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# The Spinning Equations of Motion for Objects in AP-Geometry

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**Abstract** Equations of spinning objects are obtained in Absolute Parallelism Geometry [AP], a special class of non-Riemannian geometry admitting specific types having non-vanishing curvature and torsion simultaneously. This new set of equations is the counterpart of the Papapetrou equations in the Riemannian geometry. Applying, the concept of geometerization of physics, it may give rise to describe the spin tensor as parameterized commutation relation between path and path deviation equations in both Riemannian and non-Riemannian geometries.

**Keywords:** Spinning equations, Non-Riemanian geometry ,path equations ,path deviation equations

## 1 Introduction

The problem of a rotating object in the presence of the gravitational field is essentially practical than viewing objects as a mere test particles, in order to ignore their intrinsic property due to the Orthodox General Theory of Relativity. Accordingly, several attempts were done in the last century started by Mathisson [1], followed by Papapetrou [2] and extended by Dixon [3] to include other non-gravitational fields e.g. electromagnetic. Also, there is an approach by Dixon-Souriau to include spinning motion, magnetic moment with charged objects [4]. Such of these detailed equations have been presented only in Riemannian geometry.

Now, the arising question, is based on the following:

What is the situation of the above mentioned particles in case of Non-Remannain geometries?

In order to find the above enquiry, one must take treat the situation of non-Riemanian geometries as individual cases: one of its special classes is Riemann-Cartan geometry; which considers a tetrad space $\lambda^{\mu}_{i}$ as two independent vector fields, one may be responsible for general coordinate transformation (GCT), the holonomic coordinates, labeled by Greek indices and the Latin ones are used to express the Local Lorentz transformation [LLT], mainly to describe the internal properties of the object [5], labeled by Latin letters, the anholonomic coordinates. This type of work has encouraged many authors [6-8], to relate this type of geometry with gauge theories of gravity [9], wherein there is a tetrad space for gauge translation, and spin connection to represent gauge rotation. [10-12].

Also, another trend of viewing the Non-Riemaniann geometry, is called a Teleparallel geometry- A geometry with a tetrad building blocks, which may represent a translational gauge with a vanishing curvature [13] , and treat the annholonomic coordinates as vector number. Such a tendency of neutralizing the role of annholnomic coordinates, to be a vector number, with an additional property. This may give rise to define that there are non-vanishing torsion curvatures simultaneously, due to within different types of absolute derivatives[1].

The arising notation of AP-geometry led Wanas et al (1995) to describe three different paths may act the role of geodesic in Riemanian geometry [17]. The striking features of these paths, have a step $\frac{1}{2}$ from one path into other . This gives an impression, that paths in this type of geometry are naturally quantized. Lately, Wanas(1998) obtained a parameterized absolute parallelism geometry [PAP] obtaining a spin-torsion interaction, together with defining non-vanishing curvature and torsion tensors simultaneously

[1] For more details about the underlying geometry and its application in establishing a generalized field theory see [14-16]

[18]. The existence of such an interaction has led Wanas et al to detect its presence in terms of revealing the discrepancy between theory and observation of thermal neutrons [19] and presenting a temporal model for SN1987A [20].

Accordingly, in the present work we are going to obtain the analogous of the Papapetrou equation with precession in the context of AP-geometry. This will enable us to examine, the effect of different absolute derivatives on the interaction with the torsion and spin tensors.

The paper is organized as follows; section 2 discusses the relationship between spin tensor and geodesic and geodesic deviation vectors in the Riemaniann geometry, section 3 is extending the previous relationship to become among paths and path deviation vectors with their corresponding spin tensors in AP-geometry. Section 4 deals with the Lagranagian formalism of the Papapetrou equation in AP-geometry, and finally, Section 5 presents the results obtained in the previous sections , regarding some recommendations in our future work of on this approach.

## 2 Motion in Riemanian Geometry

### 2.1 Geodesic and Geodesic Deviations : The Bazanski Approach

Equations of geodesic and geodesic deviation equations Riemannian geometry are required to examine many problems of motion for different test particles in gravitational fields. This encouraged many authors to derive them by various methods among which one of the most applicable methods is the Bazanski approach [21] in which from one single Lagrangian one can obtain simultaneously equation of geodesic and geodesic deviations in the following way:

$$L = g_{\mu\nu} U^{\mu} \frac{D\Psi^{\nu}}{Ds}, \tag{2.1}$$

where , $g_{\mu\nu}$ is the metric tensor, $U^{\mu}$, is a unit tangent vector of the path whose parameter is $s$, and $\Psi^{\nu}$ is the deviation vector associated to the path $(s)$, $\frac{D}{Ds}$ is the covariant derivative with respect to parameter $s$.

Applying the Euler Lagrange equation , by taking the variation with respect to the deviation tensor[2]:

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{\Psi}^{\mu}} - \frac{\partial L}{\partial \Psi^{\mu}} = 0 \tag{2.2}$$

to obtain the geodesic equation

$$\frac{DU^{\mu}}{Ds} = 0, \tag{2.3}$$

and taking the variation with respect the the unit vector $U^{\mu}$,

$$\frac{d}{ds}\frac{\partial L}{\partial U^{\mu}} - \frac{\partial L}{\partial x^{\mu}} = 0, \tag{2.4}$$

to obtain the geodesic deviation equation,

$$\frac{D^2\Psi^{\mu}}{Ds^2} = R^{\mu}_{\nu\rho\sigma} U^{\nu} U^{\rho} \Psi^{\sigma}, \tag{2.5}$$

where $R^{\mu}_{\nu\rho\sigma}$ is Riemann-Christoffel tensor.

### 2.2 On The Relation Between Spin Tensor and The Deviation Vector: The Riemanian Case

Equations of spinning motion ,the case of $P^{\alpha} = mU^{\alpha}$ can be related to geodesic if one follows the following transformation [23]

$$\bar{U}^{\mu} = U^{\mu} + \beta \frac{D\Psi^{\mu}}{Ds} \tag{2.6}$$

[2] See [22], to get a detailed description for deriving the geodesic and geodesic deviation equations using The Bazanski Method

 

where $\bar{U}^\alpha$ is a unit tangent vector with respect to the parameter ,such that $\bar{U}^\alpha = \frac{dx^\mu}{d\bar{s}}$ , $\bar{s}$.
By taking the covariant derivative on both sides one obtains:

$$\frac{D\bar{U}^\alpha}{D\bar{s}} = \frac{D}{Ds}(U^\mu + \beta\frac{D\Psi^\mu}{Ds})\frac{ds}{d\bar{s}}. \tag{2.7}$$

From geodesic and geodesic deviation equations one gets

$$\frac{DU^\alpha}{Ds} = 0 \tag{2.8}$$

and

$$\frac{D^2\Psi^\alpha}{Ds^2} = R^\alpha_{\mu\nu\sigma}U^\mu U^\nu \Psi^\sigma \tag{2.9}$$

Substituting equations (2.8) and (2.9) in (2.7) to get

$$\frac{D\bar{U}^\alpha}{D\bar{s}} = (\beta R^\alpha_{\mu\nu\sigma}U^\mu U^\nu \Psi^\sigma)\frac{ds}{d\bar{s}}. \tag{2.10}$$

Regarding $\beta \equiv \frac{s_*}{m}$ , where $\beta$ is the angular momentum ratio , $s_*$ is the magnitude of the spin tensor $S^{\mu\nu}$ , and $m$ is mass of the object.

Let us assume the relationship between the spin tensor and geodesic deviation vector in the following way

$$S^{\mu\nu} = s_*(U^\alpha \Psi^\beta - U^\beta \Psi^\alpha). \tag{2.11}$$

Thus, we get

$$\frac{D\bar{U}^\alpha}{D\bar{s}} = \frac{1}{2m}(R^\alpha_{\mu\nu\sigma}U^\mu U^\nu \Psi^\sigma)\frac{ds}{d\bar{s}} \tag{2.12}$$

i.e

$$\frac{D\bar{U}^\alpha}{D\bar{s}} = \frac{1}{2m}R^\alpha_{\mu\nu\sigma}S^{\nu\sigma}\bar{U}^\mu \tag{2.13}$$

which is the Papapetrou equation for short.

### 2.3 Lagrangian Formalism of Spinning Equations

Another way to derive the Papapetrou equation for short is, by applying the action principle on the following equation [24]:

$$L = g_{\mu\nu}\bar{U}^\mu\frac{D\bar{\Psi}^\nu}{D\bar{s}} + \frac{1}{2m}R_{\mu\nu\rho\sigma}S^{\rho\sigma}U^\nu\Psi^\mu, \tag{2.14}$$

taking the variation with respect to the deviation tensor $\bar{\Psi}^\alpha$ we obtain equation (2.13).

Also, by taking the variation with respect to $\bar{U}^\alpha$ after some manipulations, we get its corresponding spinning deviation equation

$$\frac{D\bar{\Psi}^2}{D^2\bar{s}} = R^\alpha_{\beta\gamma\delta}\bar{U}^\beta\bar{U}^\gamma\bar{\Psi}^\delta + \frac{1}{2m}(R^\alpha_{\beta\gamma\sigma}S^{\gamma\sigma}U^\beta)_{;\delta}\bar{\Psi}^\delta \tag{2.15}$$

Thus, we can figure out the Euler Lagrange Equations on the Bazanski-like Lagragian give an identical equation to (2.13) and its corresponding deviation equations.

### 2.4 Spinning and Spinning Deviation Equations Without Precession

The Papapetrou equation of a spinning object with precession [2]is obtained by a modified Bazanski Lagrangian [25] :

$$L = g_{\alpha\beta}(mU^\alpha + U_\beta\frac{DS^{\alpha\beta}}{Ds})\frac{D\Psi^\beta}{Ds} + \frac{1}{2}R_{\alpha\beta\gamma\delta}S^{\gamma\delta}U^\beta\Psi^\alpha$$

 

to obtain equation of a spinning object by taking the variation with respect to the deviation vector $\Psi^\alpha$

$$\frac{D}{Ds}(mU^\alpha + U_\beta \frac{DS^{\alpha\beta}}{Ds}) = \frac{1}{2} R^\alpha_{.\mu\nu\rho} S^{\rho\nu} U^\mu \tag{2.16}$$

and its deviation equation can be obtained by taking the variation with respect to $U^\alpha$ to become:

$$\frac{D^2\Psi^\alpha}{Ds^2} = R^\alpha_{.\mu\nu\rho} U^\mu (mU^\nu + U_\beta \frac{DS^{\nu\beta}}{Ds})\Psi^\rho + g^{\alpha\sigma} g_{\nu\lambda} (mU^\lambda + U_\beta \frac{DS^{\lambda\beta}}{Ds})_{;\sigma} \frac{D\Psi^\nu}{Ds}$$
$$+ \frac{1}{2}(R^\alpha_{.\mu\nu\rho} S^{\nu\rho} \frac{D\Psi^\mu}{Ds} + R^\alpha_{\mu\nu\lambda} S^{\nu\lambda}_{.;\rho} U^\mu \Psi^\rho + R^\alpha_{\mu\nu\lambda;\rho} S^{\nu\lambda} U^\mu \Psi^\rho). \tag{2.17}$$

### 2.5 Spinning and Spinning Deviation Equations with Precession

It is well known that equation of spinning charged objects in the presence of gravitational field have been studied extensively [26]. This led us to suggest its corresponding Lagrangian formalism , using a modified Bazanski Lagrangian [27], for a spinning and precessing object and their corresponding deviation equation in Riemanian geometry in the following way

$$L = g_{\alpha\beta} P^\alpha \frac{D\Psi^\beta}{Ds} + S_{\alpha\beta} \frac{D\Psi^{\alpha\beta}}{Ds} + F_\alpha \Psi^\alpha + M_{\alpha\beta} \Psi^{\alpha\beta}, \tag{2.18}$$

where

$$P^\alpha = mU^\alpha + U_\beta \frac{DS^{\alpha\beta}}{DS}. \tag{2.19}$$

Taking the variation with respect to $\Psi^\mu$ and $\Psi^{\mu\nu}$ simultaneously we obtain

$$\frac{DP^\mu}{DS} = F^\mu, \tag{2.20}$$

and

$$\frac{DS^{\mu\nu}}{DS} = M^{\mu\nu}, \tag{2.21}$$

where $P^\mu$ is the momentum vector,

$$F^\mu = \frac{1}{2} R^\mu_{\nu\rho\delta} S^{\rho\delta} U^\nu,$$

and $R^\alpha_{\beta\rho\sigma}$ is the Riemann curvature, $\frac{D}{Ds}$ is the covariant derivative with respect to a parameter $S$,$S^{\alpha\beta}$ is the spin tensor,

$$M^{\mu\nu} = P^\mu U^\nu - P^\nu U^\mu \tag{2.22}$$

$U^\mu$ is the unit tangent vector to the geodesic.

Use the following identity on both equations (20) and (21)

$$A^\mu_{;\nu\rho} - A^\mu_{;\rho\nu} = R^\mu_{\beta\nu\rho} A^\beta, \tag{2.23}$$

where $A^\mu$ is an arbitrary vector, and multiply both sides with arbitrary vectors $U^\rho \Psi^\nu$ as well as using the following condition [26]

$$U^\alpha_{;\rho} \Psi^\rho = \Psi^\alpha_{;\rho} U^\rho, \tag{2.24}$$

where $\Psi^\alpha$ is its deviation vector associated to the unit vector tangent $U^\alpha$.

Also in a similar way:

$$S^{\alpha\beta}_{;\rho} \Psi^\rho = \Phi^{\alpha\beta}_{;\rho} U^\rho, \tag{2.25}$$

one obtains the corresponding deviation equations [28]

$$\frac{D^2\Psi^\mu}{DS^2} = R^\mu_{\nu\rho\sigma} P^\nu U^\rho \Psi^\sigma + F^\mu_{;\rho} \Psi^\rho, \tag{2.26}$$

and

$$\frac{D^2\Psi^{\mu\nu}}{DS^2} = S^{\rho[\mu} R^{\nu]}_{\rho\sigma\epsilon} U^\sigma \Psi^\epsilon + M^{\mu\nu}_{;\rho} \Psi^\rho, \tag{2.27}$$

 

## 3 Motion in AP-Geometry

### A brief introduction of AP-Space

The structure of this space is defined completely by a set of n-contravariant vector fields $\underset{i}{\lambda}^{\mu}$ where $i = 1, 2, 3, ..., n)$ denotes the vector number, and $\mu(= 1, 2, 3, ..., n)$ denotes $\underset{i}{\lambda}_{\mu}$ of the vectors $\underset{i}{\lambda}^{\mu}$, in the determinant $||\underset{i}{\lambda}^{\mu}||$, is defined such that[3]

$$\underset{i}{\lambda}^{\mu}\,\underset{j}{\lambda}_{\mu} = \delta_{ij},$$

$$\underset{i}{\lambda}^{\mu}\,\underset{i}{\lambda}_{\nu} = \delta^{\mu}_{\nu}.$$

Using these vectors, the following second order symmetric tensors are defined:

$$g^{\mu\nu} \overset{\text{def}}{=} \underset{i}{\lambda}^{\mu}\,\underset{i}{\lambda}^{\nu},$$

$$g_{\mu\nu} \overset{\text{def}}{=} \underset{i}{\lambda}_{\mu}\,\underset{i}{\lambda}_{\nu}.$$

one can define Christoffel symbols and covariant derivatives using this symbol, in the usual manner. The following third order tensor, the contortion tensor, can be defined as,

$$\gamma^{\alpha}_{.\mu\nu} \overset{\text{def}}{=} \underset{i}{\lambda}^{\alpha}\,\underset{i}{\lambda}_{\mu;\nu},$$

which is non-symmetric in its last two indices $\mu, \nu$. It can be shown that $\gamma^{\alpha}_{.\mu\nu}$ is skew-symmetric in its first two indices.

**The AP-Condition**

$$\underset{i}{\lambda}^{\nu}_{|\overset{\mu}{+}} = 0$$

where $|\overset{\mu}{+}$ is the absolute +ve derivative, such that it defines $\Gamma^{\alpha}_{\mu\nu}$ a non symmetric affine connection, in which

$$\Gamma^{\alpha}_{\mu\nu} = \underset{i}{\lambda}^{\alpha}\,\underset{i}{\lambda}_{\mu,\nu}.$$

The torsion of the space time is defined by

$$\Lambda^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{.\beta\gamma} - \Gamma^{\alpha}_{.\gamma\beta}.$$

### (i)Paths and Path Deviation Equations Subject to $\boldsymbol{\Gamma^{\alpha}_{\beta\gamma}}$

Paths and Path deviations equations are the counterpart of geodesic and geodesic deviation in AP-geometry. Accordingly, we have different trajectories based on the type of the absolute derivative, with respect to $\Gamma^{\alpha}_{\beta\gamma}$ [17].

From this perspective, it has been found out that the Bazanski Lagrangian may be a good candidate to express these trajectories.

$$L = g_{\mu\nu}V^{\mu}\frac{\nabla\Phi^{\nu}}{\nabla S^{+}} \tag{3.28}$$

where

$$\frac{\nabla\Phi}{\nabla S^{+}} = \frac{d\Phi^{\alpha}}{dS^{+}} + \Gamma^{\alpha}_{\mu\nu}\Phi^{\mu}V^{\nu}.$$

Thus,taking the variation with respect to $\xi^{\mu}$ and implementing the AP-condition to find that

$$g_{\underset{+\nu+|\sigma}{\mu\ \sigma}} \equiv 0 \tag{3.29}$$

[3] for more detail see [13-16]

 

one finds out the following path equation

$$\frac{\nabla V^{\mu}}{\nabla S^{+}}=0. \tag{3.30}$$

Also, its associated deviations can be derived if one applies the following relation

$$A^{\mu}_{\underset{|++}{\nu\ \rho}}-A^{\mu}_{\underset{|++}{\rho\ \nu}}=M^{\mu}_{\sigma\nu\rho}A^{\sigma}+\Lambda^{\sigma}_{\nu\rho}A^{\mu}_{\underset{|+}{\sigma}}, \tag{3.31}$$

provided with the following condition:

$$U^{\alpha}_{\underset{|+}{\rho}}\Phi^{\rho}=\Phi^{\alpha}_{\underset{|+}{\rho}}U^{\rho}, \tag{3.32}$$

with taking into consideration, the vanishing curvature tensor

$$M^{\mu}_{\sigma\nu\rho}\equiv 0, \tag{3.33}$$

to be substituted in (3.30) to obtain the corresponding deviation equation

$$\frac{\nabla^{2}\Phi^{\alpha}}{\nabla S^{2+}}=\Lambda^{\rho}_{\mu\nu}V^{\mu}\Phi^{\nu}V^{\alpha}_{\underset{|+}{\rho}}. \tag{3.34}$$

### (ii)Paths and Path Deviation Equations subject to $\Gamma^{\alpha}_{(\beta\gamma)}$

Due to the absolute derivative with respect to $\Gamma^{\alpha}_{(\beta\gamma)}$, one can derive its associated path and path deviation equations using the following Lagrangian [17]:

$$L=g_{\mu\nu}W^{\mu}\frac{\hat{\nabla}\eta^{\nu}}{\hat{\nabla}S^{0}}, \tag{3.35}$$

where

$$\frac{\hat{\nabla}\eta^{\alpha}}{\hat{\nabla}S^{0}}=\frac{d\eta^{\alpha}}{dS^{0}}+\Gamma^{\alpha}_{(\mu\nu)}\eta^{\mu}W^{\nu}.$$

Thus, taking the variation with respect to $\eta^{\mu}$;provided that

$$g_{\underset{00|\sigma}{\mu\nu}}=\Lambda_{(\mu\nu)\sigma}, \tag{3.36}$$

to obtain its corresponding path equation:

$$\frac{\nabla W^{\mu}}{\nabla S^{0}}=\frac{1}{2}\Lambda_{(\nu\rho).}^{\ \ \ \mu}W^{\nu}W^{\rho}. \tag{3.37}$$

Using the following relation

$$A^{\mu}_{\underset{|00}{\nu\rho}}-A^{\mu}_{\underset{|00}{\rho\nu}}=L^{\mu}_{\sigma\nu\rho}A^{\sigma}+\Lambda^{\sigma}_{\nu\rho}A^{\mu}_{\underset{|0}{\sigma}}, \tag{3.38}$$

and the condition below

$$W^{\mu}_{\underset{(0)}{\rho}}\eta^{\rho}=\eta^{\mu}_{\underset{(0)}{\rho}}W^{\rho}, \tag{3.39}$$

to be substituted in (3.37), provided that its associated curvature,

$$L^{\mu}_{\sigma\nu\rho}\neq 0, \tag{3.40}$$

is non vanishing, to obtain the corresponding deviation equation

$$\frac{\hat{\nabla}^{2}\zeta^{\alpha}}{\hat{\nabla}S^{2(0)}}=\frac{1}{2}(\Lambda_{\mu\nu}^{\ \ \ \alpha}W^{\mu}W^{\nu})_{|^{\rho}_{0}}\zeta^{\rho}+L^{\alpha}_{\beta\rho\sigma}W^{\beta}W^{\rho}\zeta^{\sigma}+\Lambda^{\rho}_{\mu\nu}W^{\mu}\eta^{\nu}\zeta^{\alpha}_{\underset{|0}{\rho}} \tag{3.41}$$

 

**(iii)Paths and Path Deviation Equations Subject to $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$**

Following the same approach as explained the previous items(i) and (ii), one may derive the paths and path deviations equations subject to $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$, by introducing the following Lagrangian [17]:

$$L = g_{\mu\nu}J^{\mu}\frac{\tilde{\nabla}\zeta^{\nu}}{\tilde{\nabla}S^{-}} \tag{3.42}$$

such that

$$\frac{\tilde{\nabla}\zeta^{\nu}}{\tilde{\nabla}S^{-}} = \frac{d\zeta^{\nu}}{dS^{-}} + \tilde{\Gamma}^{\nu}_{\mu\sigma}\zeta^{\mu}J^{\sigma}.$$

Accordingly, taking the variation with respect to $\eta^{\mu}$ to derive its corresponding path equation, and provided that [16]

$$g_{\underset{--|\sigma}{\mu\ \nu}} = 2\Lambda_{(\mu\nu)\sigma} \tag{3.43}$$

we get

$$\frac{\tilde{\nabla}J^{\mu}}{\tilde{\nabla}S^{-}} = \Lambda^{\ \ \mu}_{(\alpha\beta)}J^{\alpha}J^{\beta}. \tag{3.44}$$

Also, in order to derive its corresponding path deviation equation, one must take into account the following relation:

$$A^{\mu}_{\underset{|--}{\nu\ \rho}} - A^{\mu}_{\underset{|--}{\rho\ \nu}} = N^{\mu}_{\sigma\nu\rho}A^{\sigma} + \Lambda^{\sigma}_{\nu\rho}A^{\mu}_{\underset{|-}{\ \sigma}}, \tag{3.45}$$

together with, the condition

$$J^{\mu}_{\underset{|(-)}{\ \rho}}\zeta^{\rho} = \zeta^{\mu}_{\underset{|(-)}{\ \rho}}J^{\rho}, \tag{3.46}$$

to be substituted in (3.44), provided that its associated curvature,

$$N^{\mu}_{\sigma\nu\rho} \neq 0, \tag{3.47}$$

is non vanishing curvature.

Thus we derive the corresponding path deviation equation

$$\frac{\tilde{\nabla}^2\eta^{\alpha}}{\tilde{\nabla}S^{2-}} = N^{\alpha}_{\beta\rho\sigma}J^{\beta}J^{\rho}\eta^{\sigma} + \Lambda^{\rho}_{\mu\nu}J^{\mu}\eta^{\nu}\eta^{\alpha}_{\underset{|-}{\ \rho}} \tag{3.48}$$

### 3.1 On the Relation Between Spin Tensor and The Deviation Vector: The AP-geometry

In this part, we are going to extend the relationship obtained in (2.2) to derive the corresponding spin equations and their corresponding spin deviation equations.

**Spinning equation subject to $\Gamma^{\alpha}_{\beta\gamma}$**

Equations of spinning motion ,the case of $P^{\alpha}_{+} = mV^{\alpha}$ can be related to geodesic if one follows the following transformation

$$\bar{V}^{\mu} = V^{\mu} + \beta\frac{D\Phi^{\mu}}{Ds^{+}} \tag{3.49}$$

where $\bar{V}^{\alpha}$ is a unit tangent vector with respect to the parameter ,such that $\bar{V}^{\alpha} = \frac{dx^{\mu}}{d\bar{s}^{+}}$ , $\bar{s}$. By taking the covariant derivative on both sides one obtains:

$$\frac{\nabla\bar{V}^{\alpha}}{\nabla\bar{s}^{+}} = \frac{\nabla}{\nabla s^{+}}(V^{\mu} + \beta\frac{\nabla\Phi^{\mu}}{\nabla s^{+}})\frac{ds}{d\bar{s}}. \tag{3.50}$$

Substituting equations (3.30) and (3.34) in (3.50) we get

$$\frac{\nabla\bar{V}^{\alpha}}{\nabla\bar{s}^{+}} = (\beta\Lambda^{\rho}_{\nu\sigma}V_{\underset{\mu|+}{\ \ \rho}}V^{\nu}\Phi^{\sigma})\frac{ds}{d\bar{s}} \tag{3.51}$$

 

Let us assume the following. Taking $\beta = \frac{s}{m}$

$$\bar{S}^{\mu\nu} = s(V^{\alpha}\Phi^{\beta} - V^{\beta}\Phi^{\alpha}) \tag{3.52}$$

Thus, we get

$$\frac{\nabla\bar{V}^{\alpha}}{\nabla\bar{s}^{+}} = \frac{1}{2m}(\Lambda^{\rho}_{\nu\sigma}V_{\mu|^{+}_{\rho}}\bar{S}^{\nu\sigma})\frac{ds}{d\bar{s}} \tag{3.53}$$

i.e

$$\frac{\nabla\bar{V}^{\alpha}}{\nabla\bar{s}^{+}} = \frac{1}{2m}\Lambda^{\rho}_{\nu\sigma}\bar{V}_{\mu|^{+}_{\rho}}\bar{S}^{\nu\sigma}, \tag{3.54}$$

which is the version the Papapetrou equation for absolute derivative subject to $\Gamma^{\mu\nu}$, for short.

### Spinning equation subject to $\Gamma^{\alpha}_{(\beta\gamma)}$

Equations of spinning motion, the case of $P^{\alpha}_{(0)} = mW^{\alpha}$ can be related to geodesic if one follows the following transformation

$$\bar{W}^{\mu} = W^{\mu} + \beta\frac{\bar{\nabla}\eta^{\mu}}{\bar{\nabla}\bar{s}^{(0)}} \tag{3.55}$$

where $\bar{W}^{\alpha}$ is a unit tangent vector with respect to the parameter ,such that $\bar{W}^{\alpha} = \frac{dx^{\mu}}{d\bar{s}^{(0)}}$ , $\bar{s}$. By taking the covariant derivative on both sides one obtains:

$$\frac{\bar{\nabla}\bar{W}^{\alpha}}{\bar{\nabla}\bar{s}^{(0)}} = \frac{\nabla}{\nabla s^{+}}(W^{\mu} + \beta\frac{\nabla\eta^{\mu}}{\nabla s^{(0)}})\frac{ds}{d\bar{s}}. \tag{3.56}$$

Substituting equations (3.37) and (3.41) in (3.56) to get

$$\frac{\nabla\bar{W}^{\alpha}}{\nabla\bar{s}^{(0)}} = (\frac{1}{2}\Lambda_{\mu\dot{\nu}}{}^{\cdot\alpha}W^{\mu}W^{\nu} + \beta[L^{\alpha}_{\beta\gamma\delta}W^{\beta}W^{\gamma}\eta^{\delta} + \Lambda^{\rho}_{\nu\sigma}W_{\mu|^{\rho}_{(0)}}W^{\nu}\eta^{\sigma}])\frac{ds}{d\bar{s}}. \tag{3.57}$$

Now, let us assume that $\beta = \frac{s}{m}$, and

$$\bar{S}^{\mu\nu} = s(V^{\alpha}\eta^{\beta} - W^{\beta}\eta^{\alpha}). \tag{3.58}$$

Thus, we get

$$\frac{\bar{\nabla}\bar{W}^{\alpha}}{\nabla\bar{s}^{(0)}} = (\frac{1}{2}\Lambda_{\mu\dot{\nu}}{}^{\cdot\alpha}W^{\mu}W^{\nu} + \frac{1}{2m}[L^{\alpha}_{\mu\nu\sigma}W^{\mu} + \Lambda^{\rho}_{\nu\sigma}W_{\mu|^{\rho}_{(0)}}])\bar{S}^{\nu\sigma}(\frac{ds^{(0)}}{d\bar{s}^{(0)}}), \tag{3.59}$$

i.e.

$$\frac{\nabla\bar{W}^{\alpha}}{\nabla\bar{s}^{(0)}} = \frac{1}{2}\Lambda_{\mu\dot{\nu}}{}^{\cdot\alpha}W^{\mu}\bar{W}^{\nu} + \frac{1}{2m}(L^{\alpha}_{\mu\nu\sigma}\bar{W}^{\mu} + \Lambda^{\rho}_{\nu\sigma}\bar{W}_{\mu|^{\rho}_{(0)}})\bar{S}^{\nu\sigma}. \tag{3.60}$$

If we regard

$$\frac{ds^{(0)}}{d\bar{s}^{(0)}} = 1,$$

then, equation (3.57) becomes

$$\frac{\nabla\bar{W}^{\alpha}}{\nabla\bar{s}^{(0)}} = \frac{1}{2}\Lambda_{\mu\dot{\nu}}{}^{\cdot\alpha}\bar{W}^{\mu}\bar{W}^{\nu} + \frac{1}{2m}(L^{\alpha}_{\mu\nu\sigma}\bar{W}^{\mu} + \Lambda^{\rho}_{\nu\sigma}\bar{W}_{\mu|^{\rho}_{(0)}})\bar{S}^{\nu\sigma}, \tag{3.61}$$

which is the version the Papapetrou equation subject to $\Gamma^{\alpha}_{(\beta\gamma)}$, for short.

 

**Spinning equation subject to $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$**

Equations of spinning motion ,the case of $P^{\alpha}_{-} = mJ^{\alpha}$ can be related to geodesic if one follows the following transformation

$$\bar{J}^{\mu} = J^{\mu} + \beta\frac{\tilde{\nabla}\zeta^{\mu}}{\tilde{\nabla}\bar{s}^{-}} \tag{3.62}$$

where $\bar{J}^{\alpha}$ is a unit tangent vector with respect to the parameter ,such that $\bar{J}^{\alpha} = \frac{dx^{\mu}}{d\bar{s}^{-}}$ , $\bar{s}$. By taking the covariant derivative on both sides one obtains:

$$\frac{\bar{\nabla}\bar{J}^{\alpha}}{\bar{\nabla}\tilde{s}^{(-)}} = \frac{\tilde{\nabla}}{\nabla s^{-}}(J^{\mu} + \beta\frac{\nabla\zeta^{\mu}}{\tilde{\nabla}s^{-}})\frac{ds}{d\bar{s}^{-}}. \tag{3.63}$$

Substituting equations (3.44) and (3.48) in (3.63) to get

$$\frac{\tilde{\nabla}\bar{J}^{\alpha}}{\tilde{\nabla}\tilde{s}^{-}} = (\Lambda_{\dot{\mu}\dot{\nu}}{}^{\cdot\alpha}J^{\mu}J^{\nu} + \beta[N^{\alpha}_{\beta\gamma\delta}J^{\beta}J^{\gamma}\zeta^{\delta} + \Lambda^{\rho}_{\nu\sigma}J_{\mu|-}{}^{\rho}J^{\nu}\zeta^{\sigma}])\frac{ds^{-}}{ds^{-}}. \tag{3.64}$$

Let us assume the following Taking $\beta = \frac{s}{m}$,and

$$\tilde{S}^{\mu\nu} = s(J^{\alpha}\zeta^{\beta} - J^{\beta}\zeta^{\alpha}). \tag{3.65}$$

Thus, we get

$$\frac{\tilde{\nabla}\bar{J}^{\alpha}}{\tilde{\nabla}\bar{s}^{-}} = (\Lambda_{\dot{\mu}\dot{\nu}}{}^{\cdot\alpha}J^{\mu}J^{\nu} + \frac{1}{2m}[N^{\alpha}_{\mu\nu\sigma}J^{\mu} + \Lambda^{\rho}_{\nu\sigma}J_{\mu|-}{}^{\rho}])\tilde{S}^{\nu\sigma}(\frac{ds^{(-)}}{d\bar{s}^{(-)}}), \tag{3.66}$$

i.e.

$$\frac{\tilde{\nabla}\bar{J}^{\alpha}}{\tilde{\nabla}\bar{s}^{-}} = \Lambda_{\dot{\mu}\dot{\nu}}{}^{\cdot\alpha}J^{\mu}\tilde{J}^{\nu} + \frac{1}{2m}(N^{\alpha}_{\mu\nu\sigma}\bar{J}^{\mu} + \Lambda^{\rho}_{\nu\sigma}\bar{J}_{\mu|-}{}^{\rho})\tilde{S}^{\nu\sigma}. \tag{3.67}$$

If we regard

$$\frac{ds^{(-)}}{d\tilde{s}^{(-)}} = 1,$$

then, equation (73) becomes

$$\frac{\tilde{\nabla}\bar{J}^{\alpha}}{\tilde{\nabla}\bar{s}^{-}} = \Lambda_{(\dot{\mu}\dot{\nu})}{}^{\cdot\alpha}\bar{J}^{\mu}\bar{J}^{\nu} + \frac{1}{2m}(N^{\alpha}_{\mu\nu\sigma}\bar{J}^{\mu} + \Lambda^{\rho}_{\nu\sigma}\bar{J}_{\mu|-}{}^{\rho})\tilde{S}^{\nu\sigma}. \tag{3.68}$$

which is the version the Papapetrou equation subject to $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$, for short.

## 4 Spinning and Spinning Deviation Equations in AP-geometry: Lagrangian Formalism

From the previous results, we can check the reliability of the corresponding Bazanski equation to become in the following way.

### 4.1 Spinning and Spinning Deviation Equations Subject $\Gamma^{\alpha}_{\beta\gamma}$

**i the case of $P_{+} = mV$**

$$L = g_{\mu\nu}V^{\mu}\frac{\bar{\nabla}\Phi^{\mu}}{\bar{\nabla}S^{+}} + \bar{S}_{\mu\nu}\frac{\bar{\nabla}\Phi^{\mu\nu}}{\bar{\nabla}S^{+}} + \frac{1}{2m}[\Lambda^{\rho}_{\nu\sigma}V_{\mu|+}{}^{\rho}]\bar{S}^{\nu\sigma}\Phi^{\mu}. \tag{4.69}$$

Taking the variation with respect to $\Phi^{\alpha}$ and $\Phi^{\alpha\beta}$ we obtain

$$\frac{\nabla V^{\alpha}}{\nabla S^{+}} = \frac{1}{2m}\Lambda_{\rho\delta\nu}\bar{S}^{\delta\nu}V^{\alpha}_{\sigma+} + g^{\alpha\rho}\Lambda^{\rho}_{\nu\sigma}V_{\mu|+}{}^{\rho}\bar{S}^{\nu\sigma}, \tag{4.70}$$

 

and

$$\frac{\tilde{\nabla}S^{\alpha\beta}}{\tilde{\nabla}S}=0 \tag{4.71}$$

Using the commutation relation (3.31) , conditions (3.32) and

$$\bar{S}^{\mu\nu}_{\ \ \ \ |+}{}^{\rho}\Phi^{\rho}=\Phi^{\mu\nu}_{\ \ \ \ |+}{}^{\rho}V^{\rho}, \tag{4.72}$$

to be substituted in (4.70) and (4.71) in order to derive its corresponding set of deviation equations

$$\frac{\nabla^2\Phi^{\alpha}}{\tilde{\nabla}S^{2(+)}}=\Lambda^{\rho}_{\mu\nu}V^{\mu}V^{\nu}\Phi^{\alpha}_{\ \ +}{}^{\rho}, \tag{4.73}$$

and

$$\frac{\nabla^2\Phi^{\alpha\beta}}{\hat{\nabla}S^{2+}}=\Lambda^{\rho}_{\mu\nu}V^{\mu}\Phi^{\nu}\bar{S}^{\alpha\beta}_{\ \ \ \ |+}{}^{\rho}. \tag{4.74}$$

**(ii) the case $P_+\neq mV$**

Let us suggest the following Lagrangian:

$$L=g_{\mu\nu}P^{\mu}_{+}\frac{\nabla\Phi^{\mu}}{\nabla S^{+}}+\bar{S}_{\mu\nu}\frac{\nabla\Phi^{\mu\nu}}{\nabla S^{+}}+\frac{1}{2m}g_{\mu\nu}\Lambda^{\rho}_{\delta\rho}\bar{S}^{\delta\rho}V^{\mu}_{\ \ +}{}^{\rho}\Phi^{\nu}+g_{\alpha\mu}g_{\beta\nu}[P^{\alpha}_{+}V^{\beta}-P^{\beta}_{+}V^{\alpha}]\Phi^{\mu\nu}, \tag{4.75}$$

where

$$P^{\mu}_{+}=mV^{\mu}+\frac{V_{\nu}\tilde{\nabla}\bar{S}^{\mu\nu}}{\nabla S^{(+)}}.$$

Taking the variation with respect to $\zeta^{\alpha}$ and $\zeta^{\alpha\beta}$ we obtain

$$\frac{\nabla P^{\alpha}_{+}}{\nabla S^{+}}=\frac{1}{2m}\Lambda^{\rho}_{\delta\nu}\bar{S}^{\delta\nu}V^{\alpha}_{\ \ +}{}^{\rho}+g_{\mu\rho}g_{\nu\delta}[P^{\rho}_{+}V^{\delta}-P^{\delta}_{+}V^{\rho}]\Phi^{\mu\nu}, \tag{4.76}$$

and

$$\frac{\nabla\bar{S}^{\alpha\beta}}{\tilde{\nabla}S^{+}}=[P^{\alpha}_{+}V^{\beta}-P^{\beta}_{+}V^{\alpha}]. \tag{4.77}$$

Using the commutation relation (3.31) , the conditions (3.32) and (3.32) to be substituted in (4.70)and (4.71)and (4.72) in order to derive its corresponding set of deviation equations

$$\frac{\tilde{\nabla}^2\Phi^{\alpha}}{\tilde{\nabla}S^{2+}}=(\frac{1}{2m}\Lambda^{\rho}_{\delta\sigma}\hat{S}^{\delta\sigma}V^{\alpha}_{\ \ +}{}^{\sigma})_{\ |+}{}^{\delta}\Phi^{\delta}, \tag{4.78}$$

and

$$\frac{\nabla^2\zeta^{\alpha\beta}}{\tilde{\nabla}S^{2+}}=\Lambda^{\rho}_{\mu\nu}V^{\mu}\Phi^{\nu}\tilde{S}^{\alpha\beta}_{\ \ +}{}^{\rho}+[P^{\alpha}_{+}V^{\beta}-P^{\beta}_{+}V^{\alpha}]_{\ |+}{}^{\delta}\Phi^{\delta}. \tag{4.79}$$

From the above results of spinning equations and their corresponding deviation ones, we reach to regard them as the equivalent set of equations of spinning objects in the presence of Tele-parallel gravity [13].

### 4.2 Spinning and Spinning Deviation Equations Subject $\Gamma^{\alpha}_{(\beta\gamma)}$

**ii The case of $P_{(0)}=mW$**

$$L=g_{\mu\nu}W^{\mu}\frac{\hat{\nabla}\eta^{\mu}}{\hat{\nabla}S^{(0)}}+\hat{S}_{\mu\nu}\frac{\hat{\nabla}\eta^{\mu\nu}}{\hat{\nabla}S^{(0)}}+\frac{1}{2m}L_{\mu\nu\rho\delta}\eta^{\mu}W^{\nu}S^{\rho\delta}+\Lambda^{\rho}_{\nu\sigma}W_{\mu|.(0)}{}^{\rho}\hat{S}^{\nu\sigma}\eta^{\mu}. \tag{4.80}$$

Taking the variation with respect to $\eta^{\alpha}$ and $\eta^{\alpha\beta}$, we obtain

$$\frac{\hat{\nabla}W^{\alpha}}{\hat{\nabla}S^{o}}=\frac{1}{2}\Lambda_{(\dot{\mu}\dot{\nu})}{}^{\alpha}W^{\mu}W^{\nu}+\frac{1}{2m}L^{\alpha}_{\nu\rho\sigma}W^{\nu}S^{\rho\sigma}+g^{\alpha\mu}\Lambda^{\rho}_{\nu\sigma}W_{\mu|(0)}{}^{\rho}\hat{S}^{\nu\sigma}, \tag{4.81}$$

 

and

$$\frac{\hat{\nabla}\hat{S}^{\alpha\beta}}{\hat{\nabla}S} = \frac{1}{2}\Lambda_{(\mu\nu)}^{\cdot\ \cdot\ [\alpha}S^{\beta]\mu}W^{\nu}. \tag{4.82}$$

Using the commutation relation (3.38) and conditions (3.39) and

$$\hat{S}^{\mu\nu}_{\rho\ |(0)}\eta^{\rho} = \eta^{\mu\nu}_{\rho\ |(0)}W^{\rho}, \tag{4.83}$$

to be substituted in (80) and (81) in order to derive its corresponding set of deviation equations

$$\frac{\hat{\nabla}^2\eta^{\alpha}}{\hat{\nabla}S^{2(0)}} = L^{\alpha}_{\mu\nu\rho}W^{\mu}W^{\nu}\eta^{\rho} + \Lambda^{\rho}_{\mu\nu}W^{\mu}\xi^{\nu}\eta^{\alpha}_{\rho\ (0)} + \frac{1}{2}(\Lambda_{\mu\nu}^{\cdot\ \cdot\ \alpha}W^{\mu}W^{\nu} + \frac{1}{2m}L^{\alpha}_{\nu\rho\sigma}W^{\nu}S^{\rho\sigma})_{|0^{\sigma}}\eta^{\sigma}, \tag{4.84}$$

and

$$\frac{\hat{\nabla}^2\eta^{\alpha\beta}}{\hat{\nabla}S^{2(0)}} = S^{\mu[\beta}\rho L^{\alpha]}_{\mu\nu\rho}W^{\nu}\eta^{\rho} + \Lambda^{\rho}_{\mu\nu}W^{\mu}\eta^{\nu}S^{\alpha\beta}_{\rho\ |0}. \tag{4.85}$$

### (ii) the case $P_{(0)} \neq mW$

Let us suggest the following Lagrangian:

$$L = g_{\mu\nu}P^{\mu}_{(0)}\frac{\hat{\nabla}\eta^{\mu}}{\hat{\nabla}S^{(0)}} + \hat{S}_{\mu\nu}\frac{\hat{\nabla}\eta^{\mu\nu}}{\hat{\nabla}S^{(0)}} + \frac{1}{2m}L_{\mu\nu\rho\delta}\eta^{\mu}W^{\nu}\hat{S}^{\rho\delta} + \frac{1}{2m}g_{\mu\nu}\Lambda^{\rho}_{\delta\rho}\hat{S}^{\delta\rho}W^{\mu}_{\rho\ (0)}\eta^{\nu} + g_{\mu\rho}g_{\nu\delta}[P^{\rho}_{(0)}W^{\delta} - P^{\delta}_{0}W^{\rho}]\eta^{\mu\nu}, \tag{4.86}$$

where $P^{\mu}_{(0)} = mW^{\mu} + \frac{W_{\nu}DS^{\mu\nu}_{(0)}}{DS^{(0)}}$.

Taking the variation with respect to $\eta^{\alpha}$ and $\eta^{\alpha\beta}$ we obtain

$$\frac{\hat{\nabla}P^{\alpha}_{(0)}}{\nabla\hat{S}^{(0)}} = \frac{1}{2}\Lambda^{\cdot\cdot\alpha}_{\mu\nu}P^{\mu}_{(0)}W^{\nu} + \frac{1}{2m}L^{\alpha}_{\nu\rho\sigma}W^{\nu}\hat{S}^{\rho\sigma} + \frac{1}{2m}\Lambda^{\rho}_{\delta\nu}\hat{S}^{\delta\nu}W^{\alpha}_{|\rho} \tag{4.87}$$

, and

$$\frac{\hat{\nabla}\hat{S}^{\alpha\beta}}{\hat{\nabla}S^{(0)}} = \frac{1}{2}\Lambda_{\mu\nu}^{\cdot\ \cdot[\alpha}S^{\mu\beta]}W^{\nu}. \tag{4.88}$$

Using the commutation relation (3.38) , conditions (3.39) and (4.82) to be substituted in (4.82) and (4.83) in order to derive its corresponding set of deviation equations

$$\frac{\hat{\nabla}^2\eta^{\alpha}}{\hat{\nabla}S^{2(0)}} = L^{\alpha}_{\mu\nu\rho}P^{\mu}_{(0)}W^{\nu}\eta^{\rho} + \Lambda^{\rho}_{\mu\nu}P^{\mu}_{(0)}\eta^{\nu}\eta^{\alpha}_{\rho\ (0)} + \frac{1}{2}(\Lambda^{\cdot\cdot\alpha}_{\mu\nu}P^{\mu}_{(0)}W^{\nu} + \frac{1}{2m}L^{\alpha}_{\nu\rho\sigma}W^{\nu}\hat{S}^{\rho\sigma} + \frac{1}{2m}\Lambda^{\rho}_{\delta\sigma}\hat{S}^{\delta\sigma}W^{\alpha}_{\sigma\ (0)})_{|\delta\ (0)}\eta^{\delta}, \tag{4.89}$$

and

$$\frac{\tilde{\nabla}^2\zeta^{\alpha\beta}}{\tilde{\nabla}S^{2-}} = \tilde{S}^{\mu[\beta}\rho N^{\alpha]}_{\mu\nu\rho}J^{\nu}J^{\rho} + \Lambda^{\rho}_{\mu\nu}W^{\mu}\eta^{\nu}\hat{S}^{\alpha\beta}_{\rho\ o} + (\Lambda_{\mu\nu}^{\cdot\ \cdot[\alpha}S^{\mu\beta]}J_{\nu|-\ \delta}\zeta^{\delta} + [P^{\alpha}_{(0)}W^{\beta} - P^{\beta}_{(0)}W^{\alpha}]_{|\delta\ (0)}\eta^{\delta}. \tag{4.90}$$

## 4.3 Spinning and Spinning Deviation Equations Subject to $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$

### (i) the case of $P_{-} = mJ$

$$L = g_{\mu\nu}J^{\mu}\frac{\tilde{\nabla}\zeta^{\mu}}{\tilde{\nabla}S^{-}} + \tilde{S}_{\mu\nu}\frac{\tilde{\nabla}\zeta^{\mu\nu}}{\tilde{\nabla}S^{-}} + \frac{1}{2m}N_{\mu\nu\rho\delta}\zeta^{\mu}J^{\nu}S^{\rho\delta} + \Lambda^{\rho}_{\nu\sigma}J_{\mu|-\ \rho}\hat{S}^{\nu\sigma}\zeta^{\mu}, \tag{4.91}$$

by taking the variation with respect to $\zeta^{\alpha}$ and $\zeta^{\alpha\beta}$ we obtain

$$\frac{\tilde{\nabla}J^{\alpha}}{\tilde{\nabla}S^{-}} = \Lambda^{\cdot\cdot\alpha}_{\mu\nu}J^{\mu}J^{\nu} + \frac{1}{2m}N^{\alpha}_{\nu\rho\sigma}J^{\nu}S^{\rho\sigma} + \frac{1}{2m}\Lambda_{\rho\delta\nu}S^{\delta\nu}J^{\alpha}_{\sigma\ -} + g^{\alpha\rho}\Lambda^{\rho}_{\nu\sigma}J_{\mu|-\ \rho}\tilde{S}^{\nu\sigma} \tag{4.92}$$

 

and

$$\frac{\tilde{\nabla}S^{\alpha\beta}}{\tilde{\nabla}S^{-}}=\Lambda_{(\mu\nu)}^{\cdot\ [\alpha}S^{\beta]\mu}J^{\nu}. \tag{4.93}$$

Using the commutation relation (44), conditions (48) and

$$\tilde{S}^{\mu\nu}_{\ \ \rho\ |-}\zeta^{\rho}=\zeta^{\mu\nu}_{\ \ \rho\ |-}J^{\rho} \tag{4.94}$$

to be substituted in (4.91)and (4.92) in order to derive its corresponding set of deviation equations

$$\frac{\tilde{\nabla}^2\zeta^{\alpha}}{\tilde{\nabla}S^{2(-)}}=N^{\alpha}_{\mu\nu\rho}J^{\mu}J^{\nu}\zeta^{\rho}+\Lambda^{\rho}_{\mu\nu}J^{\mu}J^{\nu}\eta^{\alpha}_{\ \underline{\rho}}+(\Lambda_{\mu\nu}^{\cdot\ \cdot\ \alpha}J^{\mu}W^{\nu}+\frac{1}{2m}N^{\alpha}_{\nu\rho\sigma}J^{\nu}\tilde{S}^{\rho\sigma})_{|-}{}^{\sigma}\zeta^{\sigma}, \tag{4.95}$$

and

$$\frac{\tilde{\nabla}^2\zeta^{\alpha\beta}}{\hat{\nabla}S^{2(-)}}=S^{\mu[\beta}\rho\Lambda^{\alpha]}_{\nu\rho}J^{\nu}\zeta^{\rho}+\Lambda^{\rho}_{\mu\nu}J^{\mu}\zeta^{\nu}\tilde{S}^{\alpha\beta}_{\ \ \underline{\rho}\ |-}. \tag{4.96}$$

**(ii) the case $P_{-}\neq mJ$**

Let us suggest the following Lagrangian:

$$L=g_{\mu\nu}P^{\mu}_{-}\frac{\tilde{\nabla}\zeta^{\mu}}{\tilde{\nabla}S^{-}}+\tilde{S}_{\mu\nu}\frac{\tilde{\nabla}\zeta^{\mu\nu}}{\tilde{\nabla}S^{-}}+\frac{1}{2m}N_{\mu\nu\rho\delta}\zeta^{\mu}J^{\nu}\tilde{S}^{\rho\delta}+\frac{1}{2m}g_{\mu\nu}\Lambda^{\rho}_{\delta\rho}\tilde{S}^{\delta\rho}J^{\mu}_{\ \underline{\rho}}\zeta^{\nu}+g_{\mu\rho}g_{\nu\delta}[P^{\rho}_{-}J^{\delta}-P^{\delta}_{-}J^{\rho}]\zeta^{\mu\nu}, \tag{4.97}$$

where

$$P^{\mu}_{(0)}=mW^{\mu}+\frac{J_{\nu}\tilde{\nabla}\tilde{S}^{\mu\nu}}{DS^{(0)}}$$

by taking the variation with respect to $\zeta^{\alpha}$ and $\zeta^{\alpha\beta}$ we obtain

$$\frac{\tilde{\nabla}P^{\alpha}_{-}}{\tilde{S}^{-}}=\frac{1}{2}\Lambda_{\mu\nu}^{\cdot\ \cdot\ \alpha}P^{\mu}_{-}J^{\nu}+\frac{1}{2m}N^{\alpha}_{\nu\rho\sigma}J^{\nu}\tilde{S}^{\rho\sigma}+\frac{1}{2m}\Lambda^{\rho}_{\delta\nu}\tilde{S}^{\delta\nu}J^{\alpha}_{\ \underline{\rho}}, \tag{4.98}$$

and

$$\frac{\tilde{\nabla}\tilde{S}^{\alpha\beta}}{\hat{\nabla}S^{-}}=\Lambda_{\mu\nu}^{\cdot\ \cdot[\alpha}S^{\mu\beta]}J^{\nu}. \tag{4.99}$$

Using commutation relation (3.44) and the conditions (3.48) and (4.93) to be substituted in (4.97) and (4.98) in order to derive its corresponding set of deviation equations

$$\frac{\tilde{\nabla}^2\zeta^{\alpha}}{\tilde{\nabla}S^{2-}}=N^{\alpha}_{\mu\nu\rho}P^{\mu}_{-}J^{\nu}\zeta^{\rho}+\Lambda^{\rho}_{\mu\nu}P^{\mu}_{-}\zeta^{\nu}\zeta^{\alpha}_{o}+(\Lambda_{\mu\nu}^{\cdot\ \cdot\ \alpha}P^{\mu}_{-}J^{\nu}+\frac{1}{2m}L^{\alpha}_{\nu\rho\sigma}J^{\nu}\tilde{S}^{\rho\sigma}+\frac{1}{2m}\Lambda^{\rho}_{\delta\sigma}\hat{S}^{\delta\sigma}J^{\alpha}_{\ \underline{\sigma}})_{\ \underline{-}\ |-}{}^{\delta}\zeta^{\delta}, \tag{4.100}$$

and

$$\frac{\tilde{\nabla}^2\zeta^{\alpha\beta}}{\tilde{\nabla}S^{2-}}=\tilde{S}^{\mu[\beta}\rho N^{\alpha]}_{\mu\nu\rho}J^{\nu}J^{\rho}+\Lambda^{\rho}_{\mu\nu}J^{\mu}\zeta^{\nu}\tilde{S}^{\alpha\beta}_{\ \ \underline{\rho}}+(\Lambda_{\mu\nu}^{\cdot\ \cdot[\alpha}\tilde{S}^{\mu\beta]}J_{\ \underline{\nu}|-}{}^{\delta}\zeta^{\delta}+[P^{\alpha}_{-}J^{\beta}-P^{\beta}_{-}J^{\alpha}]_{\ \underline{-}}{}^{\delta}\zeta^{\delta}. \tag{4.101}$$

## 5 Discussion and Concluding Remarks

The present work is related to extending the concept of geometerization of physics to explain spinning objects in a gravitational field. It has been developed the modified Bazanski Lagrangian in general relativity for spinning objects to be expressed in AP-geometry. Due to the wealth of geometric quantities, one must regard that the existence of spin tensors associated for each path is defined by a specific type of absolute derivative. Also, we have emphasized the relationship between geodesic and geodesic deviation with spinning tensors, to be viable for any type of geometries, by testing its reliability in both Riemannian and AP-geometry. Moreover, the spin tensor has been defined geometrically as a commutation relation

 

between formula between geodesic and geodesic deviation in Riemannian geometry and their counterparts in AP-geometry.

Accordingly, we have obtained three different spinning equations different from its counterpart in Riemannian geometry. One of them, can be used to describe the spinning equations and their deviation in Tele-parallel gravity, i.e. these sets of spinning equations are representing the Papapetrou equation of Hayshi-Shirifugi New General Relativity [13], while the other two paths may describe, hypnotically, a set of spinning particles subject to a class of non vanishing curvature and torsion simultaneously. This may require an efficient field theory feasible to give a physical interpretation of $\hat{S}^{\mu\nu}$ and $\tilde{S}^{\mu\nu}$, which is still an open question.

Nevertheless, equations (4.70) and (4.71) can be applied to examine the motion of neutron stars in teleparallel gravity, as taken into account their associated field equations as given in [29].

Yet, this study has also clarified the viability interaction between torsion tensor and spin deviation equations, as mentioned previously in case of Gauge theories of gravity [26].

Nevertheless, these sets of spinning equations can also be applied in PAP-geometry, to give new results. Owing to revisit, the bi-metric theories of gravity using the tetrad formalism, one may find out some promising results able to reveal the mystery of several anomalies such as dark matter and dark energy in our nature, which will be studied in our future work.

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