline{w}^m, 1 \rangle = \int_{\mathbb{D}} f\overline{w}^m dA = \langle f, w^m \rangle_2.$$

Hence

$$T_f \overline{w}^m = P(f\overline{w}^m).$$

Since $R_z = \overline{R}_z$, so

$$\overline{(T_{\bar{f}} w^m)}(z) = \overline{\langle \bar{f} w^m, R_z \rangle} = \langle f\overline{w}^m, R_z \rangle = \left(T_f \overline{w}^m\right)(z).$$

Now we compute $P(f\overline{w}^m)$.

If $f(z) = \sum_{n=N}^{\infty} a_n z^n \ (N \geq 1), 1 \leq m \leq N$, then

$$P(f\overline{w}^m)(z) = \langle f\overline{w}^m, K_z \rangle = \int_{\mathbb{D}} f\overline{w}^m dA \int_{\mathbb{D}} \overline{K_z} dA + \int \grave{f} \overline{w}^m \overline{K}_z' dA$$

$$= \langle f, w^m \rangle_2 + \langle f', w^m K_z' \rangle_2 \qquad = \langle \textstyle\sum_{n=N}^{\infty} a_n w^n, w^m \rangle_2 + \langle \sum_{n=N}^{\infty} n a_n w^{n-1}, w^m \sum_{n=1}^{\infty} \bar{z}^n w^{n-1} \rangle_2$$

$$= \langle \textstyle\sum_{n=N}^{\infty} a_n w^n, w^m \rangle_2 + \langle \sum_{n=N}^{\infty} n a_n w^{n-1}, \sum_{n=m+1}^{\infty} \bar{z}^{n-m} w^{n-1} \rangle_2.$$

So for $m = N$,

$$P(f\overline{w}^m)(z) = \frac{aN}{N+1} + \sum_{n=1}^{\infty} a_{n+N} z^n, \quad (2)$$

and for $1 \le m \le N-1$, $N \ge 2$,

$$P(f\overline{w}^m)(z) = \sum_{n=N}^{\infty} a_n z^{n-m}. \quad (2')$$

Hence, we obtain

$$z^m P(f\overline{w}^m)(z) = f(z) - \frac{N}{N+1} a_N z^N, \quad if \ \ m = N \quad (3)$$

and

$$z^m P(f\overline{w}^m)(z) = f(z) \ \ if \ \ 1 \le m \le N-1, N \ge 2. \quad (3')$$

The next lemma has been presented in [7] with a different form. But for the completeness, we include its proof here.

Lemma 2.2. If f is holomorphic in $M, \varphi \in S$, then

$$P\left(\bar{f}P(\varphi)\right) - P\left(\bar{f}P(\varphi)\right)(0) = P(\bar{f}\varphi) - P(\bar{f}\varphi)(0).$$

Proof. By definition,

$$P\left(\bar{f}P(\varphi)\right)(z) - P\left(\bar{f}P(\varphi)\right)(0) = \langle \bar{f}P(\varphi), K_z\rangle - \langle \bar{f}P(\varphi), 1\rangle$$

$$= \int_{\mathbb{D}} \bar{f}(w) \frac{\partial P(\varphi)}{\partial w}(w) \overline{\frac{\partial K_z}{\partial w}}(w) dA(w) \ = \langle \frac{\partial P(\varphi)}{\partial w}, f\frac{\partial K_z}{\partial w}\rangle_2 \quad (4)$$

and

$$P(\bar{f}\varphi)(z) - P(\bar{f}\varphi)(0) = \langle \bar{f}\varphi, K_z\rangle - \langle \bar{f}\varphi - 1\rangle$$

$$= \int_{\mathbb{D}} \bar{f}(w) \frac{\partial \varphi}{\partial w}(w) \overline{\frac{\partial K_z}{\partial w}}(w) dA(w) = \langle \frac{\partial \varphi}{\partial w}, f\frac{\partial K_2}{\partial w}\rangle_2. \quad (5)$$

Since

$$P(\varphi)(w) = \langle \varphi, K_w\rangle = \int_{\mathbb{D}} \varphi dA \int_{\mathbb{D}} \overline{K_w}\, dA + \int_{\mathbb{D}} \frac{\partial \varphi}{\partial t}\frac{\partial \overline{K_w}}{\partial t} dA(t),$$

we have

$$\frac{\partial P(\varphi)}{\partial w}(w) = \int_{\mathbb{D}} \frac{\partial \varphi}{\partial t} \frac{\partial^2 \overline{K_w}}{\partial \overline{w} \partial t} dA(t).$$

It is well known that $\frac{\partial^2 K_w}{\partial \overline{w} \partial t}(t)$ is the Bergman kernel $L_w(t) = \frac{1}{(1-\overline{w}t)^2}$.

Hence $\frac{\partial P(\varphi)}{\partial w}(w) = \langle \frac{\partial \varphi}{\partial t}, L_w\rangle_2$, which implies that

$$\langle \frac{\partial P(\varphi)}{\partial w}, f\frac{\partial K_z}{\partial w}\rangle_2 = \langle \langle \frac{\partial \varphi}{\partial t}, L_w\rangle_2, f\frac{\partial K_z}{\partial w}\rangle_2 = \langle \frac{\partial \varphi}{\partial t}, f\frac{\partial K_z}{\partial t}\rangle_2. \quad (6)$$

The conclusion follows from Eqs. (4)-(6).

The following theorem gives a necessary condition for two Toeplitz operators to be commuting.

Theorem 2.3. Let $u = f + \bar{g}, v = h + \bar{k}$ in M with f, g, h, k holomporphic such that $T_u T_v = T_v T_u$ on D_h.

(i) If both f and h are not constant, then $h = \alpha f + \gamma$ for some constants α, γ with $\alpha \ne 0$

(ii) If both g and k are not constant, then $g = \beta k + \delta$ for some constants β, δ with $\beta \ne 0$.

Proof. Without loss of generality, assume

$$f(0) = g(0) = h(0) = k(0) = 0.$$

(i) Suppose that both f and h are not constant.

Claim. For any integer $N \ge 2$, if $f'(0) = \cdots = f^{(N-1)}(0) = 0$, then

$$h'(0) = \cdots = h^{(N-1)}(0) = 0.$$

Let $f(z) = \sum_{n=N}^{\infty} a_n z^n$ and $h(z) = \sum_{n=M}^{\infty} b_n z^n$, where $M \ge 1$.

If $1 \le M \le N-1$, then since

$$h(0) = h'(0) = \cdots = h^{(M-1)}(0) = 0,$$

$$f(0) = f'(0) = \cdots = f^{(M-1)}(0) = 0,$$

by Lemma 2.1

$$T_h(\overline{w}^M) = P(h\overline{w}^M), \quad T_f(\overline{w}^M) = P(f\overline{w}^M).$$

Hence

$$T_f T_h(\overline{w}^M) = T_f\left(P(h\overline{w}^M)\right) = fP(h\overline{w}^M)$$

$$T_{\bar{g}} T_h(\overline{w}^M) = T_{\bar{g}}\left(P(h\overline{w}^M)\right) = P\left(\bar{g}P(h\overline{w}^M)\right) + \overline{P\left(g\overline{P(h\overline{w}^M)}\right)} - P\left(\bar{g}P(h\overline{w}^M)\right)(0).$$

A straightforward computation shows that

$$T_f T_{\bar{k}}(\overline{w}^M) = T_f\overline{(kw^M)} = P\left(f\overline{kw^M}\right) + \overline{P\left(\bar{f}kw^M\right)} - P\left(f\overline{kw^M}\right)(0),$$

$$T_{\bar{g}} T_{\bar{k}}(\overline{w}^M) = \overline{gkw^M}.$$

It follows that

$$T_uT_v(\overline{w}^M) = T_fT_h(\overline{w}^M) + T_fT_{\overline{k}}(\overline{w}^M) + T_{\overline{g}}T_h(\overline{w}^M) + T_{\overline{g}}T_{\overline{k}}(\overline{w}^M)$$
$$= fP(h\overline{w}^M) + P\big(f\overline{kw^M}\big) + \overline{P\big(\overline{f}kw^M\big)} - P\big(f\overline{kw^M}\big)(0)$$
$$+P\big(\overline{g}P(h\overline{w}^M)\big) + \overline{P\big(g\overline{P(h\overline{w}^M)}\big)} - P\big(\overline{g}P(h\overline{w}^M)\big)(0) + \overline{gkw^M}.$$

Similarly,

$$T_vT_u(\overline{w}^M) = T_hT_f(\overline{w}^M) + T_hT_{\overline{g}}(\overline{w}^M) + T_{\overline{k}}T_f(\overline{w}^M) + T_{\overline{k}}T_{\overline{g}}(\overline{w}^M)$$
$$= hP(f\overline{w}^M) + P\big(h\overline{gw^M}\big) + \overline{P\big(\overline{h}gw^M\big)} - P\big(h\overline{gw^M}\big)(0)$$
$$+P\left(\overline{k}P(f\overline{w}^M)\right) + \overline{P\big(k\overline{P(f\overline{w}^M)}\big)} - P\left(\overline{k}P(f\overline{w}^M)\right)(0) + \overline{kgw^M}.$$

By Lemma 2.2., we have

$$P\big(f\overline{kw^M}\big) - P\big(f\overline{kw^M}\big)(0) = P\left(\overline{k}P(f\overline{w}^M)\right) - P\left(\overline{k}P(f\overline{w}^M)\right)(0).$$
$$P\big(h\overline{gw^M}\big) - P\big(h\overline{gw^M}\big)(0) = P\big(\overline{g}P(h\overline{w}^M)\big) - P\big(\overline{g}P(h\overline{w}^M)\big)(0).$$

Since $T_uT_v = T_vT_u$ and $f(0) = h(0) = 0$, by taking the holomorphic part on both sides of $T_uT_v\overline{w}^M$ and $T_vT_u\overline{w}^M$, we have

$$fP(h\overline{w}^M) = hP(f\overline{w}^M).$$

By (3) and (3′),

$$z^Mf(z)P(h\overline{w}^M)(z) = f(z)\left(h(z) - \frac{M}{M+1}b_Mz^M\right),$$
$$z^Mh(z)P(f\overline{w}^M)(z) = h(z)f(z).$$

It follows that $b_M = 0$, completing the proof of the claim.

By the above reasoning, if $f'(0) = \cdots = f^{(N-1)}(0) = 0$, $f^{(N)}(0) \neq 0$, then

$$h'(0) = \cdots = h^{(N-1)}(0) = 0$$

and

$$fP(h\overline{w}^N) = hP(f\overline{w}^N).$$

Again by (3) and (3′), we have

$$z^Nf(z)P(h\overline{w}^N)(z) = f(z)\left(h(z) - \frac{N}{N+1}b_Nz^N\right),$$
$$z^Nh(z)P(f\overline{w}^N)(z) = h(z)\left(f(z) - \frac{N}{N+1}a_Nz^N\right).$$

Hence

$$b_Nz^Nf(z) = a_Nz^Nh(z).$$

It follows that $b_N \neq 0$. Let $\alpha = \frac{b_N}{a_N}$, then $\alpha \neq 0$ and $h(z) = \alpha f(z)$.

(ii) If both g and k are not constant, by the symmetry of the holomorphic part and the anti-holomorphic part of u, v and functions in D_h, we also have that there exist nonzero β such that $k = \beta$g.

For the proof of the main result, the following lemma is needed.

Lemma 2.4. If f is holomorphic in M,

$$f(0) = f'(0) = \cdots = f^{(N-1)}(0) = 0$$

for $N \geq 1$ and $f(z) = \sum_{n=N}^{\infty} a_nz^n$, then

(i) for $m = N$,

$$\left(T_f^*w^m\right)(z) = N\overline{a}_N + N\sum_{n=1}^{\infty}\frac{\overline{a}_{n+N}}{n}\overline{z}^n,$$

for $1 \leq m \leq N-1$, $N \geq 2$,

$$\left(T_f^*w^m\right)(z) = m\sum_{n=N}^{\infty}\frac{\overline{a}_n}{n-m}\overline{z}^{n-m}$$

and

$$\overline{\left(T_{\overline{f}}^*\overline{w}^m\right)}(z) = \left(T_f^*w^m\right)(z);$$

(ii) for $m \geq 1$, $\left(T_{\overline{f}}^*w^m\right)(z) = m\sum_{n=N+m}^{\infty}\frac{a_{n-m}}{n}z^n$.

Proof: (i) A straightforward computation shows that

$$\langle w^m, f\overline{K}_z\rangle = \int_{\mathbb{D}} w^m dA \int_{\mathbb{D}} \overline{f\overline{K}_z} dA + \int_{\mathbb{D}} mw^{m-1}\overline{f'\overline{K}_z}dA$$
$$= m\int_{\mathbb{D}} w^{m-1}K_z\overline{f'}\, dA = m\langle w^{m-1}K_z, f'\rangle_2$$
$$= m\left\langle w^{m-1}\left(\sum_{n=1}^{\infty}\frac{\overline{z}^n}{n}w^n + 1\right), \sum_{n=N}^{\infty} na_n\, w^{n-1}\right\rangle_2$$
$$= m\left\langle\sum_{n=m+1}^{\infty}\frac{\overline{z}^{n-m}}{n-m}w^{n-1}, \sum_{n=N}^{\infty} na_n\, w^{n-1}\right\rangle_2 + m\left\langle w^{m-1}, \sum_{n=N}^{\infty} na_n\, w^{n-1}\right\rangle_2.$$

So for $m = N$,

$$\langle w^m, f\overline{K}_z\rangle = N\overline{a}_N + N\sum_{n=1}^{\infty}\frac{\overline{a}_{n+N}}{n}\overline{z}^n,$$

and for $1 \leq m \leq N-1, N \geq 2$,

$$\langle w^m, f\overline{K}_z\rangle = m\sum_{n=N}^{\infty}\frac{\bar{a}_n}{n-m}\bar{z}^{n-m}.$$

Also, it is easy to verify that for $m = N$,

$$\langle w^m, fK_z\rangle = N\bar{a}_N, \quad \langle w^m, f\rangle = N\bar{a}_N,$$

and for $1 \leq m \leq N-1, N \geq 2$,

$$\langle w^m, fK_z\rangle = 0, \quad \langle w^m, f\rangle = 0.$$

Since

$$(T_f^* w^m)(z) = \langle T_f^* w^m, R_z\rangle = \langle w^m, fR_z\rangle = \langle w^m, fK_z\rangle + \langle w^m, f\overline{K}_z\rangle - \langle w^m, f\rangle$$

we have

$$(T_f^* w^m)(z) = N\bar{a}_N + N\sum_{n=1}^{\infty}\frac{\bar{a}_{n+N}}{n}\bar{z}^n$$

if $m = N$, and

$$(T_f^* w^m)(z) = m\sum_{n=N}^{\infty}\frac{\bar{a}_n}{n-m}\bar{z}^{n-m}$$

if $1 \leq m \leq N-1, N \geq 2$.

By definition,

$$(T_f^* \overline{w}^m)(z) = \langle T_f^* \overline{w}^m, R_z\rangle = \langle \overline{w}^m, \bar{f}R_z\rangle$$
$$= \overline{\langle w^m, fR_z\rangle} = \overline{\langle T_f^* w^m, R_z\rangle} = \overline{T_f^* w^m}(z).$$

(ii) For $m \geq 1$,

$$(T_{\bar{f}}^* w^m)(z) = \langle T_{\bar{f}}^* w^m, R_z\rangle = \langle w^m, \bar{f}R_z\rangle = \langle w^m, \bar{f}K_z\rangle + \langle w^m, \overline{fK_z}\rangle - \langle w^m, \bar{f}\rangle.$$

Since

$$\langle w^m, \overline{fK_z}\rangle = 0,$$
$$\langle w^m, \bar{f}\rangle = 0$$

and

$$\langle w^m, \bar{f}K_z\rangle = \int_{\mathbb{D}} w^m dA\int_{\mathbb{D}}\overline{\bar{f}K_z}dA + \int_{\mathbb{D}} mw^{m-1}\overline{\bar{f}K_z'}dA$$
$$= m\int_{\mathbb{D}} w^{m-1}f\overline{K_z'}dA = m\langle w^{m-1}f, K_z'\rangle_2 = m\,\langle w^{m-1}\sum_{n=N}^{\infty}a_n w^n, \sum_{n=1}^{\infty}\bar{z}^n\, w^{n-1}\rangle_2$$
$$= m\langle\sum_{n=N+m}^{\infty}a_{n-m}w^{n-1}, \sum_{n=1}^{\infty}\bar{z}^n\, w^{n-1}\rangle_2 = m\sum_{n=N+m}^{\infty}\frac{a_{n-m}}{n}z^n$$

we get

$$(T_{\bar{f}}^* w^m)(z) = m\sum_{n=N+m}^{\infty}\frac{a_{n-m}}{n}z^n.$$

The following theorem is a key step in the proof of Theorem 1.1

Theorem 2.5. Let f, g be holomorphic functions in M and g is not constant. If $T_f T_{\bar{g}} = T_{\bar{g}} T_f$ on D_h, then f is constant.

Proof. Without loss of generality, assume $f(0) = g(0) = 0$. We will prove that if $T_f T_{\bar{g}} = T_{\bar{g}} T_f$ on D_h with $g \neq 0$, then $f = 0$.

Let $g(z) = \sum_{n=N}^{\infty} b_n z^n$ with $b_N \neq 0, N \geq 1$, and $f(z) = \sum_{n=M}^{\infty} a_n z^n$, $M \geq 1$.

By Lemma 2.1 and Eq. (2),

$$(T_{\bar{g}} w^N)(z) = \frac{\bar{b}_N}{N+1} + \sum_{n=1}^{\infty}\bar{b}_{n+N}\bar{z}^n.$$

By Lemma 2.4,

$$(T_f^* w^M)(z) = M\bar{a}_M + M\sum_{n=1}^{\infty}\frac{\bar{a}_{n+M}}{n}\bar{z}^n,$$
$$(T_{\bar{g}}^* w^M)(z) = M\sum_{n=N+M}^{\infty}\frac{b_{n-M}}{n}z^n.$$

Hence

$$\langle T_f T_{\bar{g}} w^N, w^M\rangle = \langle T_{\bar{g}} w^N, T_f^* w^M\rangle = \langle\frac{\bar{b}_N}{N+1} + \sum_{n=1}^{\infty}\bar{b}_{n+N}\bar{z}^n, M\bar{a}_M + M\sum_{n=1}^{\infty}\frac{\bar{a}_{n+M}}{n}\bar{z}^n\rangle$$
$$= \frac{M}{N+1}\bar{b}_N a_M + M\sum_{n=1}^{\infty}\bar{b}_{n+N}a_{n+M} = \frac{M}{N+1}\bar{b}_N a_M + M\sum_{n=M+1}^{\infty}a_n\bar{b}_{n+N-M},$$
$$= \langle T_{\bar{g}} T_f w^N, w^M\rangle = \langle T_f w^N, T_{\bar{g}}^* w^M\rangle = \langle\sum_{n=M}^{\infty}a_n z^{n+N}, M\sum_{n=N+M}^{\infty}\frac{b_{n-M}}{n}z^n\rangle$$
$$= \langle\sum_{n=N+M}^{\infty}a_{n-N}z^n, M\sum_{n=N+M}^{\infty}\frac{b_{n-M}}{n}z^n\rangle = M\sum_{n=N+M}^{\infty}a_{n-N}\bar{b}_{n-M} = M\sum_{n=M}^{\infty}a_n\bar{b}_{n+N-M}.$$

Since $T_f T_{\bar{g}} = T_{\bar{g}} T_f$, we obtain

$$\frac{M}{N+1}\bar{b}_N a_M = M a_M \bar{b}_N.$$

It follows from $b_N \neq 0$ that $a_M = 0$.

By induction on M, we have $f = 0$.

Now we give the prove of Theorem 1.1.

Proof of Theorem 1.1. Since the "if" part is easy to verify, we only give the proof of the "only if" part.

Without loss of generality, assume both u and v are not constants.

Let $u = f + \bar{g}$, $v = h + \bar{k}$ with f, g, h, k holomorphic.

If at least one of f, g, h, k is constant, without loss of generality, assume f is constant, then g is not constant. It follows from $T_u T_v = T_v T_u$ that

$$T_{\bar{g}} T_{h+\bar{k}} = T_{h+\bar{k}} T_{\bar{g}} \qquad (7)$$

If k is constant, then $T_{\bar{g}} T_h = T_h T_{\bar{g}}$ by Theorem 2.5, h is constant, which contradicts to the assumption that v is not constant.

If k is not constant, then by Theorem 2.3, $k = \beta g + \delta$ with $\beta \neq 0$. By (7), we have $T_{\bar{g}} T_h = T_h T_{\bar{g}}$ and, by Theorem 2.5, h is constant, and thus a nontrivial linear combination of u and v is constant on $\mathbb{D}$.

Otherwise, none of f, g, h and k is constant, then by Theorem 2.3,

$$h = \alpha f + \gamma \quad , \quad k = \beta g + \delta$$

for some constant $\alpha, \gamma, \beta, \delta$ with $\alpha \neq 0$ and $\beta \neq 0$.

It follows from $T_u T_v = T_v T_u$ that

$$(\alpha - \bar{\beta}) T_f T_{\bar{g}} = (\alpha - \bar{\beta}) T_{\bar{g}} T_f$$

By Theorem 2.5, we must have $\alpha = \bar{\beta}$. Thus $v = \alpha u + \gamma + \bar{\delta}$. The proof is completed.

By Theorem 1.1, we have the following results. The corresponding problem in harmonic Bergman space has been described in [3].

Corollary 2.6. Let $u \in M$, then $T_u T_{\bar{u}} = T_{\bar{u}} T_u$ on D_h if and only if $u(\mathbb{D})$ is contained in a straight line.

Proof. It is enough to prove the necessity.

Let $u = f + \bar{g}$ with f, g holomorphic. If u is not constant, then by Theorem 2.5, both f and g are not constant. Assume $f(0) = g(0) = 0$.

By Theorem 1.1, there exist nonzero α such that $g = \alpha f$.

By $T_u T_{\bar{u}} = T_{\bar{u}} T_u$, we have $(1 - |\alpha|^2) T_f T_{\bar{f}} = (1 - |\alpha|^2) T_{\bar{f}} T_f$. Since f is not constant, we must have $|\alpha| = 1$, which implies that $u(\mathbb{D})$ is contained in a straight line.

Usually in the harmonic Dirichlet space D_h, $T_u^* \neq T_{\bar{u}}$ for $u \in M$, which will be showed in Theorem 2.8. So it is necessary to describe the normal Toeplitz operators with symbols in M.

Lemma 2.7. If f, g are holomorphic in M, then

$$(T_f^* 1)(z) = \langle K_z, f \rangle_2 \ , \ (T_{\bar{g}}^* 1)(z) = \langle g, K_z \rangle_2.$$

Proof. It easy to verify by definition.

Now we compute $T_f^* 1$ and $T_{\bar{g}}^* 1$ for the use in the following.

Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$(T_f^* 1)(z) = \langle 1 + \sum_{n=1}^{\infty} \frac{\bar{z}^n}{n} w^n, a_0 + \sum_{n=1}^{\infty} a_n w^n \rangle_2 = \bar{a}_0 + \sum_{n=1}^{\infty} \frac{\bar{a}_n}{n(n+1)} \bar{z}^n, \qquad (8)$$

$$(T_{\bar{g}}^* 1)(z) = \langle b_0 + \sum_{n=1}^{\infty} b_n w^n, 1 + \sum_{n=1}^{\infty} \frac{\bar{z}^n}{n} w^n \rangle_2 = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n(n+1)} z^n. \qquad (9)$$

Theorem 2.8. Let $u = f + \bar{g}$ in M with f, g holomorphic, then $T_u^* = T_{\bar{u}}$ on D_h if and only if u is constant.

Proof. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then by (8), (9).

$$(T_u^* 1)(z) = (T_f^* 1 + T_{\bar{g}}^* 1)(z) = \bar{a}_0 + \sum_{n=1}^{\infty} \frac{\bar{a}_n}{n(n+1)} \bar{z}^n + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n(n+1)} z^n.$$

On the other hand,

$$(T_{\bar{u}} 1)(z) = (T_{\bar{f}} 1 + T_g 1)(z) = \bar{a}_0 + \sum_{n=1}^{\infty} \bar{a}_n \bar{z}^n + b_0 + \sum_{n=1}^{\infty} b_n z^n.$$

Comparing the coefficients on both sides of $T_u^* 1$ and $T_{\bar{u}} 1$, for $n \geq 1$, we have,

$$\frac{a_n}{n(n+1)} = a_n \quad , \quad \frac{b_n}{n(n+1)} = b_n$$

Hence $a_n = b_n = 0$ for $n \geq 1$, and it follows that u is constant.

The sufficiency is obvious.

Theorem 2.9. Let $u = M$, then $T_u T_u^* = T_u^* T_u$ on D_h if and only if u is constant.

Proof: It suffices to prove the necessity

Let $u = f + \bar{g}$ with f, g holomorphic. Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then by (8), (9),

$$\langle T_u T_u^* 1, 1 \rangle = \langle T_{f+\bar{g}} T_{f+\bar{g}}^* 1, 1 \rangle = \langle T_f^* 1 + T_{\bar{g}}^* 1, T_f^* 1 + T_{\bar{g}}^* 1 \rangle$$

$$= |\bar{a}_0 + b_0|^2 + \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{a_n}{n+1} \right|^2 + \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{b_n}{n+1} \right|^2.$$

On the other hand,

$$\langle T_u^* T_u 1,1\rangle = \langle T_{f+\bar{g}}^* T_{f+\bar{g}} 1,1\rangle = \langle f+\bar{g}, f+\bar{g}\rangle.$$
$$= |\bar{a}_0 + b_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2 + \sum_{n=1}^{\infty} n|b_n|^2$$

Therefore

$$\sum_{n=1}^{\infty}\left(n - \frac{1}{n(n+1)^2}\right)|a_n|^2 + \sum_{n=1}^{\infty}\left(n - \frac{1}{n(n+1)^2}\right)|b_n|^2 = 0,$$

which implies $a_n = b_n = 0$ for $n \geq 1$, and thus u is constant.

In the following, we characterize semi-commuting Toeplitz operators on D_h with harmonic symbols in M.

Theorem 2.10. Let $u = f + \bar{g}$ and $v = h + \bar{k}$ in M with f, g, h, k holomorphic. If $T_u T_v = T_{uv}$ on D_h, then

(i) either f or h is constant,

(ii) either g or k is constant.

Proof. Without loss of generality, assume both u and v are not constant and $f(0) = g(0) = h(0) = k(0) = 0$.

(i) Let $f(z) = \sum_{n=M}^{\infty} a_n z^n$ and $h(z) = \sum_{n=N}^{\infty} b_n z^n$ with $M, N \geq 1$, then we can write $(fh)(z) = \sum_{n=N+M}^{\infty} C_n z^n$.

Suppose f is not constant.

By Lemma 2.1. $T_h \bar{w}^N = P(h\bar{w}^N)$ and $T_{fh}\bar{w}^N = P(fh\bar{w}^N)$, so

$$T_u T_v \bar{w}^N = T_f T_h \bar{w}^N + T_{\bar{g}} T_h \bar{w}^N + T_f T_{\bar{k}} \bar{w}^N + T_{\bar{g}} T_{\bar{k}} \bar{w}^N$$
$$= fP(h\bar{w}^N) + Q(\bar{g}P(h\bar{w}^N)) + Q(f\bar{k}\bar{w}^N) + \overline{gkw^N}$$
$$= fP(h\bar{w}^N) + P(\bar{g}P(h\bar{w}^N)) + \overline{P(g\overline{P(h\bar{w}^N)})} - P(\bar{g}P(h\bar{w}^N))(0) + Q(f\bar{k}\bar{w}^N) + \overline{gkw^N}$$

and

$$T_{uv}\bar{w}^N = T_{fh}\bar{w}^N + T_{\bar{g}h}\bar{w}^N + T_{f\bar{k}}\bar{w}^N + T_{\bar{g}\bar{k}}\bar{w}^N$$
$$= P(fh\bar{w}^N) + Q(\bar{g}h\bar{w}^N) + Q(f\bar{k}\bar{w}^N) + \overline{gkw^N}$$
$$= P(fh\bar{w}^N) + P(\bar{g}h\bar{w}^N) + \overline{P(g\bar{h}w^N)} - P(\bar{g}h\bar{w}^N)(0) + Q(f\bar{k}\bar{w}^N) + \overline{gkw^N}.$$

By Lemma 2.2,

$$P(\bar{g}P(h\bar{w}^N)) - P(\bar{g}P(h\bar{w}^N))(0) = P(\bar{g}h\bar{w}^N) - P(\bar{g}h\bar{w}^N)(0).$$

Since $P(fh\bar{w}^N)(0) = \langle fh\bar{w}^N, 1\rangle = \langle fh, w^N\rangle_2 = 0$ and $f(0) = 0$, by taking the holomorphic part on both sides of $T_u T_v \bar{w}^N$ and $T_{uv}\bar{w}^N$, we have

$$fP(h\bar{w}^N) = P(fh\bar{w}^N)$$

By (3) and (3'),

$$z^N f(z) P(h\bar{w}^N)(z) = f(z)\left(h(z) - \frac{N}{N+1} b_N z^N\right),$$
$$z^N P(fh\bar{w}^N)(z) = f(z)h(z),$$

it follows that $b_N = 0$.

By induction on N, we have $h = 0$.

(ii) By symmetry of the holomorphic part and anti-holomorphic part of u, v and functions in D_h, we have the desired conclusion.

Theorem 2.11. Let f, g be holomorphic functions in M. If

$$T_{f\bar{g}} = T_f T_{\bar{g}} \quad \text{or} \quad T_{f\bar{g}} = T_{\bar{g}} T_f$$

on D_h , then one of f and g must be constant.

Proof. By symmetry, we only give the proof that if $T_{f\bar{g}} = T_f T_{\bar{g}}$, then one of f and g is constant. Without loss of generality, assume $f(0) = g(0) = 0$.

Let $f(z) = \sum_{n=M}^{\infty} a_n z^n$ and $g(z) = \sum_{n=N}^{\infty} b_n z^n$ with $M, N \geq 1$.

Suppose g is not constant and $b_N \neq 0$.

As in the proof of Theorem 2.5, we have

$$\langle T_f T_{\bar{g}} w^N, w^M\rangle = \frac{M}{N+1}\bar{b}_N a_M + M\sum_{n=M+1}^{\infty} a_n \bar{b}_{n+N-M}. \quad (10)$$

Since

$$T_{f\bar{g}} w^N = Q(f\bar{g}w^N) = T_{fw^N}\bar{g}$$

and $(fw^N)(z) = \sum_{n=N+M}^{\infty} a_{n-N} z^n$, by Lemma 2.4(i),

$$\left(T_{fw^N}^* w^M\right)(z) = M\sum_{n=N+M}^{\infty} \frac{\bar{a}_{n-N}}{n-M}\bar{z}^{n-M} = M\sum_{n=N}^{\infty}\frac{\bar{a}_{n+M-N}}{n}\bar{z}^N,$$

and hence

$$\langle T_{f\bar{g}} w^N, w^M\rangle = \langle \bar{g}, T_{fw^N}^* w^M\rangle = \langle \sum_{n=N}^{\infty}\bar{b}_n \bar{z}^n, M\sum_{n=N}^{\infty}\frac{\bar{a}_{n+M-N}}{n}\bar{z}^N\rangle$$
$$= M\sum_{n=N}^{\infty}\bar{b}_n a_{n+M-N} = M\sum_{n=M}^{\infty} a_n \bar{b}_{n+N-M}. \quad (11)$$

Since $T_f T_{\bar{g}} = T_{f\bar{g}}$ and $b_N \neq 0$, by (10) and (11), we get $a_M = 0$.
By induction on $M, f = 0$, the proof is completed.
Now we present the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume both u and v are not constant. Let $u = f + \bar{g}$ and $v = h + \bar{k}$ with f, g, h, k holomorphic.

By Theorem 2.10, either f or h is constant. Without loss of generality, assume f is constant.

If f is constant, then g is not constant and, by Theorem 2.10, k is constant. Hence $T_{\bar{g}} T_h = T_{\bar{g}h}$. By Theorem 2.11, we must have h is a constant, a contradiction.

Corollary 2.12 If f is holomorphic in $M, \varphi_1, \varphi_2 \in S$, then

$$P\left(\bar{f}P(\varphi_1 + \varphi_2)\right) - P\left(\bar{f}P(\varphi_1 + \varphi_2)\right)(0) = P\left(\bar{f}(\varphi_1 + \varphi_2)\right) - P\left(\bar{f}(\varphi_1 + \varphi_2)\right)(0).$$

Proof. By definition, in [11], we have,

$$P\left(\bar{f}P(\varphi_1 + \varphi_2)\right)(z) - P\left(\bar{f}P(\varphi_1 + \varphi_2)\right)(0) = \langle \bar{f}P(\varphi_1 + \varphi_2), K_z\rangle - \langle \bar{f}P(\varphi_1 + \varphi_2), 1\rangle = \int_{\mathbb{D}} \bar{f}(w) \frac{\partial P(\varphi_1+\varphi_2)}{\partial w}(w) \frac{\overline{\partial K_z}}{\partial w}(w) dA(w) = \langle \frac{\partial P(\varphi_1+\varphi_2)}{\partial w}, f\frac{\partial K_z}{\partial w}\rangle_2$$

and

$$P\left(\bar{f}(\varphi_1 + \varphi_2)\right)(z) - P\left(\bar{f}(\varphi_1 + \varphi_2)\right)(0) = \langle \bar{f}(\varphi_1 + \varphi_2), K_z\rangle - \langle \bar{f}(\varphi_1 + \varphi_2) - 1\rangle = \int_{\mathbb{D}} \bar{f}(w) \frac{\partial(\varphi_1+\varphi_2)}{\partial w}(w) \frac{\overline{\partial K_z}}{\partial w}(w) dA(w) = \langle \frac{\partial(\varphi_1+\varphi_2)}{\partial w}, f\frac{\partial K_z}{\partial w}\rangle_2.$$

Since

$$P(\varphi_1 + \varphi_2)(w) = \langle(\varphi_1 + \varphi_2), K_w\rangle = \int_{\mathbb{D}} (\varphi_1 + \varphi_2) dA \int_{\mathbb{D}} \overline{K_w}\, dA + \int_{\mathbb{D}} \frac{\partial(\varphi_1+\varphi_2)}{\partial t} \frac{\partial \overline{K_w}}{\partial t} dA(t),$$

we have

$$\frac{\partial P(\varphi_1+\varphi_2)}{\partial w}(w) = \int_{\mathbb{D}} \frac{\partial(\varphi_1+\varphi_2)}{\partial t} \frac{\partial^2 \overline{K_w}}{\partial \bar{w} \partial t} dA(t).$$

It is well known that $\frac{\partial^2 K_w}{\partial w \partial t}(t)$ is the Bergman kernel $L_w(t) = \frac{1}{(1-\bar{w}t)^2}$.

Hence $\frac{\partial P(\varphi_1+\varphi_2)}{\partial w}(w) = \langle \frac{\partial(\varphi_1+\varphi_2)}{\partial t}, L_w\rangle_2$, which implies that

$$\langle \frac{\partial P(\varphi_1+\varphi_2)}{\partial w}, f\frac{\partial K_z}{\partial w}\rangle_2 = \langle\langle \frac{\partial(\varphi_1+\varphi_2)}{\partial t}, L_w\rangle_2, f\frac{\partial K_z}{\partial w}\rangle_2 = \langle \frac{\partial(\varphi_1+\varphi_2)}{\partial t}, f\frac{\partial K_z}{\partial t}\rangle_2.$$

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