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Prime labeling in the context of duplication of graph elements in $K_{2,n}$

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Abstract

In this paper, we investigate prime labeling for some graphs obtained by duplication of graph elements and also we derive some result for $K_{2,n}$.

Keywords: Graph labeling, prime labeling, prime graph.
AMS Subject Classification(2010): 05C78.

1 Introduction

We begin with finite, undirected and non-trivial graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The elements of $V(G)$ and $E(G)$ are commonly termed as graph elements. Throughout this paper $|V(G)|$ and $|E(G)|$ denote the cardinality of the vertex set and edge set respectively. Throughout this work $K_{2,n}$ denotes the bipartite graph in which $M = \{u_1, u_2\}$ and $N = \{v_1, v_2, ..., v_n\}$ are two partite sets of $K_{2,n}$ such that each edge has one end in M and the other end in N, C_n denotes the cycle with n vertices and P_n denotes the path on n vertices. For various graph theoretic notation and terminology we follow West [12] and for number theory we follow Burton [1]. We give a brief summary of definitions and other information which are useful for the present investigation.

Definition 1.1. For a graph $G = (V, E)$ a function f having domain V, E or $V \cup E$ is said to be a *graph labeling* of G. If the domain is V, E or $V \cup E$ then the corresponding labeling is said to be a *vertex labeling*, an *edge labeling* or a *total labeling*.

Definition 1.2. A *prime labeling* of a graph G is an injective function $f : V(G) \longrightarrow \{1, 2, ..., |V(G)|\}$ such that for every pair of adjacent vertices u and v, $gcd(f(u), f(v)) = 1$. The graph which admits a prime labeling is called a *prime graph*.

The notion of a prime labeling was originated by Entringer and discussed by Tout et al [7]. Fu and Huang [3] proved that P_n and $K_{1,n}$ are prime graphs. Seoud et al [5] proved that $K_{2,n}$ is a prime graph. Deretsky et al [2] proved that C_n is a prime graph. Vaidya and Prajapati discussed prime labeling in the context of duplication of graph elements in P_n, $K_{1,n}$ and C_n [8]. The switching invariance of various graphs was discussed by Vaidya and Prajapati [9] and the same authors introduced the concept of strongly prime graph [10]. A variant of prime labeling known as vertex-edge prime labeling was also introduced by Venkatachalam and Antoni Raj [11].

Definition 1.3. *Duplication of a vertex v* of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.

Definition 1.4. *Duplication of a vertex v_k by a new edge $e = v'_k v''_k$* in a graph G produces a new graph G' such that $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$.

Definition 1.5. *Duplication of an edge $e = uv$ by a new vertex w* in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.

Definition 1.6. *Duplication of an edge $e = uv$* of a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$.

Bertrand's Postulate: For every positive integer $n > 1$ there is a prime p such that $n < p < 2n$.

2 Duplication of Graph elements in $K_{2,n}$

Throughout this section we consider $M = \{u_1, u_2\}$ and $N = \{v_1, v_2, ..., v_n\}$ are two partite sets of $K_{2,n}$ so that each edge has one end in M and the other end in N.

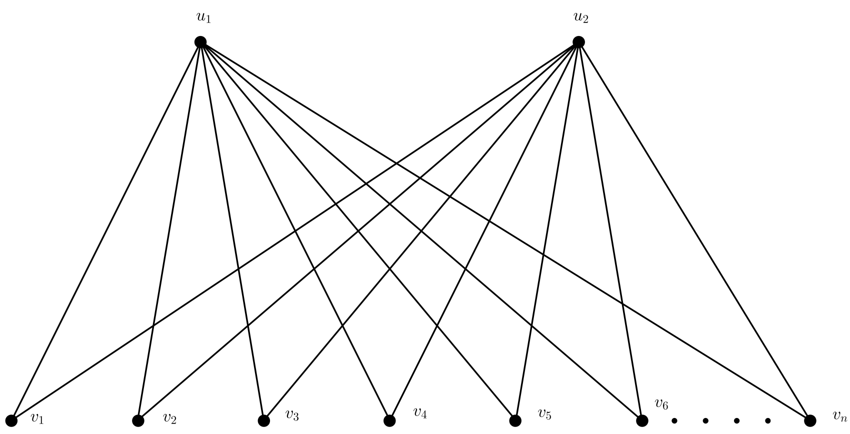

Figure 1

Theorem 2.1. The graph obtained by duplication of a vertex from M in $K_{2,n}$ is a prime graph except $n = 3, 7$.

Proof: The result is obvious for $n = 1$ as when we duplicate one of the vertices of u_1 and u_2, the resulting graph will be a star graph, which is a prime graph [3].
Let G be a graph obtained by duplication of one of the vertices of M. Without loss of generality we duplicate u_1. Then G will be $K_{3,n}$, which is a prime graph except $n = 3, 7$ [5]. ■

Theorem 2.2. The graph obtained by duplication of a vertex from N in $K_{2,n}$ is a prime graph.

Proof: The result is obvious for $n = 1$ as when we duplicate v_1, the resulting graph will be a cycle C_4, which is a prime graph [2]. Let G be a graph obtained by duplication of one of the vertices of N. Let p be the largest prime $\leq n + 1$. Define a function $f : V(G) \longrightarrow \{1, 2, ..., n, n + 1\}$ as,

$$f(x) = \begin{cases} 1 & \text{if } x = u_1; \\ p & \text{if } x = u_2; \\ j+1 & \text{if } x = v_j; \forall j = 1, 2, \cdots, p-2; \\ j+2 & \text{if } x = v_j; \forall j = p-1, \cdots, n. \end{cases}$$

Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ■

Illustration 2.3. A prime labeling of the graph obtained by duplication of a vertex from N in $K_{2,7}$ is shown in Figure 2.

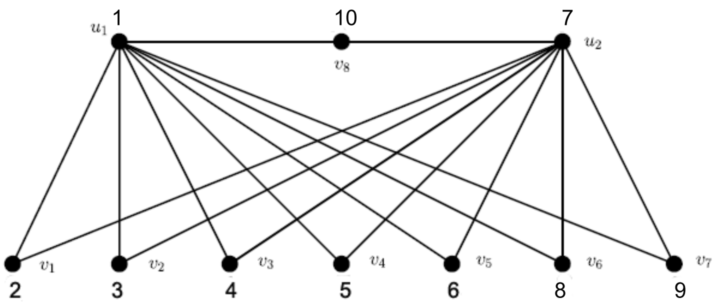

Figure 2: The graph obtained by duplication of a vertex from N in $K_{2,7}$ and its prime labeling.

Theorem 2.4. The graph obtained by duplication of a vertex by an edge from M in $K_{2,n}$ is a prime graph.

Proof: Let G be a graph obtained by duplication of one of the vertices from M in $K_{2,n}$ by an edge $e = u_1^{'}u_1^{''}$. Without loss of generality we duplicate u_1 by an edge $e = u_1^{'}u_1^{''}$. Let p be the largest prime $\leq n+2$.
Define a function $f : V(G) \longrightarrow \{1, 2, ..., n, n+1, n+2\}$ as,

$$f(x) = \begin{cases} 1 & \text{if } x = u_1; \\ p & \text{if } x = u_2; \\ 2 & \text{if } x = u_1^{'}; \\ 3 & \text{if } x = u_1^{''}; \\ j+3 & \text{if } x = v_j; \forall j = 1, 2, \cdots, p-4; \\ j+4 & \text{if } x = v_j; \forall j = p-3, \cdots, n. \end{cases}$$

Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ■

Illustration 2.5. A prime labeling of the graph obtained by duplication of a vertex by an edge $e = u_1^{'}u_1^{''}$ from M in $K_{2,4}$ is shown in Figure 3.

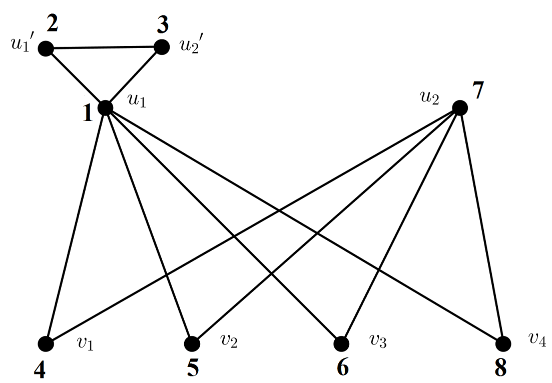

Figure 3: The graph obtained by duplication of a vertex by an edge $e = u_1^{'}u_1^{''}$ from M in $K_{2,4}$ and its prime labeling.

Theorem 2.6. The graph obtained by duplication of a vertex by an edge from N in $K_{2,n}$ is a prime graph.

Proof: Let G be a graph obtained by duplication of one of the vertices of N by an edge. Without loss of generality we duplicate v_1 by an edge $e = v_1^{'}v_1^{''}$.Let p be the largest prime $\leq n+2$.

Define a function $f : V(G) \longrightarrow \{1, 2, ..., n, n+1, n+2\}$ as,

$$f(x) = \begin{cases} 1 & \text{if } x = u_1; \\ p & \text{if } x = u_2; \\ 3 & \text{if } x = v_1; \\ 4 & \text{if } x = v_1^{'}; \\ 5 & \text{if } x = v_1^{''}; \\ j+4 & \text{if } x = v_j; \forall j = 2, \cdots, p-5; \\ j+5 & \text{if } x = v_j; \forall j = p-4, \cdots, n-1; \\ 2 & \text{if } x = v_n. \end{cases}$$

Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ■

Illustration 2.7. A prime labeling of the graph obtained by duplication of a vertex by an edge e from N in $K_{2,10}$ is shown in Figure 4.

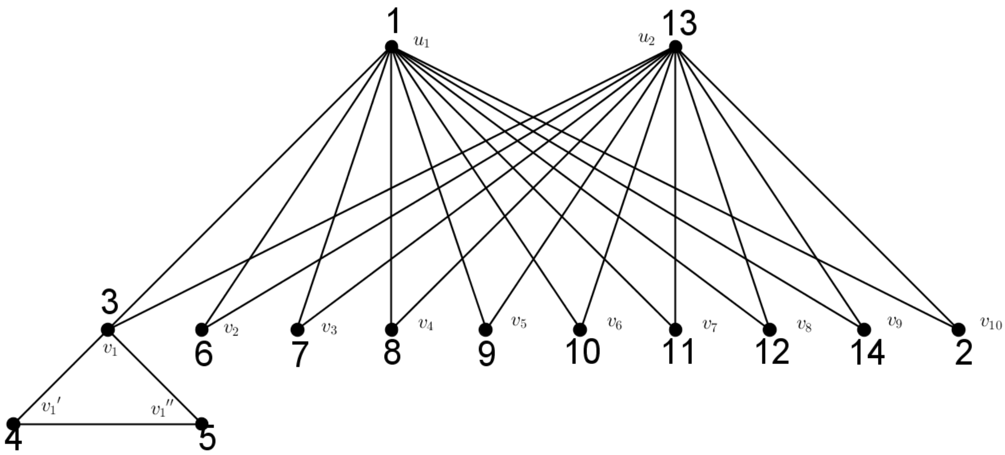

Figure 4: The graph obtained by duplication of a vertex by an edge e from N in $K_{2,10}$ and its prime labeling.

Theorem 2.8. The graph obtained by duplication of an edge by a vertex in $K_{2,n}$ is a prime graph.

Proof: Let G be a graph obtained by duplication of an edge by a vertex. Without loss of generality we duplicate an edge $e = u_1v_1$ by a vertex w. Let p be the largest prime $\leq n+1$.

Define a function $f : V(G) \longrightarrow \{1, 2, ..., n, n+1\}$ as,

$$f(x) = \begin{cases} 1 & \text{if } x = u_1; \\ p & \text{if } x = u_2; \\ 2 & \text{if } x = w; \\ 3 & \text{if } x = v_1; \\ j+2 & \text{if } x = v_j; \forall j = 2, \cdots, p-3; \\ j+3 & \text{if } x = v_j; \forall j = p-2, \cdots, n. \end{cases}$$

Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ■

Illustration 2.9. A prime labeling of the graph obtained by duplication of an edge by a vertex in $K_{2,7}$ is shown in Figure 5.

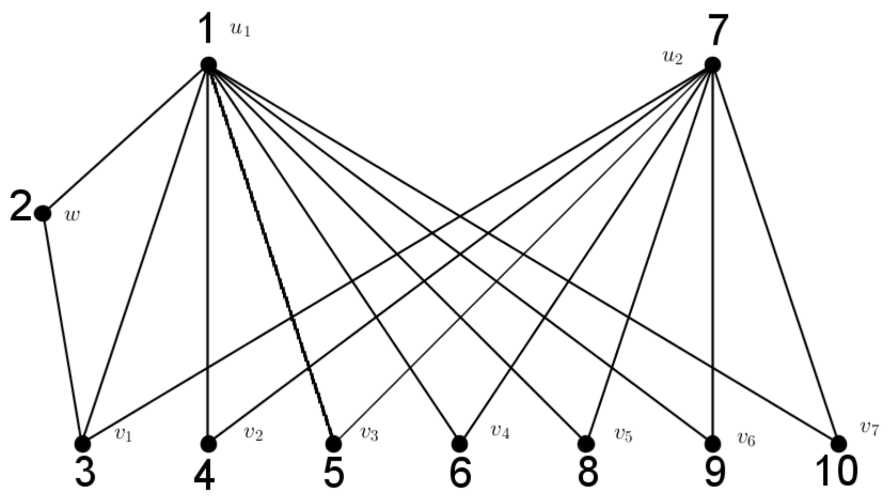

Figure 5: The graph obtained by duplication of an edge by a vertex in $K_{2,7}$ and its prime labeling.

Theorem 2.10. The graph obtained by duplication of both the vertices u_1, u_2 from M in $K_{2,n}$ is not a prime graph for $n \geq 4$.

Proof: For $n = 1$ if we duplicate both the vertices u_1, u_2 from M in $K_{2,1}$ then the resulting graph will be $K_{4,1}$ which is a prime graph.
For $n = 2$ if we duplicate both the vertices u_1, u_2 from M in $K_{2,2}$ then the resulting graph will be $K_{4,2}$ which is a prime graph.
For $n = 3$ if we duplicate both the vertices u_1, u_2 from M in $K_{2,3}$ then the resulting graph will be $K_{4,3}$ which is a prime graph.

For $n = j$ if we duplicate both the vertices u_1, u_2 from M in $K_{2,j}$ then the resulting graph will be $K_{4,j}; \forall j \geq 4$, which is not a prime graph [6]. ■

Theorem 2.11. The graph obtained by duplication of all the vertices from N in $K_{2,n}$ is a prime graph.

Proof: Let G be a graph obtained by duplication of all the vertices from N in $K_{2,n}$ and let $v_j; \forall j = n+1, ..., 2n$ be the vertices which we got after duplication of the vertices $v_j; \forall j = 1, ..., n$. The result is obvious for $n = 1$ as when we duplicate v_1, the resulting graph will be a cycle C_4, which is a prime graph.So we start with $n \geq 2$. Let p be the largest prime $\leq 2n + 2$.
Define a function $f : V(G) \longrightarrow \{1, 2, ..., 2n + 2\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = u_1; \\ p & \text{if } x = u_2; \\ j+1 & \text{if } x = v_j; \forall j = 2, \cdots, p-2; \\ j+2 & \text{if } x = v_j; \forall j = p-1, \cdots, 2n. \end{cases}$$

Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ■

Illustration 2.12. A prime labeling of the graph obtained by duplication of all the vertices from N in $K_{2,4}$ is shown in Figure 6.

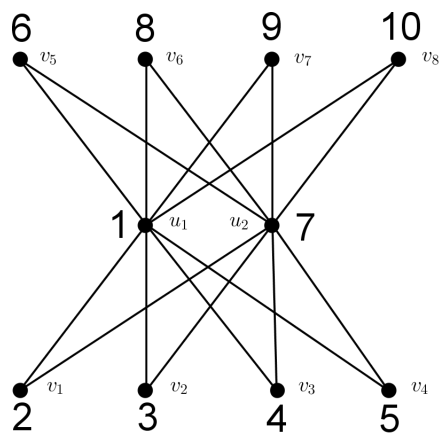

Figure 6: The graph obtained by duplication of all the vertices from N in $K_{2,4}$ and its prime labeling.

Theorem 2.13. The graph obtained by duplication of both the vertices u_1, u_2 from M in $K_{2,n}$ by edge is a prime graph.

Proof: Let G be a graph obtained by duplication of both the vertices u_1, u_2 from M in $K_{2,n}$ by edges $e_1 = u_1^{'}u_1^{''}$ and $e_2 = u_2^{'}u_2^{''}$ Let p be the largest prime $\leq n+6$.
Define a function $f : V(G) \longrightarrow \{1, 2, ..., n+6\}$ as,

$$f(x) = \begin{cases} 1 & \text{if } x = u_1; \\ p & \text{if } x = u_2; \\ 2 & \text{if } x = u_1^{'}; \\ 3 & \text{if } x = u_1^{''}; \\ 4 & \text{if } x = u_2^{'}; \\ 5 & \text{if } x = u_2^{''}; \\ j+5 & \text{if } x = v_j; \forall j = 1, 2, \cdots, p-6; \\ j+6 & \text{if } x = v_j; \forall j = p-5, \cdots, n. \end{cases}$$

Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ■

Illustration 2.14. A prime labeling of the graph obtained by duplication of both the vertices from M in $K_{2,6}$ by edge is shown in Figure 7.

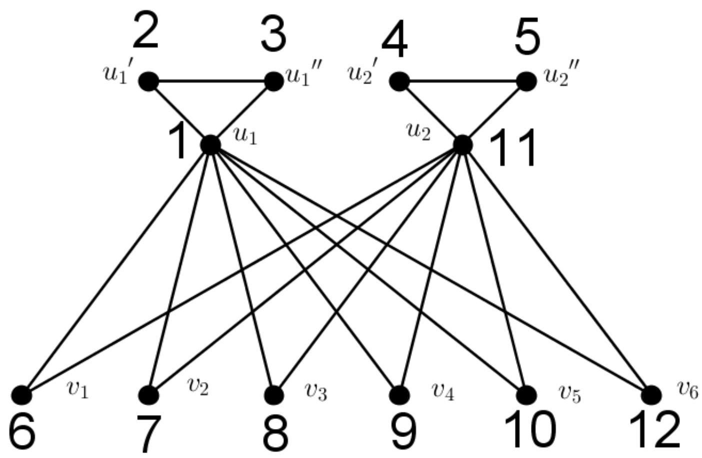

Figure 7: The graph obtained by duplication of of both the vertices from M in $K_{2,6}$ by edge and its prime labeling.

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