An Optimal Control Problem for a Predator-prey Model with a General Monotonic or Non-monotonic Functional Response for Prey

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Abstract An optimal control problem is studied for a predator-prey model with a general monotonic and non-monotonic functional response for prey. The control functions represent the rate of mixture of the populations and the cost functional is of Mayer type. The goal of this paper is to maximize the total density of the two populations at a fixed time moment. The number of switching points of the optimal control is determined by the threshold value c, which is the rate of translating the prey population into the predator population. It is shown that general monotonic or non-monotonic functional response has no influence on the number of switching points of the optimal control.

Keywords: Predator-prey model, general functional response, Pontryagin's maximum principle, bang-bang control, switching point.

1 Introduction

In [1], Apreutesei found necessary optimality conditions for a control problem related to the predator-prey model

$$\begin{cases} \dfrac{dy_1}{dt} = ry_1(1 - \dfrac{y_1}{k}) - y_2 F(y_1, y_2), \ t \in [0, T], \\ \dfrac{dy_2}{dt} = y_2[-d + cF(y_1, y_2)], \qquad t \in [0, T], \end{cases} \tag{1.1}$$

where $y_1(t)$ and $y_2(t)$ stand for the densities of prey and predators at the moment $t \in [0, T]$ respectively, r is intrinsic growth rate and k is the prey carrying capacity in the absence of predation, d is the death rate of predator, c is the rate of translating the prey population into the predator population.

The predator functional response $F(y_1, y_2)$ signifies the number of prey consumed per predator in unit time, and the function $F(y_1, y_2)$ is assumed to satisfy the following assumptions:

(i) F is well defined and $F(y_1, y_2) > 0$, for all $y_1, y_2 > 0$.

(ii) $F \in C^1([0, +\infty)^2)$; F is bounded with respect to y_2.

(iii)

$$\frac{\partial F(y_1, y_2)}{\partial y_1} > 0, \tag{1.2}$$

and

$$F(y_1, y_2) + y_2 \frac{\partial F(y_1, y_2)}{\partial y_2} > 0, \tag{1.3}$$

for all $y_1, y_2 > 0$.

In fact, function $F(y_1, y_2)$ above includes as particular cases of various classical functional responses (see [2-4]). Obviously, the following functional responses satisfy the above assumptions (i)- (iii):

$F(y_1, y_2) = by_1, b > 0$, (Holling type I).

$F(y_1, y_2) = \dfrac{by_1}{1 + my_1}, b > 0, m > 0$, (Holling type II).

$F(y_1, y_2) = \dfrac{by_1^2}{1 + my_1^2}, b > 0, m > 0$, (Holling type III).

$F(y_1, y_2) = k(1 - e^{-by_1}), b > 0, k > 0$, (Ivlev functional response).

$F(y_1, y_2) = \dfrac{by_1^n}{y_2^m + ly_1^n}, n, b, l > 0, 0 < m \le 1$, (Hassell-Valery).

$F(y_1, y_2) = \dfrac{by_1}{a + y_1 + ly_2}, a, b, l > 0$, (DeAngelis-Beddington).

In [1], Apreutesei assumed that the functional response $F(y_1, y_2)$ satisfies the condition (1.2) of (iii), that is $\dfrac{\partial F(y_1, y_2)}{\partial y_1} > 0$. It implies that, as the prey population increases, the consumption rate of prey per predator increases. But some experiments and observations indicate that a non-monotonic response occurs at this level: when the nutrient concentration reaches a high level an inhibitory effect on the specific growth rate may occur. To model such an inhibitory effect, Andrews[5] suggested a function

$$\varphi(x) = \frac{mx}{a + bx + x^2},$$

called the Monod-Haldane function, and also called a Holling type-IV function. Sokol and Howell[6] proposed a simplified Holling Type-IV function of the form

$$\varphi(x) = \frac{mx}{a + x^2}.$$

Motivated by the above question: a functional response may be monotonic or non-monotonic, we reconsider the predator-prey model with general nonlinear functional response (1.1) without condition (1.2).

As same as the method used in [1], one may separate the prey from the predators with the aid of a control function $u : [0, T] \to R$, $0 \le u(t) \le 1$ a.e. on $[0, T]$. Then the functional response $F(y_1, y_2)$ is multiplied by u. The separation rate at the moment t is $1 - u(t)$. If $u(t) = 0$, then prey and predators are completely separated from each other at the moment t; if $u(t) = 1$, then they are not separated at all, that is the ecosystem coincides with the original one. Similarly, one separates the prey individuals from each other by a control function $v : [0, T] \to R$. Then the second term of the growth rate of the prey population will be multiplied by v. Suppose that the prey individuals cannot be completely isolated from each other, i.e. $v(t) > 0$. More exactly, assume that $0 < v_0 \le v(t) \le 1$ a.e. on $[0, T]$, where v_0 is a fixed value in $(0, 1)$. The control functions represent the rate of mixture of the populations: u is the rate of mixture between prey and predators, while v is the rate of mixture between prey individuals. The dynamics of the controlled ecosystem (1.1) is given by

$$\begin{cases} \dfrac{dy_1}{dt} = ry_1(1 - \dfrac{vy_1}{k}) - uy_2 F(y_1, y_2), \ t \in [0, T], \\ \dfrac{dy_2}{dt} = y_2[-d + cuF(y_1, y_2)], \qquad t \in [0, T]. \end{cases} \tag{1.4}$$

Initial value conditions of the form

$$y_1(0) = y_1^0 > 0, y_2(0) = y_2^0 > 0 \tag{1.5}$$

are associated with system (1.4). Here the predator functional response $F(y_1, y_2)$ is assumed to satisfy the following assumptions:

(1) F is well defined and $F(y_1, y_2) > 0$, for all $y_1, y_2 > 0$.

(2) $F \in C^1([0, +\infty)^2)$; F is bounded with respect to y_2.

(3) $F(y_1, y_2) + y_2 \dfrac{\partial F(y_1, y_2)}{\partial y_2} > 0$, for all $y_1, y_2 > 0$.

Assumptions (1)-(3) assure the existence and uniqueness of a local solution $y = (y_1, y_2)$ of problem (1.4) and (1.5), defined on a maximal interval $[0, \delta), \delta > 0$. Since system (1.4) admits the zero solution and $y_1^0 > 0, y_2^0 > 0$, it follows by a comparison theorem that $y_1 > 0, y_2 > 0$ on $[0, \delta)$. Condition (1) implies the boundedness of y_1 and y_2 and, consequently, the solution of (1.4) and (1.5) is defined on the whole of $[0, T]$.

The goal of the work is to find the optimal control (u, v) such that, at the end of the time interval $[0, T]$, the total density of the two populations is maximal. The optimal control problem associated with system (1.4) and (1.5) is

$$\min \left\{ -y_1(T) - y_2(T) \right\}, \tag{1.6}$$

where $u : [0, T] \to [0, 1]$, $v : [0, T] \to [v_0, 1]$, and (y_1, y_2) verifies (1.4) and (1.5).

There is a vast literature on optimality conditions for biology control problems. Basic results on the optimal control theory can be found in [7-9]. The aim of this paper is, by further developing the analysis technique of [1], to discuss that whether or not the functional response is monotonic has influence on the control problem. Section 2 of the present work is devoted to the maximum principle for our problem. One finds that the control variable u is bang-bang, while v is v_0 or 1 on the entire interval $[0, T]$. In section 3, we establish the number of switching points of u which is determined by the threshold value c. It is shown that general monotonic or non-monotonic functional response has no influence on the number of switching points of the optimal control. In section 4, we give some conclusions which describe conclusion and other innovation in this paper compared to previous literature.

2   The Maximum Principle

The boundedness of the solution y of the control system (1.4) and (1.5) permits us to take a compact target set at $t = T$. Then, according to [7, Theorem 1.2, pp. 43], it follows that our optimal control problem admits at least one solution $(y, (u, v))$, where $y = (y_1, y_2)$.

We apply Pontryagin's maximum principle to find the form of the optimal control (u, v) for problem (1.4)-(1.6). To this end, we associate the Hamiltonian function

$$H(y, p, u, v) = r y_1 p_1 - \frac{r}{k} y_1^2 v p_1 - d y_2 p_2 + u y_2 F(y_1, y_2)(c p_2 - p_1), \tag{2.1}$$

where $p = (p_1, p_2)$, p_1 and p_2 are the adjoint variables. If (u, v) is the optimal control, $y = (y_1, y_2)$ is the optimal state, then p_1 and p_2 verify the adjoint system

$$\begin{cases} \dfrac{dp_1}{dt} = -r p_1 + \dfrac{2 v r y_1}{k} p_1 - u y_2 \dfrac{\partial F(y_1, y_2)}{\partial y_1}(c p_2 - p_1), \\ \dfrac{dp_2}{dt} = d p_2 - u(c p_2 - p_1)[F(y_1, y_2) + y_2 \dfrac{\partial F(y_1, y_2)}{\partial y_2}], \end{cases} \tag{2.2}$$

and the transversality condition

$$p_1(T) = p_2(T) = 1. \tag{2.3}$$

The optimal control (u, v) should maximize the Hamiltonian function H for fixed y_1, y_2, p_1, p_2. In the sequel, we will discuss the form of the optimal variables u and v. Now, from hypothesis (1), we can easily obtain the following form of the optimal variables u and v:

$$u(t) = \begin{cases} 1, & (c p_2 - p_1)(t) \geq 0, \\ 0, & (c p_2 - p_1)(t) < 0, \end{cases} \quad v(t) = \begin{cases} v_0, & p_1(t) \geq 0, \\ 1, & p_1(t) < 0. \end{cases} \tag{2.4}$$

3   The Number of Switching Points

In section 3, we will discuss the form of u and v which is determined by the threshold value c. Firstly, we show a result for the sign of $p_1(t)$ and $p_2(t)$.

Lemma 3.1. For system (2.2), $p_1(t) \geq 0$ and $p_2(t) > 0$, $\forall t \in [0, T]$.

Proof. From (2.4), we can easily obtain that

$$u(t)(p_1 - c p_2)(t) \leq 0, a.e.on[0, T]. \tag{3.1}$$

Regarding the second equation from (2.2) as a linear equation in p_2 of the form $p_2' = -\gamma(t)p_2 + \alpha(t)$, with $\gamma(t) = -d$, $\alpha(t) = -u(c p_2 - p_1)[F(y_1, y_2) + y_2 \frac{\partial F(y_1, y_2)}{\partial y_2}]$, and the end-point value $p_2(T) = 1$, we can write

$$p_2(t) = e^{-d(T-t)} \left\{ 1 - \int_t^T [u(p_1 - c p_2)(F(y_1, y_2) + y_2 \frac{\partial F(y_1, y_2)}{\partial y_2})](s) e^{d(T-s)} ds \right\}. \tag{3.2}$$

Using hypothesis (3), together with the estimates (3.1), one observes that $p_2(t) > 0, \forall t \in [0, T]$.

Similarly, if $p_1(t) \neq 0$, it follows from the first equation of (2.2) that $p_1' = -\eta(t)p_1$, where $\eta(t) = r - v\frac{2ry_1}{k} - uy_2\frac{\partial F(y_1, y_2)}{\partial y_1} - (cp_2 uy_2 \frac{\partial F(y_1, y_2)}{\partial y_1})/p_1$. From $p_1' = -\eta(t)p_1$, and the end-point value $p_1(T) = 1$, we can get

$$p_1(t) = e^{\int_t^T \eta(s)ds} > 0. \tag{3.3}$$

Now, we will discuss the form of the optimal variables u and v. From Lemma 3.1, it follows that $p_1(t) \geq 0$. Therefore, $v(t) = v_0, \forall t \in [0, T]$.

To establish the number of switching points of u, we analyze the sign of $p_1 - cp_2$. One specifies three cases according to the sign of $c - 1$. Recall that $p_1 \geq 0, p_2 > 0$ on $[0, T]$ (see Lemma 3.1). Using system (2.2) and (2.3) with $v = v_0$ on $[0, T]$, one finds that for any $t \in [0, T]$

$$\begin{aligned}(cp_2 - p_1)' &= cdp_2 + rp_1 - v_0\frac{2ry_1p_1}{k} \\ &\quad - u(cp_2 - p_1)[cF(y_1, y_2) + cy_2\frac{\partial F(y_1, y_2)}{\partial y_2} - y_2\frac{\partial F(y_1, y_2)}{\partial y_1}].\end{aligned} \tag{3.4}$$

Define $y_1^{max} = \max_{t \in [0, T]} y_1(t)$. In the sequel, we choose v_0 such that

$$0 < v_0 < \min\left\{1, \frac{k}{2y_1^{max}}\right\}. \tag{3.5}$$

Case 1. $c < 1$. From (2.3), we have $p_1(T) = p_2(T) = 1$. Therefore, $(cp_2 - p_1)(T) = c - 1 < 0$, it follows that $cp_2 - p_1 < 0$ in a neighborhood $(\tau, T]$ of T. Suppose it is maximal with respect to this property. If $\tau \in (0, T)$, then $(cp_2 - p_1)(\tau) = 0$ and the optimal control u is 0 on $(\tau, T]$. Equalities (3.4) and (3.5) lead to

$$(cp_2 - p_1)'|_{t=\tau} = cdp_2(\tau) + rp_1(\tau)(1 - v_0\frac{2y_1(\tau)}{k}) > 0, \tag{3.6}$$

i.e. $cp_2 - p_1$ is monotonically increasing in a neighborhood of τ, and $(cp_2 - p_1)(t) < 0, t \in (\tau, T]$. Therefore, $(cp_2 - p_1)(\tau) < 0$. This contradicts the condition $(cp_2 - p_1)(\tau) = 0$. Hence $\tau = 0$ and $u(t) = 0$, for every $t \in [0, T]$.

Case 2. $c = 1$. From (2.3), we have $p_1(T) = p_2(T) = 1$. Therefore, $(cp_2 - p_1)(T) = c - 1 = 0$. By the similar proof of (3.6), we have $(cp_2 - p_1)'|_{t=T} = cdp_2(T) + rp_1(T)(1 - v_0\frac{2y_1(T)}{k}) > 0$. It follows that $cp_2 - p_1$ is monotonically increasing in a left neighborhood of T. As in Case 1, we infer again that $u(t) = 0$ for each $t \in [0, T]$.

Case 3. $c > 1$. From (2.3), we have $p_1(T) = p_2(T) = 1$. In this case, $(cp_2 - p_1)(T) = c - 1 > 0$, so $cp_2 - p_1 > 0$ in a neighborhood $(\tau, T]$ of T, which can be chosen maximal. Then $u(t) = 1$ on $(\tau, T]$. There are two subcases:

(a)If $\tau = 0$, then $u(t) = 1$ on the whole interval $[0, T]$.

(b)If $\tau \in (0, T)$, then $cp_2 - p_1 > 0$ on $(\tau, T]$ and $(cp_2 - p_1)(\tau) = 0$. With the aid of (3.4)and by the similar proof of (3.6), we can obtain that the function $cp_2 - p_1$ is increasing in τ, that is $cp_2 - p_1 < 0$ at least in a left neighborhood of τ. According to (2.4), here $u(t) = 0$. As long as there exists a τ' such that $(cp_2 - p_1)(\tau') = 0$, function $cp_2 - p_1$ is increasing in a neighborhood of τ', so we can repeat the reasoning of Case 2 with τ' instead of T, to deduce that $u(t) = 0$, for all $t \in [0, \tau)$. Therefore in Case 3, the optimal control u either equals 1 on $[0, T]$, or has a unique switching time $\tau \in (0, T)$.

From the above discussion, we can easily obtain the following result.

Theorem 3.1. Assume that (3.5) holds and function F satisfies assumptions(1)-(3). If (u, v) is the optimal control for problem(1.6), then $v = v_0$ on $[0, T]$ and u is bang-bang, namely it has at most one switching time. More exactly, we can easily obtain the following cases:

(1)If $c \leq 1$, then $u(t) = 0, \forall t \in [0, T]$.

(2)If $c > 1$, then u admits at most one switching time τ, which is the solution in $(0, T)$ of the equation $cp_2 - p_1 = 0$. Here $p = (p_1, p_2)$ is the solution of the adjoint system (2.2) and (2.3). If equation $cp_2 - p_1 = 0$ has no solution in $(0, T)$, then $u(t) = 1, \forall t \in [0, T]$. If equation $cp_2 - p_1 = 0$ has a unique solution τ in $(0, T)$, then u has the form

$$u(t) = \begin{cases} 0, t \in [0, \tau), \\ 1, t \in [\tau, T]. \end{cases}$$

4   Conclusion

In [1], Apreutesei found necessary optimality conditions for a problem related to the prey-predator system (1.1). Under the assumptions (i)-(iii), he obtained that if (u, v) is the optimal control, then $v = v_0$ on $[0, T]$ and u is bang-bang, namely it has at most one switching time. However, we reconsider necessary optimality conditions for the prey-predator system (1.1) without assumption (1.2), that is the functional response $F(y_1, y_2)$ for system (1.4) may be monotonic or non-monotonic. It is shown that general monotonic or non-monotonic functional response has no influence on the number of switching points of the optimal control.

From Theorem 3.1, we can see that the number of switching points of the optimal control is determined by the threshold value c, which is the rate of translating the prey population into the predator population. It is shown that if $c \leq 1$, then $u(t) = 0$, it follows that prey and predators are not separated from each other at all, that is the ecosystem coincides with the original one. If $c > 1$, then u admits at most one switching time τ, and has the form

$$u(t) = \begin{cases} 0, t \in [0, \tau), \\ 1, t \in [\tau, T]. \end{cases}$$

It follows that prey and predators are completely separated from each other at the moment $t \in [0, \tau)$ and not separated at all for $t \in [\tau, T]$.

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