International Journal of Mathematics and Soft Computing
Vol.7, No.1 (2017), 109 - 116.

ISSN Print : 2249 - 3328
ISSN Online : 2319 - 5215

Strong domination number of some path related graphs

S K Vaidya[1], S H Karkar[2]

[1]Department of Mathematics, Saurashtra University
Rajkot - 360005, Gujarat, India.
samirkvaidya@yahoo.co.in

[2]Government Engineering College
JRajkot - 360 005, Gujarat, India.
sdpansuria@gmail.com

Abstract

Let $G = (V, E)$ be a graph and $uv \in E$ be an edge then u strongly dominates v if $deg(u) \geq deg(v)$. A set S is a *strong dominating set* ($sd - set$) if every vertex $v \in V - S$ is strongly dominated by some u in S. We investigate strong domination number of some path related graphs.

Keywords: Domination number, Independent domination number, strong domination number, independent strong domination number.
AMS Subject Classification(2010): 05C69, 05C76.

1 Introduction

We consider simple, finite, connected and undirected graph G with vertex set V and edge set E. For all standard terminology and notations we follow West [15] while the terms related to the theory of domination in graphs are used in the sense of Haynes *et al.* [6].

Definition 1.1. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is a *minimal dominating set* if no proper subset $S' \subset S$ is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in graph G.

Definition 1.2. A set $S \subseteq V$ is an *independent set* of G, if $\forall u, v \in S$, $N(u) \cap \{v\} = \phi$. A dominating set which is independent, is called an *independent dominating set*. The minimum cardinality of an independent dominating set in G is called the *independent domination number* $i(G)$ of a graph G.

The theory of independent domination was formalized by Berge [2] and Ore [8] in 1962. Allan and Laskar [1] have identified the graphs G for which $\gamma(G) = i(G)$ whereas bounds on the independent domination number are determined by Goddard and Henning [5]. Vaidya and

Pandit [14] have investigated the exact value of independent domination number of some wheel related graphs.

We denote the degree of a vertex v in a graph G by $deg(v)$ while the maximum and minimum degree of the graph G are denoted by $\triangle(G)$ and $\delta(G)$ respectively.

Definition 1.3. Let $G = (V, E)$ be a graph and $uv \in E$. Then, u strongly dominates v if $deg(u) \geq deg(v)$. A set S is a *strong dominating set* $(sd - set)$ if every vertex $v \in V - S$ is strongly dominated by some u in S. Analogously, one can define a weak dominating set $(wd - set)$.

The concept of strong (weak) domination was introduced by Sampathkumar and Pushpa Latha [11]. Rautenbach[10] has derived a new bound on $\gamma_{st}(G)$ and Meena *et al.*[7] have found the classes of graphs which are strong efficient. Domke *et al.*[4] have proved that the problems of computing i_w and i_{st} are NP-hard. Bounds on strong domination number are also reported by Rautenbach[9]. Swaminathan and Thangaraju [12] have established the relation between strong domination and maximum degree of the graph as well as weak domination and minimum degree of the graph.

Definition 1.4. The *independent strong*$(weak)$ *domination number* of a graph G is the minimum cardinality of a strongly (weak) dominating set which is independent. The independent strong domination number and the independent weak domination number are denoted by $i_{st}(G)$ and $i_w(G)$ respectively.

2 Main Results

Proposition 2.1. [13] If $S \subseteq V$ is a strong dominating set and $v \in V$ is the only vertex of maximum degree in G then $v \in S$.

Proposition 2.2. [13] Let v be a vertex with deg$(v) = \Delta(G) = k$ and v is not adjacent to any other vertex of degree k then v must be in $sd - set$.

Proposition 2.3. [3] $\gamma_{st}(P_n) = \left\lceil \dfrac{n}{3} \right\rceil$.

Definition 2.4. Let G be a graph with $V(G) = S_1 \cup S_2 \cup \ldots S_t \cup T$ where S_i is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$ where $i = 1, 2, \ldots, t$. The *degree splitting graph* $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \ldots, w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$.

Theorem 2.5. $\gamma_{st}(DS(P_n)) = i_{st}(DS(P_n)) = 2$ for $n \geq 5$.

Proof: The path P_n has two pendant vertices and the remaining $n - 2$ vertices are of degree 2. Thus, $V(P_n) = \{v_i; 1 \leq i \leq n\} = S_1 \cup S_2$ where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i; 2 \leq i \leq n - 1\}$.

To obtain $DS(P_n)$ from P_n, add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively.
Thus, $V(DS(P_n)) = V(P_n) \cup \{w_1, w_2\}$ and $|V(DS(P_n))| = n + 2$.
As w_2 is adjacent to $n - 2$ vertices of degree 3, it strongly dominates them and the vertex w_1 strongly dominates both the pendant vertices v_1 and v_n. Thus all the vertices of the graph including w_1 and w_2 are strongly dominated by $\{w_1, w_2\}$. Thus, $S = \{w_1, w_2\}$ is the strong dominating set as well as independent set of minimum cardinality. Hence $\gamma_{st}(DS(P_n)) = i_{st}(DS(P_n)) = 2$. ■

Definition 2.6. A *uniform t-ply* is vertex disjoint union of t paths of same length having common end points. A uniform ply has t number of paths of length k. It is denoted by $P_t(k)$. The paths are called threads.

Theorem 2.7. $\gamma_{st}(P_t(k)) = 2 + t\left\lceil \frac{k-3}{3} \right\rceil$ for $t \geq 3$ and $k \geq 3$.

Proof: Let $P_t(k)$ be the graph with t number of paths of length k, where u and v are the common points with degree t. So, $|V(P_t(k)| = t(k-1) + 2$.

By Proposition 2.2, the vertices u and v must be in strong dominating set. As there are t paths between u and v, the vertex u strongly dominates t vertices which are adjacent to it and the vertex v strongly dominates t vertices which are adjacent to it. Therefore, total $2t+2$ vertices are strongly dominated by u and v including themselves. Now, there are $t(k-1)+2-(2t+2) = tk - 3t = (k-3)t$ vertices which are not strongly dominated. That is, from each path there are $k - 3$ vertices which are not strongly dominated. Sincet the strongly domination number of the path P_n is $\left\lceil \frac{n}{3} \right\rceil$, we need $\left\lceil \frac{k-3}{3} \right\rceil$ vertices from each path to strongly dominate remaining vertices of the graph. Therefore, $t\left\lceil \frac{k-3}{3} \right\rceil$ vertices are enough to strongly dominate remaining vertices of the graph. So, total $2 + t\left\lceil \frac{k-3}{3} \right\rceil$ vertices are enough to strongly dominate all the vertices of the graph.

Hence, $\gamma_{st}(P_t(k)) = 2 + t\left\lceil \frac{k-3}{3} \right\rceil$. ■

Theorem 2.8. 1. $\gamma_{st}(P_1(k)) = \left\lceil \frac{k+1}{3} \right\rceil$ for $k \in N$.

2. $\gamma_{st}(P_2(k)) = \left\lceil \frac{2k}{3} \right\rceil$ for $k \in N$.

3. $\gamma_{st}(P_t(1)) = 1$

4. $\gamma_{st}(P_t(2)) = 2$

Proof: In $\gamma_{st}(P_t(k))$, if $t = 1$, $k \in N$ then $P_1(k)$ becomes path of length k with $k+1$ vertices. Hence, $\gamma_{st}(P_1(k)) = \left\lceil \frac{k+1}{3} \right\rceil$.

If $t = 2$, $k \in N$ then $P_2(k)$ becomes cycle with $(k+1)+(k+1)-2 = 2k$ vertices. Hence, $\gamma_{st}(P_2(k)) = \left\lceil \frac{2k}{3} \right\rceil$ for $k \in N$.

If $k = 1$, $t \in N$ then there are only two vertices u and v which can strongly dominate each other. Therefore, the minimal strong dominating set contains either u or v. Hence, $\gamma_{st}(P_t(1)) = 1$.

If $k = 2$, $t \in N$ then from $P_t(2)$ the vertices u and v strongly dominate all the vertices of the graph. Hence, $\gamma_{st}(P_t(2)) = 2$. ■

Definition 2.9. The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices in $M(G)$ are adjacent whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Theorem 2.10. $\gamma_{st}(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil$ for $n \geq 5$.

Proof: Let $v_1, v_2 \ldots v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. Then, $V(M(P_n)) = \{v_1, v_2, v_3, \ldots, v_n, e_1, e_2, e_3, \ldots, e_{n-1}\}$. Therefore, $|V(M(P_n))| = 2n-1$. Let $V(M(P_n)) = P \cup P'$ where $P = \{v_1, v_2, v_3, \ldots, v_n\}$ and $P' = \{e_1, e_2, e_3, \ldots, e_{n-1}\}$. Every $v_i (i = 1, 2, \ldots n)$ strongly dominates itself and every $e_i (i = 2, 3, \ldots, n-2)$ strongly dominates four vertices $(e_{i-1}, e_{i+1}, v_i, v_{i+1})$ other than itself. We can observe that for any $n \in N$, v_1 and v_n are strongly dominated by only e_1 and e_{n-1} respectively except themselves. Therefore, to strongly dominate v_1 and v_n the vertices e_1 and e_{n-1} must be in $sd-set$ S. In order to prove the result we consider following cases.

Case (i): n is odd.

If $n = 5$ then by the argument given in the beginning of the proof, $e_1, e_4 \in S$. Now e_1 strongly dominates v_1 and v_2 other than itself whereas e_4 strongly dominates v_4 and v_5 other than itself. The vertices e_2, e_3 and v_4 are not strongly dominated by e_1 or e_4. We know that v_3 is strongly dominated by e_2, e_3 or itself while e_2 and e_3 strongly dominate each other. Therefore, we must consider at least one vertex from $\{e_2, e_3\}$ in S to obtain strong dominating set. As S is an $sd-set$ of minimum cardinality, $\gamma_{st}(M(P_5)) = 3$.

If $n = 7$ then by the argument given in the beginning of the proof, $e_1, e_6 \in S$. Now e_1 strongly dominates v_1 and v_2 other than itself while e_6 strongly dominates v_6 and v_7 other than itself. We observe that the vertices $e_2, v_3, e_3, v_4, e_4, v_5, e_5$ are not strongly dominated by e_1 or e_6. If $e_3 \in S$ then the vertices of the set $\{e_1, e_3, e_6\}$ strongly dominate all the vertices except v_5 and e_5 while if $e_4 \in S$ then the vertices of the set $\{e_1, e_4, e_6\}$ strongly dominate all the vertices except v_3, e_2. If e_3 is included in S then e_5 or e_4 must be in S or if e_4 is included in S then e_2 or e_3 must be in S. Hence, we need to include two vertices in S other than e_1 and e_7 to obtain

a strong dominating set of minimum cardinality. Hence, $\gamma_{st}(M(P_7)) = 4$.
In general, if $n = 2k + 1$ where $k = 2, 3, ...$ then $e_1, e_{n-1} \in S$. That is, only six vertices are strongly dominated by e_1 and e_{n-1}. Therefore, to strongly dominate remaining vertices of $M(P_n)$ we consider $\dfrac{n-3}{2}$ alternate vertices from P'. Therefore, to obtain a strong dominating set S of minimum cardinality, we have to include e_1, e_{n-1} and $\dfrac{n-3}{2}$ vertices from P'. Hence, $\gamma_{st}(M(P_n)) = 2 + \dfrac{n-3}{2} = \dfrac{n+1}{2}$.

Case (ii): n is even.

If $n = 6$ then by the argument given in the beginning of the proof, $e_1, e_5 \in S$. Now e_1 strongly dominates v_1 and v_2 other than itself and e_5 strongly dominates v_5 and v_6 other than itself. The vertices e_2, v_3, e_3, v_4 and e_4 are not strongly dominated by e_1 or e_5. By including e_3 in S the remaining vertices e_2, v_3, e_3, v_4 and e_4 are strongly dominated by e_3. Therefore, e_3 must be in S. So, $S = \{e_1, e_3, e_5\}$ becomes a strong dominating set of minimum cardinality. Hence, $\gamma_{st}(M(P_6)) = 3$.

If $n = 8$ then by the argument given in the beginning of the proof, $e_1, e_7 \in S$. Now e_1 strongly dominates v_1 and v_2 other than itself and e_7 strongly dominates v_7 and v_8 other than itself. The vertices $e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6$ are not strongly dominated by e_1 or e_7. If we consider any one vertex from $\{e_2, e_3, e_4, e_5, e_6\}$ then some vertices of $M(P_8)$ are not strongly dominated.

Therefore, we must include at least two vertices in S from $\{e_3, e_4, e_5\}$ such that S will become a strong dominating set of minimum cardinality. If we consider any two successive vertices e_3, e_4 then v_6, e_6 are not dominated and e_4, e_5 are considered then v_3, e_3 are not dominated. In the same way we can not consider e_2 and e_4. Therefore, we must consider e_3 and e_5 in S. So, $S = \{e_1, e_3, e_5, e_7\}$ and S becomes an $sd - set$ of minimum cardinality. Hence, $\gamma_{st}(M(P_8)) = 4$.

In general, if $n = 2k$ where $k = 3, 4, ...$ then $e_1, e_{n-1} \in S$. That is, only eight vertices are strongly dominated by e_1 and e_{n-1}. Therefore, to strongly dominate remaining vertices of $M(P_n)$ we consider $\dfrac{n-4}{2}$ alternate vertices from P'. Therefore, $S = \{e_1, e_3, e_5 \dots e_{n-1}\}$ is an $sd - set$. As $S - e_i$ will no longer remain an $sd - set$, S is minimal $sd - set$. Since S is the only strong dominating set of minimum cardinality, $\gamma_{st}(M(P_n)) = 2 + \dfrac{n-4}{2} = \dfrac{n}{2}$. Hence, in each case $\gamma_{st}(M(P_n)) = \left\lceil \dfrac{n}{2} \right\rceil$. ∎

Definition 2.11. [11] Let $G = (V, E)$ be a graph and $D \subset V$. Then,

1. D is full if every $u \in D$ is adjacent to some $v \in V - D$.

2. D is s-full (w-full) if every $u \in D$ strongly (weakly) dominates some $v \in V - D$.

Definition 2.12. [11] A graph G is *domination balanced* ($d - balanced$) if there exists an sd-set D_1 and a wd-set D_2 such that $D_1 \cap D_2 = \phi$.

Proposition 2.13. [11] For a graph G, the following statements are equivalent.

1. G is d-balanced.

2. There exists an sd-set D which is s-full.

3. There exists an wd-set D which is w-full.

Theorem 2.14. For the complete bipartite graph $K_{m,n}$,

1. $i_{st}(K_{m,n}) + i_w(K_{m,n}) = m + n$, for $m \neq n$.

2. $\gamma(K_{m,n}) + i_w(K_{m,n}) = m + n$, for $m \neq n$.

Proof: (1) If $m > n$ then by the definition of independent strong domination number and independent weak domination number $i_s(K_{m,n}) = n$ and $i_w(K_{m,n}) = m$. Therefore, $i_s(K_{m,n}) + i_w(K_{m,n}) = m + n$.
If $m < n$ then by the definition of independent strong domination number and independent weak domination number $i_s(K_{m,n}) = m$ and $i_w(K_{m,n}) = n$. Therefore, $i_s(K_{m,n}) + i_w(K_{m,n}) = m+n$.
(2) If $m > n$ then by the definition of domination number and independent weak domination number $\gamma(K_{m,n}) = n$ and $i_w(K_{m,n}) = m$. Therefore, $\gamma(K_{m,n}) + i_w(K_{m,n}) = m + n$.
If $m < n$ then by the definition of domination number and independent weak domination number $\gamma(K_{m,n}) = m$ and $i_w(K_{m,n}) = n$. Therefore, $\gamma(K_{m,n}) + i_w(K_{m,n}) = m + n$. ■

Theorem 2.15. The degree splitting graph $DS(P_n)$ is d-balanced for $n \geq 4$.

Proof: In Theorem 2.5, we obtained an $sd - set$ $S = \{w_1, w_2\}$ for $DS(P_n)$ for $n \geq 4$ which is s-full as every vertex of S strongly dominates some $v \in V - S$. Therefore, by Proposition 2.13, $DS(P_n)$ is a d-balanced graph $n \geq 4$. ■

Proposition 2.16. [13] If there exists an isolated vertex in graph G then G is not d- balanced.

Definition 2.17. The *switching of a vertex* v of G means removing all the edges incident to v and adding edges joining v to every vertex which is not adjacent to v in G. The resultant graph is denoted by $\widetilde{G}$.

Theorem 2.18. Let G be any graph with order p and there is at least one vertex v such that $\triangle(v) = p - 1$. If $\widetilde{G}$ is the graph obtained by switching of a vertex v of degree $p - 1$ from G then, $\widetilde{G}$ is not d-balanced.

Proof: Since G has a vertex with degree $p - 1$, there is an isolated vertex in $\widetilde{G}$, Hence, by Proposition 2.16 it is not a d-balanced graph. ■

3 Concluding Remarks

The concept of strong domination in graphs relates dominating sets and the degree of vertices. The strong domination numbers of some standard graphs are already available in the literature while we investigate the strong domination number for the larger graphs obtained from path P_n by means of some graph operations to derive similar results for other graph families as well as in the context of various domination models are open areas of research.

Acknowledgement

The authors are highly thankful to the anonymous referees for their critical comments and constructive suggestions for the improvement in the first draft of this paper.

References

[1] R. B. Allan and R. Laskar, *On domination and independent domination numbers of a graph*, Discrete Mathematics, Vol. 23 (1978), 73-76.

[2] C. Berge, *Theory of graphs and its applications*, Methuen, London, 1962.

[3] R. Boutrig and M. Chellali, *A note on a relation between the weak and strong domination numbers of a graph*, Opuscula Mathematica, Vol. 32(2) (2012), 235-238.

[4] G. S. Domke, J. H. Hattingh, L. R. Markus and E.Ungerer, *On parameters related to strong and weak domination in graphs*, Discrete Mathematics, Vol. 258 (2002), 1-11.

[5] W. Goddard and M. Henning, *Independent domination in graphs: A survey and recent results*, Discrete Math., Vol. 313 (2013), 839-854.

[6] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.

[7] N. Meena, A. Subramanian and V. Swaminathan, *Strong efficient domination in graphs*, International Journal of Innovative Science , Engineering & Technology, Vol.1, Issue 4(2014), 172-177.

[8] O. Ore, *Theory of graphs*, Amer. Math. Soc. Transl., Vol. 38 (1962), 206-212.

[9] D. Rautenbach, *Bounds on the strong domination number*, Discrete Mathematics, Vol. 215 (2000), 201-212.

[10] D. Rautenbach, *The influence of special vertices on the strong domination*, Discrete Mathematics, Vol. 197/198 (1999), 683-690.

[11] E. Sampathkumar and L. Pushpa Latha, *Strong weak domination and domination balance in a graph*, Discrete Mathematics, Vol. 161 (1996), 235-242.

[12] V. Swaminathan and P. Thangaraju, *Strong and weak domination in graphs*, Electronics Notes in Discrete mathematics, Vol. 15(2003), 213-215.

[13] S. K. Vaidya and S. H. Karkar, *On strong domination number of graphs*, Preprint.

[14] S. K. Vaidya and R. M. Pandit, *Independent domination in some wheel related graphs*, Applications and Applied Mathematics, Vol. 11 (2016), 397-407.

[15] D. B. West, *Introduction to graph theory, 2/e*, Prentice - Hall of India, New Delhi, 2003.