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Inverse Nodal Problems for Impulsive Sturm-Liouville Equation with Boundary Conditions Depending on the Parameter

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Abstract In this work, the Sturm–Liouville problem with boundary conditions depending rationally on the spectral parameter is studied. We give a uniqueness theorem and algorithm to reconstruct the potential of the problem from nodal points (zeros of eigenfunctions).

Keywords: Sturm-Liouville equation, inverse nodal problem, parameter dependent boundary condition, discontinuity condition.

1 Introduction

We consider the boundary value problem L generated by the regular Sturm–Liouville equation

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in (0,1) \tag{1}$$

subject to the boundary conditions

$$U(y) := a(\lambda)y'(0) - b(\lambda)y(0) = 0 \tag{2}$$
$$V(y) := c(\lambda)y'(1) - d(\lambda)y(1) = 0 \tag{3}$$

and the jump conditions

$$\begin{cases} y(\frac{1}{2}+0) = \alpha y(\frac{1}{2}-0) \\ y'(\frac{1}{2}+0) = \alpha^{-1} y'(\frac{1}{2}-0) \end{cases}, \tag{4}$$

where λ is the spectral parameter; $q(x)$ is a real-valued function from the class $L_2(0,1)$; α is a positive real constant; $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ and $d(\lambda)$ are real polynomials such that

$$a(\lambda) = \sum_{j=0}^{m} a_j \lambda^j,\ b(\lambda) = \sum_{j=0}^{m} b_j \lambda^j,$$
$$c(\lambda) = \sum_{j=0}^{r} c_j \lambda^j,\ d(\lambda) = \sum_{j=0}^{r} d_j \lambda^j,$$

Without loss of generality, we assume that $a_m = c_r = 1$ and $\int_0^1 q(x)\mathrm{d}x = 0$, and define $f = \dfrac{a(\lambda)}{b(\lambda)}$.

The values of the parameter λ for which L has nonzero solutions, are called eigenvalues and the corresponding nontrivial solutions are called eigenfunctions.

Spectral problems for various differential equation with the eigen-dependent-boundary conditions have been well studied. Inverse problems for the special case when f is an affine function on λ were solved in [11]. The case when f is a more general rational function of λ is difficult. In [1]-[4], [8], [16], [13], [19] and [23], various spectral problems with rational conditions were studied.

Inverse spectral problems for Sturm-Liouville operator with the discontinuity conditions, like (4), were studied in [7], [12] and references therein.

The inverse nodal problem, which is different from the classical inverse spectral theory of Gelfand and Levitan [10], was initiated by McLaughlin [15]. Later, Hald and McLaughlin [13] and Browne and

Sleeman [5] proved that it is sufficient to know the nodal points to uniquely determine the potential function of the regular Sturm–Liouville problem. Yang gave an algorithm to recover q from dense subset of nodal points[20]. Recently, the inverse nodal Sturm–Liouville problems has been investigated by several authors [5], [6], [13], [15], [17], [18], [21] and [22].

In the present paper, we investigate an impulsive Sturm-Liouville operator and give a uniqueness theorem to reconstruct the potential of the problem from nodal points.

2 Preliminaries

Let $\varphi(x,\lambda)$ be the solution of (1), satisfying the initial conditions

$$\varphi(0,\lambda)=a(\lambda),\quad \varphi'(0,\lambda)=b(\lambda) \tag{5}$$

and the jump conditions (4). Moreover, the following integral equations of the solution hold for $x<\frac{1}{2}$

$$\begin{aligned}\varphi(x,\lambda)&=a(\lambda)\cos\sqrt{\lambda}x+b(\lambda)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\\&+\int_0^x\frac{\sin\sqrt{\lambda}\,(x-t)}{\sqrt{\lambda}}q(t)\varphi(t,\lambda)\mathrm{d}t,\end{aligned} \tag{6}$$

for $x>\frac{1}{2}$

$$\begin{aligned}\varphi(x,\lambda)=\alpha^+&\left[a(\lambda)\cos\sqrt{\lambda}x+b(\lambda)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right]\\&+\alpha^-\left[a(\lambda)\cos\sqrt{\lambda}(1-x)+b(\lambda)\frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}}\right]\\&+\int_0^{1/2}\left[\alpha^+\frac{\sin\sqrt{\lambda}\,(x-t)}{\sqrt{\lambda}}+\alpha^-\frac{\sin\sqrt{\lambda}\,(1-x-t)}{\sqrt{\lambda}}\right]q(t)\varphi(t,\lambda)\mathrm{d}t\\&+\int_{1/2}^x\frac{\sin\sqrt{\lambda}\,(x-t)}{\sqrt{\lambda}}q(t)\varphi(t,\lambda)\mathrm{d}t\end{aligned} \tag{7}$$

where $\alpha^{\pm}=\dfrac{1}{2}\left(\alpha\pm\dfrac{1}{\alpha}\right)$. Using these equations, we prove that the following asymptotic relations are valid for $|\lambda|\to\infty$,
for $x<\frac{1}{2}$

$$\varphi(x,\lambda)=\lambda^m\left\{\cos\sqrt{\lambda}x+\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\left(b_m+\frac{1}{2}\int_0^x q(t)\mathrm{d}t\right)+o\left(\frac{1}{\sqrt{\lambda}}\exp\tau x\right)\right\}, \tag{8}$$

for $x>\frac{1}{2}$

$$\begin{aligned}\varphi(x,\lambda)&=\lambda^m\left\{\alpha^+\cos\sqrt{\lambda}x+\alpha^-\cos\sqrt{\lambda}\,(1-x)\right\}+\\&+\lambda^{m-\frac{1}{2}}\left\{\alpha^+I_1(x)\sin\sqrt{\lambda}x+\alpha^-I_2(x)\sin\sqrt{\lambda}\,(1-x)\right\}\\&+o\left(\lambda^{m-\frac{1}{2}}\exp\tau x\right)\end{aligned} \tag{9}$$

where

$$\begin{aligned}I_1(x)&=b_m+\frac{1}{2}\int_0^x q(t)\mathrm{d}t,\\I_2(x)&=b_m+\frac{1}{2}\int_0^{1/2}q(t)\mathrm{d}t-\frac{1}{2}\int_{1/2}^x q(t)\mathrm{d}t.\end{aligned}$$

and $\tau = \left| Im\sqrt{\lambda} \right|$.

Consider the function

$$\Delta(\lambda) := c(\lambda)\varphi'(1,\lambda) - d(\lambda)\varphi(1,\lambda). \tag{10}$$

$\Delta(\lambda)$ is called characteristic function of the problem L. It is obvious that $\Delta(\lambda)$ is an entire function and its zeros, namely $\{\lambda_n\}_{n\geq 0}$, are eigenvalues of the problem L. Moreover, the following relation holds.

$$\Delta(\lambda) = -\alpha^+\lambda^{m+r}\left\{\sqrt{\lambda}\sin\sqrt{\lambda} - w_1\cos\sqrt{\lambda} + w_2 + o\left(\exp\tau\right)\right\}. \tag{11}$$

It can be shown using classical methods in the similar studies that the sequence $\{\lambda_n\}_{n\geq 0}$ satisfies the following asymptotic relation for $n \to \infty$:

$$\sqrt{\lambda_n} = (n-m-r)\pi + \frac{(w_1 - (-1)^{n-m-r}w_2)}{(n-m-r)\pi} + o(\frac{1}{n}) \tag{12}$$

where $w_1 = I_1(1) - dr$ and $w_2 = \dfrac{\alpha^-}{\alpha^+}\left(I_2(1) + d_r\right)$.

Let $\varphi(x,\lambda_n)$ be the eigenfunction corresponding to the eigenvalue λ_n.

Lemma 2.1. $\varphi(x,\lambda_n)$ has exactly $n-m-r$ nodes $\left\{x_n^j : j = \overline{0, n-m-r-1}\right\}$ in $(0,1)$ for sufficiently large n. The numbers $\left\{x_n^j\right\}$ satisfy the following asymptotic formulae

for $x_n^j \in (0, \dfrac{1}{2})$

$$x_n^j = \begin{cases} \frac{(j+1/2)}{n-m-r} + \frac{I_1(x_n^j)}{(n-m-r)^2\pi^2} - \frac{(w_1-w_2)}{(n-m-r)^2\pi^2}\frac{(j+1/2)}{n-m-r} \\ \qquad +o\left(\frac{1}{n^2}\right), \text{ for } n-m-r = 2k \\ \frac{(j+1/2)}{n-m-r} + \frac{I_1(x_n^j)}{(n-m-r)^2\pi^2} - \frac{(w_1+w_2)}{(n-m-r)^2\pi^2}\frac{(j+1/2)}{n-m-r} \\ \qquad +o\left(\frac{1}{n^2}\right), \quad \text{for } n-m-r = 2k+1 \end{cases} \tag{13}$$

and for $x_n^j \in (\dfrac{1}{2}, 1)$

$$x_n^j = \begin{cases} \frac{(j+1/2)}{n-m-r} + \frac{w_1-w_2}{(n-m-r)^2\pi^2}\frac{(j+1/2)}{n-m-r} + \frac{1}{2(n-m-r)^2\pi^2}\int_0^x q(t)\mathrm{d}t \\ \qquad +\frac{\rho_0}{(n-m-r)^2\pi^2} + o\left(\frac{1}{n^2}\right), \quad \text{for } n-m-r = 2k \\ \frac{(j+1/2)}{n-m-r} + \frac{w_1+w_2}{(n-m-r)^2\pi^2}\frac{(j+1/2)}{n-m-r} + \frac{1}{2(n-m-r)^2\pi^2}\int_0^x q(t)\mathrm{d}t \\ \qquad +\frac{\rho_1}{(n-m-r)^2\pi^2} + o\left(\frac{1}{n^2}\right), \quad \text{for } n-m-r = 2k+1 \end{cases} \tag{14}$$

where

$$\rho_0 = \frac{\alpha^-}{2\alpha^+}\left(\int_{1/2}^1 q(t)\mathrm{d}t - \int_0^{1/2} q(t)\mathrm{d}t - 2d_r\right) + \frac{\alpha^+ - \alpha^-}{\alpha^+}b_m,$$

$$\rho_1 = \frac{\alpha^-}{2\alpha^+}\left(\frac{\alpha^+ + \alpha^-}{\alpha^+ - \alpha^-}\right)\left(\int_0^{1/2} q(t)\mathrm{d}t - \int_{1/2}^1 q(t)\mathrm{d}t + 2d_r\right) + \frac{\left(\alpha^+\right)^2 - \left(\alpha^-\right)^2}{\left(\alpha^+ - \alpha^-\right)\alpha^+}b_m.$$

Proof. It can be seen from (8), (9) and oscilation theorem that the function $\varphi(x,\lambda_n)$ has exactly $n-m-r$ zeros in the interval $(0,1)$ for sufficiently large n. Using (8) and (9) again, we get the following asymptotic formulae

$$\varphi(x,\lambda_n) = \lambda_n^m\left\{\cos\sqrt{\lambda_n}x + \frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}I_1(x) + o\left(\frac{\exp\tau_n x}{\sqrt{\lambda_n}}\right)\right\} \quad \text{for } x < \frac{1}{2},$$

$$\begin{aligned} \varphi(x,\lambda_n) &= \lambda_n^m\left\{\alpha^+\cos\sqrt{\lambda_n}x + \alpha^-\cos\sqrt{\lambda_n}\left(1-x\right)\right\} \\ &+\lambda^{m-\frac{1}{2}}\left\{\alpha^+ I_1(x)\sin\sqrt{\lambda_n}x + \alpha^- I_2(x)\sin\sqrt{\lambda_n}\left(1-x\right)\right\} \\ &+o\left(\lambda^{m-\frac{1}{2}}\exp\tau_n x\right) \quad \text{for } \quad x > \tfrac{1}{2}. \end{aligned} \tag{15}$$

From

$$\begin{aligned}
0 &= \varphi(x_n^j, \lambda_n) \\
&= \lambda_n^m \left\{ \alpha^+ \cos\sqrt{\lambda_n} x_n^j + \alpha^- \cos\sqrt{\lambda_n}\left(1 - x_n^j\right) \right\} \\
&+ \lambda^{m-\frac{1}{2}} \left\{ \alpha^+ I_1(x_n^j) \sin\sqrt{\lambda_n} x_n^j + \alpha^- I_2(x_n^j) \sin\sqrt{\lambda_n}\left(1 - x_n^j\right) \right\} + o\left(\lambda^{m-\frac{1}{2}} \exp \tau_n x_n^j\right),
\end{aligned}$$

we get

$$\begin{aligned}
&\tan(\sqrt{\lambda_n} x - \tfrac{\pi}{2}) \\
&= \frac{(-1)^{n-m-r}(w_1 - (-1)^{n-m-r} w_2)\alpha^- + \alpha^+ I_1(x) - (-1)^{n-m-r}\alpha^- I_2(x)}{\left(\alpha^+ + (-1)^{n-m-r}\alpha^-\right)(n-m-r)\pi} + o\left(\frac{1}{n}\right),
\end{aligned}$$

for $x_n^j > \frac{1}{2}$. Taylor's formula for the arctangent yields

$$\begin{aligned}
x_n^j &= \frac{(j+1/2)}{n-m-r} + \frac{w_1 - (-1)^{n-m-r} w_2}{(n-m-r)^2 \pi^2} \frac{(j+1/2)}{n-m-r} \\
&+ \frac{(-1)^{n-m-r}(w_1 - (-1)^{n-m-r} w_2)\alpha^- + \alpha^+ I_1(x_n^j) - (-1)^{n-m-r}\alpha^- I_2(x_n^j)}{\left(\alpha^+ + (-1)^{n-m-r}\alpha^-\right)(n-m-r)^2 \pi^2} + o\left(\frac{1}{n^2}\right).
\end{aligned}$$

The last equality is the proof of (14). The equation (13) can be proved similarly.

Let $X = X_0 \cup X_1$ be the set of nodal points such that $X_0 = \left\{x_n^j : n-m-r = 2s,\ s \in \mathbb{Z}\right\}$, $X_1 = \left\{x_n^j : n-m-r = 2s+1,\ s \in \mathbb{Z}\right\}$. For each fixed $x \in [0,1]$ and $k \in \{0,1\}$, there exists a sequence $\left(x_n^{j(n)}\right) \subset X_k$ which converges to x. Therefore, from Lemma 2.1, we can show the following limits ~~are~~ exist and finite

$$\lim_{n\to\infty} (n-m-r)^2 \pi^2 \left(x_n^{j(n)} - \frac{\left(j(n) + \frac{1}{2}\right)\pi}{n-m-r} \right) = f_k(x), \tag{16}$$

where

$$f_k(x) = \begin{cases} \dfrac{1}{2}\int_0^x q(t)\mathrm{d}t - \left(w_1 - (-1)^k w_2\right)x + b_m \text{ for } x < \dfrac{1}{2}, \\ \dfrac{1}{2}\int_0^x q(t)\mathrm{d}t - \left(w_1 - (-1)^k w_2\right)x + \rho_k \text{ for } x > \dfrac{1}{2}. \end{cases}$$

Theorem 2.2. The given nodal sets X_0 or X_1 uniquely determine the potential $q(x)$, a.e. on $(0,1)$ and the coefficients b_m and d_r of the boundary conditions. The potential $q(x)$ and the constants b_m and d_r can be constructed by the following formulae:

1- For each fixed $x \in [0,1]$, choose a sequence $\left(x_n^{j(n)}\right) \subset X$ such that $\lim\limits_{n\to\infty} x_n^{j(n)} = x$;

2- Find the function $f_k(x)$ from the equation (16) and calculate

$$q(x) = 2\left[f_k'(x) - f_k(1) + f_k(0) + f_k(\frac{1}{2}+0) - f_k(\frac{1}{2}-0)\right] \tag{17}$$

$$b_m = f_k(0) \tag{18}$$

$$\begin{aligned}
d_r &= f_k(1) - f_k(\frac{1}{2}+0) + f_k(\frac{1}{2}-0) \\
&\quad - (-1)^k \frac{\alpha^-}{\alpha^+}\left(b_m + \int_0^{1/2} q(t)\mathrm{d}t\right)
\end{aligned} \tag{19}$$

Proof. Direct calculations in (13), (14) and (16) yield

$$\begin{aligned} b_m &= f_k(0), \\ q(x) &= 2\left[f_k'(x) - \left(w_1 - (-1)^k w_2\right)\right], \\ w_1 - (-1)^k w_2 &= f_k(1) - c_k \\ c_k &= f_k(\frac{1}{2}+0) - f_k(\frac{1}{2}-0) + b_m, \\ d_r &= f_k(1) - f_k(\frac{1}{2}+0) + f_k(\frac{1}{2}-0) + \\ &\qquad - (-1)^k \frac{\alpha^-}{\alpha^+}\left(b_m + \int\limits_0^{1/2} q(t)\mathrm{d}t\right) \end{aligned}$$

This completes the proof.

Example 2.3. Consider the BVP

$$L: \begin{cases} \ell y := -y'' + q(x)y = \lambda y, \quad x \in (0,1), \\ a(\lambda)y'(0) - b(\lambda)y(0) = 0, \\ c(\lambda)y'(1) - d(\lambda)y(1) = 0, \\ y(\frac{1}{2}+0) = \alpha y(\frac{1}{2}-0), \\ y'(\frac{1}{2}+0) = \alpha^{-1} y'(\frac{1}{2}-0) \end{cases}$$

where $q(x) \in L_2(0,1)$ and $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ and $d(\lambda)$ are unknown coefficients of the problem L. Let $\Omega = \left\{x_n^j\right\} \subset X_0$ be the dense subset of nodal points in $(0,1)$ satisfies the following asimptotics

If $x_n^j \in \left(0, \frac{1}{2}\right)$,

$$x_n^j = \frac{(j+1/2)}{n-m-r} + \frac{2+\sin\pi(\frac{j+1/2}{n-m-r})}{2\left(n-m-r\right)^2\pi^2} +$$

$$+\frac{2\alpha^-}{\alpha^+\left(n-m-r\right)^2\pi^2}\frac{(j+1/2)}{n-m-r} + o\left(\tfrac{1}{n^2}\right),$$

If $x_n^j \in \left(\frac{1}{2}, 1\right)$,

$$x_n^j = \frac{(j+1/2)}{n-m-r} + \frac{\sin\pi(\frac{j+1/2}{n-m-r})}{2\left(n-m-r\right)^2\pi^2}$$

$$+\frac{2\alpha^-}{\alpha^+\left(n-m-r\right)^2\pi^2}\frac{(j+1/2)}{n-m-r} + \frac{1-\frac{3\alpha^-}{\alpha^+}}{\left(n-m-r\right)^2\pi^2} + o\left(\tfrac{1}{n^2}\right).$$

It can be calculated that

$$f_0(x) = \begin{cases} 1 + \dfrac{1}{2}\sin\pi x + \dfrac{2\alpha^-}{\alpha^+}x, \text{ for } x < \dfrac{1}{2}, \\ \dfrac{1}{2}\sin\pi x + \dfrac{2\alpha^-}{\alpha^+}x + 1 - \dfrac{3\alpha^-}{\alpha^+} \text{ for } x > \dfrac{1}{2}, \end{cases}$$

$$\begin{aligned}
q(x) &= 2\left[f_0'(x) - f_0(1) + f_0(0) + f_0(\frac{1}{2}+0) - f_0(\frac{1}{2}-0)\right] \\
&= \pi\cos\pi x, \\
b_m &= f_0(0) = 1, \\
d_r &= f_0(1) - f_0(\frac{1}{2}+0) + f_0(\frac{1}{2}-0) \\
&\qquad\qquad -\frac{\alpha^-}{\alpha^+}\left(b_m + \int_0^{1/2} q(t)\mathrm{d}t\right) = 1.
\end{aligned}$$

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