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Research Article

A Necessary and Sufficient Condition for the Equation $x^3+1=2py^2$ Has Positive Integer Solutions

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Abstract Let p be an odd prime with $p \equiv 1 \pmod 6$. In this paper, using some elementary number theory methods, a necessary and sufficient condition for the equation $x^3+1=2py^2$ has positive integer solutions (x,y) is given. Thus it can be seen that if $p \equiv 13 \pmod{24}$, then equation has no positive integer solution.

Keywords Cubic diophantine equation; Positive integer solution; Necessary and sufficient condition.

1. Introduction

Let ℕ be the set of all integers and D is an positive integer with no square factors. For a long time, to solve of the equation

$$x^3-1=Dy^2,\ x,y \in \mathbb{N} \tag{1.1}$$

is a very interesting problem in number theory. In the history, many scholars of number theory such as T. Nagell [1] and W. Ljunggren [2] et al. have researched such problems in depth. In 1981, Ke and Sun [3] prove that if D>6 and prime factor P of D satisfies $p \not\equiv 1 \pmod 6$, equation (1.1) has no solution (x,y). Hereafter, J.H.E. Cohn [4] prove above results again together with the case $D \le 6$.Therefore, so far we only need to consider the case that the prime factor P of D satisfies $p \equiv 1 \pmod 6$. At this time, to find the solution of the equation is a very difficult problem.

Let p is an odd prime number satisfying $p \equiv 1 \pmod 6$. This paper will discuss the equation (1.1) under the case $D=2p$, this time the equation can be expressed as follows

$$x^3+1=2py^2,\ x,y \in \mathbb{N}. \tag{1.2}$$

For smaller number p, the equation(1.2) have been solved for the following cases:

1 (Luo M. [5]) If $p=7$, (1.2) only has a solution $(x,y)=(5,3)$.

2 (Wang Y. [6]) If $p=13$, (1.2) has no solution.

3 (Duan H. [7]) If $p=19$, (1.2) has and only has a solution $(x,y)=(31,28)$.

4 (Duan H. [8]) If $p=43$, (1.2) has a solution $(x,y)=(7,2)$. Besides, if p satisfies the following conditions, the equation(1.2)has no solution:

(1) (Zhou W. [9]) $p=12r^2+1$, where r is a positive odd number.

(2) (Du X., Zhao D. and Zhao J. [10]) $p=3r(r+1)+1$ and $p\equiv 13(\bmod 24)$, where r is a positive integer.

(3) (Guan X. [11]) $p=6(4r+2)+1$, where r is a nonnegative integer.

For a given positive integer n, n can be only expressed as the form $n=dm^2$, where d and m are positive integer, d has no square factor. Such d called $quadratfrei$ of n, denoted by $Q(n)$. In this paper, we will apply the method of elementary number theory to prove the following generalized results:

Theorem If $p\equiv 1(\bmod 24)$, equation (1.2) has solution if and only if

$$p=Q(4r^4-6r^2+3),\ r\in\mathbb{N}, \gcd(6,r)=1 \quad (1.3)$$

or

$$p=Q(192r^4-24r^2+1),\ r\in\mathbb{N}. \quad (1.4)$$

If the condition (1.3) or (1.4) hold, then (1.2) has solution $(x,y)=(2r^2-1,rs)$ or $(x,y)=(24r^2-1,6rs)$ respectively, where s is a positive integer satisfying

$$ps^2=4r^4-6r^2+3 \quad (1.5)$$

or

$$ps^2=192r^4-24r^2+1 \quad (1.6)$$

respectively. If $p\equiv 7(\bmod 24)$, (1.2) has solution if and only if

$$p=Q(12r^4-6r^2+1),\ r\in\mathbb{N},\ 2\nmid r. \quad (1.7)$$

If condition (1.7) holds, then (1.2) has solution $(x,y)=(6r^2-1,3rs)$, where s is a positive integer satisfying

$$ps^2=12r^4-6r^2+1. \quad (1.8)$$

If $p\equiv 13(\bmod 24)$, (1.2) has no solution. If $p\equiv 19(\bmod 24)$, (1.2) has solution if and only if

$$p=Q(64r^4-24r^2+3),\ r\in\mathbb{N},\ 3\nmid r. \quad (1.9)$$

If condition (1.9) holds, then (1.2) has solution $(x,y)=(8r^2-1,2rs)$, where s is a positive integer satisfying

$$ps^2=64r^4-24r+3. \quad (1.10)$$

Due to the discussion in paper [6], [9], [10] and [11], all odd prime numbers p satisfy $p\equiv 13(\bmod 24)$, thus from above theorem, we can directly to know that the equation (1.2) has no solution. Therefore, all these results are the particular case of the theorem of this paper.

2. The Proof of Theorem

Assume that (x,y) is a group solution of equation (1.2). From the analysis of the papers [3] and [12], we know that x and y are sure to satisfy

$$x+1=2a^2,\ x^2-x+1=pb^2,\ y=ab,\ a,b\in\mathbb{N},\ 3\nmid a, 2\nmid b. \quad (2.1)$$

or

$$x+1=6a^2,\ x^2-x+1=3pb^2,\ y=3ab,\ a,b\in\mathbb{N},\ 2\nmid b. \quad (2.2)$$

When (2.1) holds, by

$$x \equiv 2a^2 - 1 \equiv \begin{cases} 7(\bmod 8), & \text{if } 2\,|a, \\ 1(\bmod 8), & \text{if } 2 \nmid a \end{cases} \tag{2.3}$$

then from (2.1) and (2.3), we have

$$p \equiv pb^2 \equiv x^2 - \mathrm{x} - 1 \equiv \begin{cases} 3(\bmod 8), & \text{if } 2\,|a, \\ 1(\bmod 8), & \text{if } 2 \nmid a \end{cases} \tag{2.4}$$

When (2.2) holds, because of

$$x \equiv 6a^2 - 1 \equiv \begin{cases} 7(\bmod 8), & \text{if } 2\,|a, \\ 5(\bmod 8), & \text{if } 2 \nmid a \end{cases} \tag{2.5}$$

then, by (2.2) and (2.5), we get

$$p \equiv pb^2 \equiv \frac{1}{3}(x^2 - \mathrm{x} - 1) \equiv \begin{cases} 1(\bmod 8), & \text{if } 2\,|a, \\ 7(\bmod 8), & \text{if } 2 \nmid a \end{cases} \tag{2.6}$$

Due to $p \equiv 1(\bmod 6)$, then according to the definition of no existence of *quadratfrei* for positive integers, and from (2.1), (2.2), (2.4) and (2.6), we obtain the theorem. The proof of theorem is complete.

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