A New Type of Tripled Fixed Point Theorem in Partially Ordered Complete Metric Space

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Abstract In this paper, we introduce a new type of tripled fixed point theorem in partially ordered complete metric space. We give an example to support our result.

Keywords: Tripled fixed point, partially ordered set, mixed monotone mappings.

1 Introduction

Fixed point theory in recent years has developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. The first result in this direction was obtained by Ran and Reurings [6]. They presented some applications of results of matrix equations. In [3], Nieto and Lopez extended the result of Ran and Reurings [4], for non decreasing mappings and applied their result to get a unique solution for a first order differential equation. Agrawal et al. [1] and O'Regan and Petrutel [5] studied some results for generalized contractions in ordered metric spaces.

Berinde and Borcut [2] introduced the concept of triple fixed point and proved some related fixed point theorem. After that various results on tripled fixed point have been obtained. The following definitions are from [2].

Definition 1. *Let $(X, \preceq)$ be a partially ordered set, $F : X^3 \to X$ mapping. The mapping F is said to have the mixed monotone property if for any $x, y, z \in X$,*

(i) $x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z)$,
(ii) $y_1, y_2 \in X, \quad y_1 \succeq y_2 \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z)$,
(iii) $z_1, z_2 \in X, \quad z_1 \preceq z_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2)$.

Definition 2. *An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if*

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad \text{and} \quad F(z, y, x) = z.$$

Definition 3. *Let $(X, \preceq)$ be a partially ordered set, $F : X^3 \to X$ and $g : X \to X$ two mappings. The mapping F is said to have the mixed $g-$ monotone property if for any $x, y, z \in X$.*

i. $x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z)$,
ii. $y_1, y_2 \in X, \quad g(y_1) \succeq g(y_2) \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z)$,
iii. $z_1, z_2 \in X, \quad g(z_1) \preceq g(z_2) \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2)$.

Definition 4. *An element $(x, y, z) \in X^3$ is called a tripled coincidence point of the mappings $F : X^3 \to X$ and $g : X \to X$ if*

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \quad \text{and} \quad F(z, y, x) = gz.$$

Definition 5. *An element $(x, y, z) \in X^3$ is called a tripled common fixed point of the mappings $F : X^3 \to X$ and $g : X \to X$ if*

$$F(x, y, z) = gx = x, \quad F(y, x, y) = gy = y \quad \text{and} \quad F(z, y, x) = gz = z.$$

Definition 6. *An element $x \in X$ is called a common fixed point of the mappings $F : X^3 \to X$ and $g : X \to X$ if*

$$F(x, x, x) = gx = x.$$

Definition 7. *Let X be a non empty set. The mappings $F : X^3 \to X$ and $g : X \to X$ are commuting if for all $x, y, z \in X$,*

$$g(F(x, y, z)) = F(g(x), g(y), g(z)).$$

Definition 8. *Let (X, d) be a metric space. The mappings F and g where $F : X^3 \to X$ and $g : X \to X$ are said to be compatible if*

$$\lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n))) = 0$$

$$\lim_{n\to\infty} d(g(F(y_n, x_n, y_n)), F(g(y_n), g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n))) = 0$$

whenever $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n, y_n) = \lim_{n\to\infty} g(y_n) = y$ and $\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} g(z_n) = z$ for some $x, y, z \in X$.

In [2] Berinde and Borcut proved the following theorem.

Theorem 9. *Let $(X, \preceq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist constants $a, b, c \in [0, 1)$ such that $a + b + c \prec 1$ for which,*

$$d(F(x, y, z), F(u, v, w)) \preceq ad(x, u) + bd(y, v) + cd(z, w) \tag{1.1}$$

For all $x \succeq u, y \preceq v, z \succeq w$. Assume either,

1. *F is continuous,*
2. *X has the following properties:*
   (a) *if non decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
   (b) *if non increasing sequence $y_n \to y$, then $y_n \succeq x$ for all n,*

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0, z_0), \quad y_0 \succeq F(y_0, x_0, y_0), \quad \textit{and} \quad z_0 \preceq F(z_0, y_0, x_0)$$

Then there exist $x, y, z \in X$ such that,

$$F(x,y,z) = x,\ F(y,x,y) = y, \text{ and } F(z,y,x) = z$$

Inspired by above works, we derive new tripled fixed point theorems for mapping having the mixed monotone property $F : X \times X \times X \to X$ in partially ordered metric space and we give an example to support our result.

Theorem 10. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that F satisfies the following condition:*

$$d(F(x, y, z), F(u, v, w)) \preceq \delta(x, y, z, u, v, w)[d(x, u) + d(y, v) + d(z, w)]. \tag{1.2}$$

where

$$\delta(x, y, z, u, v, w) = \frac{\begin{pmatrix} d(x, F(u, v, w)) + d(y, F(v, u, v)) + d(z, F(w, v, u)) \\ +d(u, F(x, y, z)) + d(v, F(y, x, y)) + d(w, F(z, y, x)) \end{pmatrix}}{1 + 3\begin{pmatrix} d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) \\ +d(u, F(u, v, w)) + d(v, F(v, u, v)) + d(w, F(w, v, u)) \end{pmatrix}}$$

for all $x, y, z, u, v, w \in X$ with $x \preceq u$, $y \preceq v$ and $z \preceq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0, z_0), y_0 \preceq F(y_0, x_0, y_0) \quad \textit{and} \quad z_0 \preceq F(z_0, y_0, x_0)$$

then

a) F has at least a tripled fixed point there exist $(x, y, z) \in X$ such that

$$x = F(x, y, z), y = F(y, x, y) \quad \text{and} \quad z = F(z, y, x).$$

b) if $(x, y, z), (u, v, w)$ are two distinct tripled fixed points of F, then

$$d(x, u) + d(y, v) + d(z, w) \geq \frac{1}{9}.$$

Proof.

Proof of (a). Consider the two sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that,

$$x_{n+1} = F(x_n, y_n, z_n), \quad y_{n+1} = F(y_n, x_n, y_n) \quad \text{and} \quad z_{n+1} = F(z_n, y_n, x_n) \tag{1.3}$$

for all $n = 0, 1, 2, \ldots$. Now, we claim that $\{x_n\}$ is nondecreasing, $\{y_n\}$ is nonincreasing and $\{z_n\}$ is nondecreasing i.e.,

$$x_n \leq x_{n+1}, \quad y_n \geq y_{n+1} \quad \text{and} \quad z_n \leq z_{n+1} \tag{1.4}$$

for all $n = 0, 1, 2, \ldots$. From statement of theorem, we know that $x_0, y_0, z_0 \in X$ with

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq F(z_0, y_0, x_0) \tag{1.5}$$

By using the mixed monotone property of F, we write

$$x_1 = F(x_0, y_0, z_0), \quad y_1 = F(y_0, x_0, y_0) \quad \text{and} \quad z_0 = F(z_0, y_0, x_0). \tag{1.6}$$

Therefore $x_0 \leq x_1$, $y_0 \geq y_1$ and $z_0 \leq z_1$. That is, the inequality 1.4 is true for $n = 0$. Assume $x_n \leq x_{n+1}$, $y_n \geq y_{n+1}$ and $z_n \leq z_{n+1}$ for some n. Now we shall prove that 1.4 is true for $n + 1$. Indeed, from 1.4 and the mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}, z_{n+1}) \geq F(x_n, y_{n+1}, z_{n+1}) \geq F(x_n, y_n, z_n) = x_{n+1}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}, y_{n+1}) \leq F(y_n, x_{n+1}, z_{n+1}) \leq F(y_n, x_n, z_n) = y_{n+1},$$

and

$$z_{n+2} = F(z_{n+1}, y_{n+1}, x_{n+1}) \geq F(z_n, y_{n+1}, x_{n+1}) \geq F(z_n, y_n, x_n) = z_{n+1}.$$

Hence, by induction, $x_n \leq x_{n+1}$, $y_n \geq y_{n+1}$ and $z_n \leq z_{n+1}$ for all n. Since 1.2, $x_{n-1} \leq x_n$,$y_{n-1} \geq y_n$ and $z_{n-1} \leq z_n$, we have

$$\begin{aligned}
& d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\
\preceq & \frac{\begin{pmatrix} d(x_n, F(x_{n-1}, y_{n-1}, z_{n-1})) + d(y_n, F(y_{n-1}, x_{n-1}, y_{n-1})) + d(z_n, F(z_{n-1}, y_{n-1}, x_{n-1})) \\ +d(x_{n-1}, F(x_n, y_n, z_n)) + d(y_{n-1}, F(y_n, x_n, y_n)) + d(z_{n-1}, F(z_n, y_n, x_n)) \end{pmatrix}}{1 + 3\begin{pmatrix} d(x_n, F(x_n, y_n, z_n)) + d(y_n, F(y_n, x_n, y_n)) + d(z_n, F(z_n, y_n, x_n)) + d(x_{n-1}, F(x_{n-1}, y_{n-1}, z_{n-1})) \\ +d(y_{n-1}, F(y_{n-1}, x_{n-1}, y_{n-1})) + d(z_{n-1}, F(z_{n-1}, y_{n-1}, x_{n-1})) \end{pmatrix}} \\
& [d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1})] \\
= & \frac{\begin{pmatrix} d(x_n, x_n) + d(y_n, y_n) + d(z_n, z_n) \\ +d(x_{n-1}, x_{n+1}) + d(y_{n-1}, y_{n+1}) + d(z_{n-1}, z_{n+1}) \end{pmatrix}}{1 + 3\begin{pmatrix} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(x_{n-1}, x_n) \\ +d(y_{n-1}, y_n) + d(z_{n-1}, z_n) \end{pmatrix}} [d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1})] \\
= & \frac{\left(d(x_{n-1}, x_{n+1}) + d(y_{n-1}, y_{n+1}) + d(z_{n-1}, z_{n+1}) \right)}{1 + 3\begin{pmatrix} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(x_{n-1}, x_n) \\ +d(y_{n-1}, y_n) + d(z_{n-1}, z_n) \end{pmatrix}} [d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1})] \\
\leq & \frac{\begin{pmatrix} d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(y_{n-1}, y_n) \\ +d(y_n, y_{n+1}) + d(z_{n-1}, z_n) + d(z_n, z_{n+1}) \end{pmatrix}}{1 + 3\begin{pmatrix} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\ +d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) \end{pmatrix}} [d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1})].
\end{aligned}$$

This implies

$$d(x_{n+1},x_n) \preceq \frac{\begin{pmatrix} d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(y_{n-1},y_n) \\ +d(y_n,y_{n+1})+d(z_{n-1},z_n)+d(z_n,z_{n+1}) \end{pmatrix}}{1+3\begin{pmatrix} d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1}) \\ +d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n) \end{pmatrix}} [d(x_n,x_{n-1})+d(y_n,y_{n-1})+d(z_n,z_{n-1})]. \tag{1.7}$$

Similarly, from 1.2, $y_{n-1} \geq y_n$, $x_{n-1} \leq x_n$ and $z_{n-1} \leq z_n$ we obtain

$$d(y_{n+1},y_n) \preceq \frac{\begin{pmatrix} d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(y_{n-1},y_n) \\ +d(y_n,y_{n+1})+d(y_{n-1},y_n)+d(y_n,y_{n+1}) \end{pmatrix}}{1+3\begin{pmatrix} d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(y_n,y_{n+1}) \\ +d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(y_{n-1},y_n) \end{pmatrix}} [d(x_n,x_{n-1})+d(y_n,y_{n-1})+d(y_n,y_{n-1})] \tag{1.8}$$

and

$$d(z_{n+1},z_n) \preceq \frac{\begin{pmatrix} d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(y_{n-1},y_n) \\ +d(y_n,y_{n+1})+d(z_{n-1},z_n)+d(z_n,z_{n+1}) \end{pmatrix}}{1+3\begin{pmatrix} d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1}) \\ +d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n) \end{pmatrix}} [d(x_n,x_{n-1})+d(y_n,y_{n-1})+d(z_n,z_{n-1})] \tag{1.9}$$

From these inequalities 1.7-1.9, we get

$$d(x_{n+1},x_n)+d(y_{n+1},y_n)+d(z_{n+1},z_n) \preceq 3\frac{\begin{pmatrix} d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(y_{n-1},y_n) \\ +d(y_n,y_{n+1})+d(z_{n-1},z_n)+d(z_n,z_{n+1}) \end{pmatrix}}{1+3\begin{pmatrix} d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1}) \\ +d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n) \end{pmatrix}} [d(x_n,x_{n-1})+d(y_n,y_{n-1})+d(z_n,z_{n-1})].$$

Now, let

$$\beta_n = 3\frac{\begin{pmatrix} d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(y_{n-1},y_n) \\ +d(y_n,y_{n+1})+d(z_{n-1},z_n)+d(z_n,z_{n+1}) \end{pmatrix}}{1+3\begin{pmatrix} d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1}) \\ +d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n) \end{pmatrix}}.$$

Then

$$\begin{aligned} d(x_{n+1},x_n)+d(y_{n+1},y_n)+d(z_{n+1},z_n) &\preceq \beta_n[d(x_n,x_{n-1})+d(y_n,y_{n-1})+d(z_n,z_{n-1})] \qquad (1.10) \\ &\preceq \beta_n\beta_{n-1}[d(x_{n-1},x_{n-2})+d(y_{n-1},y_{n-2})+d(z_{n-1},z_{n-2})] \\ &\vdots \\ &\preceq \beta_n\beta_{n-1}\ldots\beta_1[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)] \end{aligned}$$

Observe that (β_n) is nonincreasing, with positive terms. So, $\beta_n\beta_{n-1}\ldots\beta_1 \leq \beta_1^n$ and $\beta_1^n \to 0$. It follows that

$$\lim_{n\to\infty}(\beta_n\beta_{n-1}\ldots beta_1) = 0.$$

Hence, this implies that

$$\lim_{n\to\infty}[d(x_{n+1},x_n)+d(y_{n+1},y_n)+d(z_{n+1},z_n)]=0.$$

From this limit, we have

$$\lim_{n\to\infty}d(x_{n+1},x_n)=\lim_{n\to\infty}d(y_{n+1},y_n)=\lim_{n\to\infty}d(z_{n+1},z_n)=0.$$

We claim that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are a Cauchy sequence in X. Let $n<m$. Then, from the triangle inequality and 1.7-1.10, we have

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)\\ &\le \frac{\beta_1^n}{3}[d(x_1,x_0))+d(y_1,y_0)+d(z_1,z_0)]+\frac{\beta_1^{n+1}}{3}[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)]\\ &\quad+\cdots+\frac{\beta_1^{m-1}}{3}[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)]\\ &=\frac{\beta_1^n}{3}\left(\frac{1-\beta_1^{m-n}}{1-\beta_1}\right)[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)]\\ &\le\frac{\beta_1^n}{3}\left(\frac{1-\beta_1}{1-\beta_1}\right)[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)]\\ &=\frac{\beta_1^n}{3}[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)], \end{aligned}$$

$$\begin{aligned} d(y_n,y_m) &\le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\cdots+d(y_{m-1},y_m)\\ &\le \frac{\beta_1^n}{3}[d(x_1,x_0))+d(y_1,y_0)+d(y_1,y_0)]+\frac{\beta_1^{n+1}}{3}[d(x_1,x_0)+d(y_1,y_0)+d(y_1,y_0)]\\ &\quad+\cdots+\frac{\beta_1^{m-1}}{2}[d(x_1,x_0)+d(y_1,y_0)+d(y_1,y_0)]\\ &=\frac{\beta_1^n}{3}\left(\frac{1-\beta_1^{m-n}}{1-\beta_1}\right)[d(x_1,x_0)+d(y_1,y_0)+d(y_1,y_0)]\\ &\le\frac{\beta_1^n}{3}\left(\frac{1-\beta_1}{1-\beta_1}\right)[d(x_1,x_0)+d(y_1,y_0)+d(y_1,y_0)]\\ &=\frac{\beta_1^n}{3}[d(x_1,x_0)+d(y_1,y_0)+d(y_1,y_0)] \end{aligned}$$

and

$$\begin{aligned} d(z_n,z_m) &\le d(z_n,z_{n+1})+d(z_{n+1},z_{n+2})+\cdots+d(z_{m-1},z_m)\\ &\le \frac{\beta_1^n}{3}[d(x_1,x_0))+d(y_1,y_0)+d(z_1,z_0)]+\frac{\beta_1^{n+1}}{3}[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)]\\ &\quad+\cdots+\frac{\beta_1^{m-1}}{3}[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)]\\ &=\frac{\beta_1^n}{3}\left(\frac{1-\beta_1^{m-n}}{1-\beta_1}\right)[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)]\\ &\le\frac{\beta_1^n}{3}\left(\frac{1-\beta_1}{1-\beta_1}\right)[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)]\\ &=\frac{\beta_1^n}{3}[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)] \end{aligned}$$

By adding these two inequalities, we obtain

$$d(x_n,x_m)+d(y_n,y_m)+d(z_n,z_m)\le\frac{\beta_1^n}{3}[d(x_1,x_0)+d(y_1,y_0)+d(z_1,z_0)].$$

This implies that

$$\lim_{n,m\to\infty}[d(x_n,x_m)+d(y_n,y_m)+d(z_n,z_m)]=0.$$

So, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are indeed a Cauchy sequence in the complete metric space X and hence, convergent: there exist $x,y\in X$ such that

$$\lim_{n\to\infty}x_n=x,\quad \lim_{n\to\infty}y_n=y \quad \textit{and} \quad \lim_{n\to\infty}z_n=z.$$

Taking limit for both sides of 1.3 and using continuity of F, we get

$$x=\lim_{n\to\infty}x_n=\lim_{n\to\infty}F(x_{n-1},y_{n-1},z_{n-1})=F\left(\lim_{n\to\infty}(x_{n-1},y_{n-1},z_{n-1})\right)=F(x,y,z)$$

$$y=\lim_{n\to\infty}y_n=\lim_{n\to\infty}F(y_{n-1},x_{n-1},y_{n-1})=F\left(\lim_{n\to\infty}(y_{n-1},x_{n-1},y_{n-1})\right)=F(y,x,y)$$

and

$$z=\lim_{n\to\infty}x_n=\lim_{n\to\infty}F(z_{n-1},y_{n-1},x_{n-1})=F\left(\lim_{n\to\infty}(z_{n-1},y_{n-1},x_{n-1})\right)=F(z,y,x)$$

Therefore, $x=F(x,y,z)$, $y=F(y,x,y)$ and $z=F(z,y,x)$, that is, (x,y,z) is a tripled fixed point of F.
Proof of (b). If there exist two distinct tripled fixed points $(x,y,z),(u,v,w)$ of F, then

$$\begin{aligned}
d(x,u)+d(y,v)+d(z,w) &= d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+(F(z,y,x),F(w,v,u))\\
&\le [d(x,F(u,v,w))+d(y,F(v,u,v))+d(z,F(w,v,u))+d(u,F(x,y,z))\\
&\quad +d(v,F(y,x,y))+d(w,F(z,y,x))][d(x,u)+d(y,v)+d(z,w)]\\
&\quad +[d(y,F(v,u,v))+d(x,F(u,v,w))+d(y,F(v,u,v))+d(v,F(y,x,y))\\
&\quad +d(u,F(x,y,z))+d(v,F(y,x,y))][d(x,u)+d(y,v)+d(z,w)]\\
&\quad +[d(x,F(u,v,w))+d(y,F(v,u,v))+d(z,F(w,v,u))+d(u,F(x,y,z))\\
&\quad +d(v,F(y,x,y))+d(w,F(z,y,x))][d(x,u)+d(y,v)+d(z,w)]\\
&= [d(x,u)+d(y,v)+d(z,w)][9d(x,u)+9d(y,v)+9d(z,w)]\\
&= 9[d(x,u)+d(y,v)+d(z,w)]^2.
\end{aligned}$$

Therefore, we obtain that $d(x,u)+d(y,v)+d(z,w)\ge\frac{1}{9}$. □

Example 11. *Let* $X=\{0,1\}$ *and* $x\le y,\quad x,y\in\{0,1\}$ *and* $x\le y$ *where* $a\le\ a$ *be usual ordering then* $(X,\le)$ *is a partially ordered set. Let* $d:X\times X\to[0,1)$ *be defined by*

$$d(0,1)=3,d(0,0)=d(1,1)=0,d(a,b)=d(b,a),\forall a,b\in X.$$

Then (X,d) *is a complete metric space. Let*

$$S=\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,1,0),(1,0,1),(1,1,1)\}$$

$$S_1=\{(0,0,0),(0,0,1),(0,1,0),(1,0,0)\}$$

$$S_2=\{(0,1,1),(1,1,0),(1,0,1),(1,1,1)\}.$$

We define $F:X\times X\times X\to X$ *as*

$$F(x,y,z)=0,\quad \forall(x,y,z)\in S_1$$

and

$$F(x,y,z)=1,\quad \forall(x,y,z)\in S_2.$$

Then F *is continuous and has the mixed monotone property. It is obvious that*

$$(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,1,0),(1,0,1),(1,1,1)$$

are the tripled fixed points of F. We have following possibilities for values of (x, y, z) and (u, v, w) such that $x \geq u$, $y \leq v$ and $z \geq w$.
Case 1: *If we take $(x, y, z) = (u, v, w) = r$ where $r \in S$, then*

$$d(F(x, y, z), F(u, v, w)) = 0.$$

Thus, the inequality 1.2 holds.
Case 2: *If we take $(x, y, z) \neq (u, v, w) = r$ where $r \in S_1$ or S_2, then*

$$d(F(x, y, z), F(u, v, w)) = 0.$$

Thus, the inequality 1.2 holds.
Case 3: *If we take $(x, y, z) \in S_1$ and $(u, v, w) \in S_2$ then all the conditions of Theorem 10 are satisfied. Also, F has eight distinct tripled fixed points in X that are*

$$(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 1, 0), (1, 0, 1), (1, 1, 1)$$

and

$$d(x, u) + d(y, v) + d(z, w) \geq \frac{1}{9}$$

where $(x, y, z), (u, v, w)$ are two distinct tripled fixed points of F.

Remark 12. *The ratio*

$$\frac{\begin{pmatrix} d(x, F(u, v, w)) + d(y, F(v, u, v)) + d(z, F(w, v, u)) \\ +d(u, F(x, y, z)) + d(v, F(y, x, y)) + d(w, F(z, y, x)) \end{pmatrix}}{1 + 3\begin{pmatrix} d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) \\ +d(u, F(u, v, w)) + d(v, F(v, u, v)) + d(w, F(w, v, u)) \end{pmatrix}} \tag{1.11}$$

might be greater or less than $\frac{1}{3}$ and has not introduced an upper bound. If

$$d(x, u) + d(y, v) + d(z, w) < \frac{1}{9}$$

for every $x, y, z \in X$, then we have

$$\begin{aligned}
& d(x, F(u, v, w)) + d(y, F(v, u, v)) + d(z, F(w, v, u)) + d(u, F(x, y, z)) + d(v, F(y, x, y)) + d(w, F(z, y, x)) \\
\leq & \; d(x, u) + d(u, F(u, v, w)) + d(y, v) + d(v, F(v, u, v)) + d(z, w) + d(w, F(w, v, u)) \\
& +d(u, x) + d(x, F(x, y, z)) + d(v, y) + d(y, F(y, x, y)) + d(w, z) + d(z, F(z, y, x) \\
< & \; 3d(x, u) + 3d(y, v) + 3d(z, w) \\
& +d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) + d(u, F(u, v, w)) + d(v, F(v, u, v)) + d(w, F(w, v, u)) \\
< & \; \frac{1}{3} + d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) + d(u, F(u, v, w)) + d(v, F(v, u, v)) + d(w, F(w, v, u)) \\
= & \; \frac{1}{2}(1 + 3[d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) + d(u, F(u, v, w)) + d(v, F(v, u, v)) + d(w, F(w, v, u))]).
\end{aligned}$$

It means that

$$\left(\frac{d(x, F(u, v, w)) + d(y, F(v, u, v)) + d(z, F(w, v, u)) + d(u, F(x, y, z)) + d(v, F(y, x, y)) + d(w, F(z, y, x))}{1 + 3[d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) + d(u, F(u, v, w)) + d(v, F(v, u, v)) + d(w, F(w, v, u))]}\right) < \frac{1}{3}.$$

which is a special case of the Theorem 9. Therefore, when (X, d) is a complete metric space such that, for all $x, y \in X, d(x, u) + d(y, v) + d(z, w) \preceq \frac{1}{3}$, the above Theorem is valuable because 1.11 might be greater than $\frac{1}{3}$.

Remark 13. *The example 11 does not satisfy the conditions of Theorem 9. That is, we can not say F has a tripled fixed point in X or not. But, we can see that F has a tripled fixed point in X from Theorem 10. In other words the Theorem 10 is a generalization of Theorem 9.*

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