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# Geometric mean cordial labeling of graphs

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**Abstract**

Let $G = (V, E)$ be a graph and $f$ be a mapping from $V(G) \rightarrow \{0, 1, 2\}$. For each edge $uv$ assign the label $\left\lceil \sqrt{f(u)f(v)} \right\rceil$, $f$ is called a geometric mean cordial labeling if $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with $x$, $x \in \{0, 1, 2\}$ respectively. A graph with a geometric mean cordial labeling is called geometric mean cordial graph. In this paper geometric mean cordiality of some standard graphs such as path, star, cycle, complete graph, complete bipartite graph, wheel are discussed.

**Keywords**: Cordial Labeling, cordial graphs, geometric mean cordial labeling, geometric mean cordial graphs.
**AMS Subject Classification(2010):** 05C78.

## 1 Introduction

We use the symbol $\lceil x \rceil$ denote the smallest integer greater than or equal to $x$. Some terms are used in the sense of Harary, Bondy and Murthy [2, 5]. The concept of cordial labeling [3] was introduced by Cahit in the year 1987. The labeled graphs are applied mostly in the areas of radar, circuit design, communication network, astronomy, cryptography etc [1]. For a graph $G = (V, E)$, let $f : V(G) \rightarrow \{0, 1\}$ be a function. For each edge $uv$ assign the label $| f(u) - f(v) |$, $f$ is called a cordial labeling if $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with $x$, $x \in \{0, 1\}$ respectively. A graph which admits cordial labeling is called is a cordial graph. Mean cordial labeling was introduced by Raja Ponraj, Muthirulan Sivakumar and Murugesan Sundaram [6].

**Definition 1.1.** For a graph $G = (V, E)$, let $f$ be function from $V(G) \rightarrow \{0, 1, 2\}$. For each edge $uv$ of G assign the label $\lceil \frac{f(u)+f(v)}{2} \rceil$, $f$ is called a mean cordial labeling if $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with $x$, $x \in \{0, 1, 2\}$ respectively. A graph which admits mean cordial labeling is called a mean cordial graph.

In the paper [4, 6], they checked the mean cordiality of some standard graphs and other graphs. Also, they established its properties.

## 2 Main Results

Motivated by the concept of mean cordial labeling, we introduce a new labeling called geometric mean cordial labeling as follows.

**Definition 2.1.** Let $G = (V, E)$ be a $(p, q)$ graph. Let $f$ be a function from $V(G) \to \{0, 1, 2\}$. For each edge $uv$ assign the label $\left\lceil \sqrt{f(u)f(v)} \right\rceil$, $f$ is called a geometric mean cordial labeling if $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$ where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with $x$, $x \in \{0, 1, 2\}$ respectively.

A graph which admits a geometric mean cordial labeling is called **geometric mean cordial graph**.

We illustrate the definition using the following example.

**Example 2.2.** A graph that admits a geometric mean cordial labeling is given below.

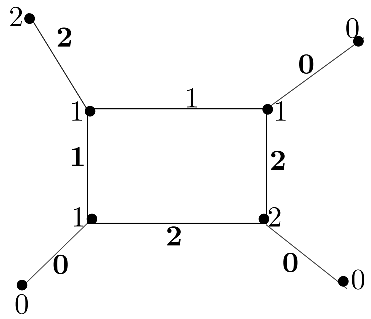


**Figure 1:** A geometric mean cordial graph.

Here $v_f(0) = v_f(1) = 3$, $v_f(2) = 2$ and $e_f(0) = e_f(2) = 3$, $e_f(1) = 2$ and $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$ for all $i, j \in \{0, 1, 2\}$.

We check the geometric mean cordiality of some standard graphs.

**Theorem 2.3.** Any path $P_n$ is geometric mean cordial.

**Proof:** Let $P_n : u_1 u_2 \cdots u_n$ be a path. Define $f : V(P_n) \to \{0, 1, 2\}$ as follows:

**Case(i):** $n \equiv 0 \pmod 3$. Let $n = 3t$.

$f(u_i) = 0$; $1 \leq i \leq t$,

$f(u_{t+i}) = 1$; $1 \leq i \leq t$,

$f(u_{2t+i}) = 2$; $1 \leq i \leq t$.

Then $v_f(0) = v_f(1) = v_f(2) = t$ and $e_f(0) = e_f(2) = t$, $e_f(1) = t - 1$.

**Case(ii):** $n \equiv 1 \pmod 3$. Let $n = 3t + 1$.

$f(u_i) = 2;\ 1 \le i \le t,$
$f(u_{t+i}) = 1;\ 1 \le i \le t+1,$
$f(u_{2t+1+i}) = 0;\ 1 \le i \le t+1.$
Then $v_f(0) = v_f(2) = t$, $v_f(1) = t+1$ and $e_f(0) = e_f(1) = e_f(2) = t$.
**Case(iii):** $n \equiv 2 \pmod 3$. Let $n = 3t+2$.
$f(u_i) = 2;\ 1 \le i \le t,$
$f(u_{t+i}) = 1;\ 1 \le i \le t+1,$
$f(u_{2t+1+i}) = 0;\ 1 \le i \le t+1.$
Then $v_f(0) = v_f(1) = t+1$, $v_f(2) = t$ and $e_f(0) = t+1$, $e_f(1) = e_f(2) = t$.

From all the three cases above, we see that $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$ for all $i, j \in \{0, 1, 2\}$ and hence $f$ is a geometric mean cordial labeling. ■

**Example 2.4.** Geometric mean cordial labeling of the path $P_6$ is given below. Here $v_f(0) = v_f(1) = v_f(2) = 2$ and $e_f(0) = e_f(2) = 2$, $e_f(1) = 1$.

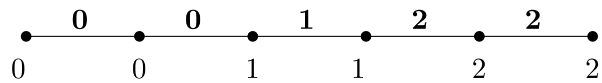


**Figure 2a:** Geometric mean cordial labeling of the path $P_6$.

**Example 2.5.** Geometric mean cordial labeling of the path $P_7$ is given below. Here $v_f(0) = v_f(2) = 2$, $v_f(1) = 3$ and $e_f(0) = e_f(1) = e_f(2) = 2$.

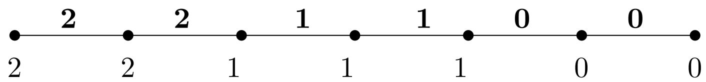


**Figure 2b:** Geometric mean cordial labeling of the path $P_7$.

**Example 2.6.** Geometric mean cordial labeling of the path $P_8$ is given below. Here $v_f(0) = v_f(1) = 3$, $v_f(2) = 2$ and $e_f(0) = 3$, $e_f(1) = e_f(2) = 2$.

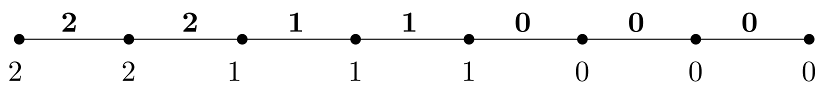


**Figure 2c:** Geometric mean cordial labeling of the path $P_8$.

**Theorem 2.7.** The star $K_{1,n}$ is geometric mean cordial.

**Proof:** Let $V(K_{1,n}) = \{u, u_i : 1 \le i \le n\}$ and $E(K_{1,n}) = \{uu_i : 1 \le i \le n\}$. $K_{1,n}$ has $n+1$ vertices and $n$ edges. Let $u$ is the centre of $K_{1,n}$.
Define $f : V(K_{1,n}) \to \{0, 1, 2\}$ as follows: Let $f(u) = 1$.
**Case(i):** $n \equiv 0 \pmod 3$. Let $n = 3t$.

Assign the labels 0, 1, 2 to each of the $t$ vertices respectively. Then $v_f(0) = v_f(2) = t$, $v_f(1) = t+1$ and $e_f(0) = e_f(1) = e_f(2) = t$.

**Case(ii):** $n \equiv 1 \pmod 3$. Let $n = 3t+1$.

Assign the label 0 to $t+1$ vertices and the labels 1 and 2 to the remaining each of $t$ vertices respectively. Then $v_f(0) = v_f(1) = t+1, v_f(2) = t$ and $e_f(0) = t+1, e_f(1) = e_f(2) = t$.

**Case(iii):** $n \equiv 2 \pmod 3$. Let $n = 3t+2$.

Assign the label 1 to $t$ vertices and the labels 0 and 2 to the remaining each of $t+1$ vertices respectively. Then $v_f(0) = v_f(1) = v_f(2) = t+1$ and $e_f(0) = e_f(2) = t+1$, $e_f(1) = t$.

From all the three cases above we see that $\mid v_f(i) - v_f(j) \mid leq1$, $\mid e_f(i) - e_f(j) \mid \leq 1$ for all $i, j \in \{0, 1, 2\}$ and hence $f$ is a geometric mean cordial labeling. ■

**Example 2.8.** Geometric mean cordial labeling of the star $K_{1,8}$ is given below. Here $v_f(0) = v_f(1) = v_f(2) = 3$ and $e_f(0) = e_f(2) = 3$, $e_f(1) = 2$.

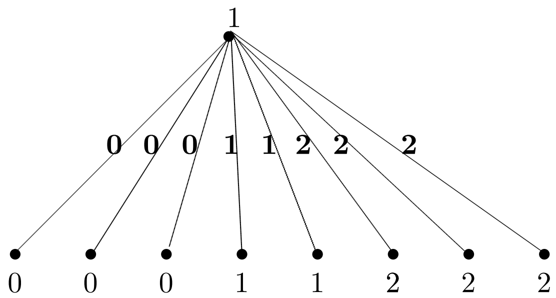


**Figure 3:** Geometric mean cordial labeling of the path $K_{1,8}$.

**Theorem 2.9.** The cycle $C_n$ is geometric mean cordial if $n \equiv 1, 2 \pmod 3$.

**Proof:** Let $C_n$ be the cycle $u_1 u_2 \cdots u_n u_1$. It has $n$ vertices and $n$ edges.

**Case(i):** $n \equiv 0 \pmod 3$. Let $n = 3t$.

If $C_n$ admits geometric mean cordial labeling $f$, then the only possibility is $v_f(0) = v_f(1) = v_f(2) = t$ and $e_f(0) = e_f(1) = e_f(2) = t$. If we assign 0's to $t$ number of vertices in $C_n$, then we get $e_f(0) > t$. Hence $f$ is not a geometric mean cordial labeling.

For the remaining two cases, define $f : V(C_n) \to \{0, 1, 2\}$ as follows:

**Case(ii):** $n \equiv 1 \pmod 3$. Let $n = 3t+1$.

Assign the label 1 to $t+1$ vertices and the labels 0 and 2 to the remaining each of $t$ vertices. Then $v_f(0) = v_f(2) = t$, $v_f(1) = t+1$ and $e_f(0) = t+1, e_f(1) = e_f(2) = t$. Hence $f$ is a geometric mean cordial labeling.

**Case(iii):** $n \equiv 2 \pmod 3$. Let $n = 3t+2$.

Assign the label 0 to $t$ vertices and the labels 1 and 2 to remaining each of $t+1$ vertices respectively. Then $v_f(0) = t$, $v_f(1) = v_f(2) = t+1$ and $e_f(0) = e_f(2) = t+1$, $e_f(1) = t$. Hence $f$ is a geometric mean cordial labeling. ■

**Example 2.10.** Geometric mean cordial labeling of the star $C_7$ is given below. Here $v_f(0) = v_f(2) = 2, v_f(1) = 3$ and $e_f(0) = 3, e_f(1) = e_f(2) = 2$.

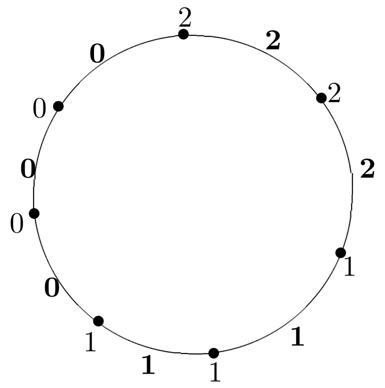


**Figure 4:** Geometric mean cordial labeling of the path $C_7$.

**Example 2.11.** Geometric mean cordial labeling of the star $C_7$ is given below. Here $v_f(0) = 2$, $v_f(1) = v_f(2) = 3$ and $e_f(0) = e_f(2) = 3$, $e_f(1) = 2$.

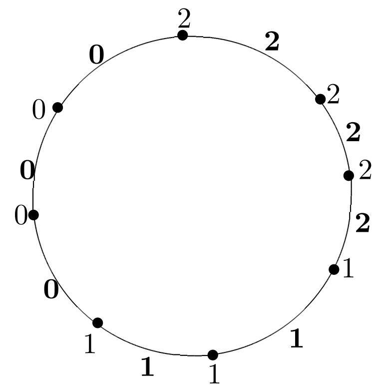


**Figure 5:** Geometric mean cordial labeling of the path $C_8$.

**Theorem 2.12.** The complete graph $K_n$ is geometric mean cordial if $n \leq 2$.

**Proof:** From Theorem 2.3, it follows that $K_1$ and $K_2$ are geometric mean cordial. Assume that $n > 2$. If possible, let there be a geometric mean cordial labeling $f : V(K_n) \to \{0, 1, 2\}$.
**Case(i):** $n \equiv 0 \pmod 3$. Let $n = 3t$, $t \geq 1$.

Then we must have $v_f(0) = v_f(1) = v_f(2) = t$.

Consider the edges having end vertices with label 0 only. We see that each of this kind of edges contributes 1 to $e_f(0)$ and clearly there are $\binom{t}{2}$ edges having label 0. Now consider the vertices having label 1, which are adjacent to $t$ vertices having label 0. Each of these edges contributes 1 to $e_f(0)$ and clearly $t^2$ edges are having the label 0. The same is true for the vertices having label 2 only. Then $e_f(0) = \binom{t}{2} + t^2 + t^2$.

Consider the edges having end vertices with label 1 only. We see that each of these edges contributes 1 to $e_f(1)$ and clearly $\binom{t}{2}$ edges are having label 1. The edges incident with the vertices having the label 2 have no contribution to $e_f(1)$ and clearly $e_f(1) = 0$. The same is true for the edges incident with the vertices having the labels 0 and $e_f(1) = 0$. Then $e_f(1) = \binom{t}{2}$.

Consider the edges having end vertices with label 2 only. We see that each of these edges contributes 1 to $e_f(2)$ and there are $\binom{t}{2}$ edges having label 1. Consider the edges having end vertices with labels 1 and 2. These edges contribute $t^2$ 1 's to $e_f(2)$. The edges having ends with label 0 contribute 0 to $e_f(2)$. Thus $e_f(2) = \binom{t}{2} + t^2$. Hence $e_f(0) - e_f(1) = 2t^2 > 1$.

**Case(ii):** $n \equiv 1 \pmod 3$. Let $n = 3t + 1$.

**Subcase (i):** $v_f(0) = v_f(2) = t$, $v_f(1) = t + 1$.

Then by the argument as in case (i), we have $e_f(0) = \binom{t}{2} + t^2 + t + t^2$, $e_f(1) = \binom{t+1}{2}$, $e_f(2) = t^2 + t\binom{t}{2}$. Hence $e_f(0) - e_f(2) = t^2 > 1$.

**Subcase (ii):** $v_f(0) = t + 1$, $v_f(1) = v_f(2) = t$.

Then we have $e_f(0) = \binom{t+1}{2} + (t^2 + t) + (t^2 + t)$, $e_f(1) = \binom{t}{2}$, $e_f(2) = t^2 + \binom{t}{2}$ and so $e_f(1) - e_f(2) = t^2 > 1$.

**Subcase (iii):** $v_f(0) = v_f(1) = t$, $v_f(2) = t + 1$.

Then we have $e_f(0) = \binom{t}{2} + t^2 + (t^2 + t)$, $e_f(1) = \binom{t}{2}$, $e_f(2) = (t^2 + t) + \binom{t+1}{2}$ and so $e_f(0) - e_f(1) = 2t^2 + t > 1$.

**Case(iii):** $n \equiv 2 \pmod 3$. Let $n = 3t + 2$.

**Subcase (i):** $v_f(0) = t$, $v_f(1) = v_f(2) = t + 1$.

$e_f(0) = \binom{t}{2} + (t^2 + t) + (t^2 + t)$, $e_f(1) = \binom{t+1}{2}$, $e_f(2) = (t + 1)^2 + \binom{t+1}{2}$ and so $e_f(1) - e_f(2) = (t + 1)^2 > 1$.

**Subcase (ii):** $v_f(0) = v_f(2) = t + 1$, $v_f(1) = t$.

$e_f(0) = \binom{t+1}{2} + (t^2 + t) + (t + 1)^2$, $e_f(1) = \binom{t}{2}$, $e_f(2) = (t^2 + t) + \binom{t+1}{2}$ and so $e_f(0) - e_f(2) = (t + 1)^2 > 1$.

**Subcase (iii):** $v_f(0) = v_f(1) = t + 1$, $v_f(2) = t$.

$e_f(0) = \binom{t+1}{2} + (\ t+1\ )^2 + (\ t^2 + t\ ),\ e_f(1) = \binom{t+1}{2},\ e_f(2) = (\ t^2 + t\ ) + \binom{t}{2}$ and so $e_f(0) - e_f(1) = (\ t+1\ )^2 + t^2 + t > 1.$

In all the above cases, we see that $K_n$ is not geometric mean cordial. ■

**Theorem 2.13.** The complete bipartite graph $K_{2,n}$ is not geometric mean cordial for $n > 2$.

**Proof:** Let $V(K_{2,n}) = A \cup B$, where $A = \{u, v\}$ and $B = \{u_1, u_2, u_3, \ldots, u_n\}$. Then $E(K_{2,n}) = \{uu_i, vu_i : 1 \leq i \leq n\}$.

From Theorem 2.3 and Theorem 2.9 it follows that $K_{1,1}$ and $K_{2,2}$ are geometric mean cordial. Assume that $n > 2$. We see that $K_{2,n}$ has $2 + n = m$ (say) vertices and $2n = 2(m-2)$ edges.

**Case(i):** $m \equiv 2 \pmod 3$. Let $m = 3t$.

Then $K_{2,n}$ has $6t - 4$ edges. The only possibility of assigning labels to the vertices without violating the vertex label difference is $v_f(0) = v_f(1) = v_f(2) = t$.

**Subcase (i):** $f(u) = f(v) = 0$. Then we see that $e_f(0) = (t-2)+t+t+(t-2)+t+t = 6t-4$, $e_f(1) = e_f(2) = 0$.

**Subcase (ii):** $f(u) = 0,\ f(v) = 1$.

Then we see that $e_f(0) = (t-1) + (2t-2) + t = 4t - 3$, $e_f(1) = t - 1$, $e_f(2) = t$.

**Subcase (iii):** $f(u) = 1,\ f(v) = 0$. Similar to Subcase (ii).

**Subcase (iv):** $f(u) = 0,\ f(v) = 2$.

Then we see that $e_f(0) = (t-1) + (2t-2) + t = 4t - 3$, $e_f(1) = 0$, $e_f(2) = 2t - 1$.

**Subcase (v):** $f(u) = 2$,f(v) $= 0$.

Similar to Subcase (iv).

**Subcase (vi)** : $f(u) = 1,\ f(v) = 1$.

Then we see that $e_f(0) = t + t = 2t$, $e_f(1) = t - 2 + t - 2 = 2t - 4$, $e_f(2) = t + t = 2t$.

**Subcase (vii):** $f(u) = 1,\ f(v) = 2$.

Then we see that $e_f(0) = t + t = 2t$, $e_f(1) = t - 1$, $\mathrm{e}_f(2) = t - 1 + t - 1 + t - 1 = 3t - 3$.

**Subcase (viii):** $f(u) = 2,\ f(v) = 1$.

Similar to subcase (vii).

**Subcase (ix):** $f(u) = 2,\ f(v) = 2$.

Then $e_f(0) = t + t = 2t$, $e_f(1) = 0$, $e_f(2) = t + t - 2 + t + t - 2 = 4t - 4$.

In all the above subcases, we observe that if atleast any one of the labeling of the vertices $u$ and $v$ is zero, then $e_f(0) \geq 3t - 2$. Also if either or both of the vertices $u$ and $v$ are having the label 2, then $e_f(2) \geq 3t - 3$. So in all the subcases, we see that $K_{2,n}$ is not geometric mean cordial. Thus in the following cases, we consider only the subcases in which both the vertices $u$ and $v$ are having label 1.

**Case(ii):** $m \equiv 1 \pmod 3$. Let $m = 3t + 1$.

Then $K_{2,n}$ has $6t - 2$ edges. Suppose $f(u) = f(v) = 1$.

**Subcase(a):** $v_f(0) = t + 1$, $v_f(1) = v_f(2) = t$.

Then $e_f(0) = t + 1 + t + 1 = 2t + 2$, $e_f(1) = t - 2 + t - 2 = 2t - 4$, $e_f(2) = 2t$.

**Subcase(b):** $v_f(0) = v_f(2) = t$, $v_f(1) = t + 1$.

Then $e_f(0) = t + t = 2t$, $e_f(1) = t - 1 + t - 1 = 2t - 2$, $e_f(2) = t + t = 2t$.

**Subcase(c):** $v_f(0) = v_f(1) = t$, $v_f(2) = t + 1$.

Then $e_f(0) = t + t = 2t$, $e_f(1) = t - 2 + t - 2 = 2t - 4$, $e_f(2) = t + 1 + t + 1 = 2t + 2$.

**Case(iii):** $m \equiv 2 \pmod 3$. Let $m = 3t + 2$.

Then $K_{2,n}$ has $6t$ edges. Suppose $f(u) = f(v) = 1$.

**Subcase(a):** $v_f(0) = v_f(1) = t + 1$, $v_f(2) = t$.

Then $e_f(0) = t + 1 + t + 1 = 2t + 2$, $e_f(1) = t - 1 + t - 1 = 2t - 2$, $e_f(2) = 2t$.

**Subcase(b):** $v_f(0) = t$, $v_f(1) = v_f(2) = t + 1$.

Then $e_f(0) = t + t = 2t$, $e_f(1) = t - 1 + t - 1 = 2t - 2$, $e_f(2) = t + 1 + t + 1 = 2t + 2$.

From all the above cases, we have $K_{2,n}$ is not a geometric mean cordial graph for $n > 2$. ■

**Theorem 2.14.** The wheel $W_n$ is not geometric mean cordial for $n \geq 4$.

**Proof:** Let $u$ be the central vertex and $u_1, u_2, u_3, \ldots, u_{n-1}$ be the rim vertices of $W_n$. $W_n$ has $n$ vertices and $2n - 2$ edges.

**Case (i):** $n \equiv 0 \pmod 3$. Let $n = 3t$, $t \geq 2$.

Then $W_n$ has $3t$ vertices and $6t - 2$ edges. Suppose $W_n$ admits a geometric mean cordial labeling, then $v_f(0) = v_f(1) = v_f(2) = t$ and we have the following three possibilities.

1. $e_f(0) = e_f(1) = 2t - 1, e_f(2) = 2t$,

2. $e_f(0) = e_f(2) = 2t - 1, e_f(1) = 2t$,

3. $e_f(1) = e_f(2) = 2t - 1, e_f(0) = 2t$.

**Subcase (i):** $f(u) = 0$.

Then spokes contribute $3t-1$ to $e_f(0)$ and from Theorem 2.4 case (ii), it follows that the rim edges of $W_n$ contribute $t$ to $e_f(0)$. Thus $e_f(0) = (3t - 1) + t = 4t - 1 > 2t - 1$, a contradiction.

**Subcase (ii):** Let $f(u) = 1$.

Then spokes contribute $t$ to $e_f(0)$ and from Theorem 2.4 case (ii), it follows that the rim edges of $W_n$ contribute $t+1$ to $e_f(0)$. Thus $e_f(0) = t + (t+1) = 2t+1$, we get a contradiction.

**Subcase (iii):** Let $f(u) = 2$.

Similar to subcase (ii).

**Case (ii):** $n \equiv 1 \pmod 3$.

Let $n = 3t + 1$, then $W_n$ has $3t + 1$ vertices and $6t$ edges. Suppose $W_n$ admits a geometric mean cordial labeling, then we have three possibilities.

$(a)$ $v_f(0) = t + 1$, $v_f(1) = v_f(2) = t$

(b) $v_f(0) = v_f(2) = t$, $v_f(1) = t+1$
(c) $v_f(0) = v_f(1) = t$, $v_f(2) = t+1$
and $e_f(0) = 2t$, $e_f(1) = 2t$, $e_f(2) = 2t$.
**Subcase (i):** $f(u) = 0$.

Consider (a) $v_f(0) = t+1$, $v_f(1) = v_f(2) = t$.
Then spokes contribute $3t$ to $e_f(0)$ and from Theorem 2.4 case (iii), it follows that the rim edges of $W_n$ contribute $t+1$ to $e_f(0)$. Thus $e_f(0) = 3t + (t+1) = 4t+1 > 2t$, we get a contradiction.

Consider (b) $v_f(0) = v_f(2) = t$, $v_f(1) = t+1$. Then spokes contribute $3t$ to $e_f(0)$ and from Theorem 2.4 case (iii), it follows that the rim edges of $W_n$ contribute $t$ to $e_f(0)$. Thus $e_f(0) = 4t$, we get a contradiction.

Consider (c) $v_f(0) = v_f(1) = t$, $v_f(2) = t+1$. This case is similar to the previous case.
**Subcase (ii):** $f(u) = 1$.

Consider (a) $v_f(0) = t+1$, $v_f(1) = v_f(2) = t$. Then spokes contribute $t+1$ to $e_f(0)$ and from Theorem 2.4 case (iii), it follows that the rim edges of $W_n$ contribute $t+2$ to $e_f(0)$. Thus $e_f(0) = (t+1) + (t+2) = 2t+3$, we get a contradiction.

Consider (b) $v_f(0) = v_f(2) = t$, $v_f(1) = t+1$. Then spokes contribute $t$ to $e_f(0)$ and from Theorem 2.4 case (iii), it follows that the rim edges of $W_n$ contribute $t+1$ to $e_f(0)$. Thus $e_f(0) = t + (t+1) = 2t+1$, we get a contradiction.

Consider (c) $v_f(0) = v_f(1) = t$, $v_f(2) = t+1$. This case is similar to the previous case.
**Subcase (iii):** $f(u) = 2$.
Similar to subcase (ii).
**Case (iii):** $n \equiv 2 \pmod 3$

Let $n = 3t+2$, then $W_n$ has $3t+2$ vertices $6t+2$ edges. Suppose $W_n$ admits a geometric mean cordial labeling, then we have three possibilities.
(a) $v_f(0) = v_f(1) = t+1$, $v_f(2) = t$,
(b) $v_f(1) = v_f(2) = t+1$, $v_f(0) = t$,
(c) $v_f(0) = v_f(2) = t+1$, $v_f(1) = t$ and
$e_f(0) = e_f(1) = 2t+1$, $e_f(2) = 2t$.
**Subcase(i):** $f(u) = 0$.

Consider (a) $v_f(0) = v_f(1) = t+1$, $v_f(2) = t$. Then spokes contribute $3t+1$ to $e_f(0)$ and from Theorem 2.4 case (i), it follows that the rim edges of $W_n$ contribute $t+1$ to $e_f(0)$. Thus $e_f(0) = (3t+1) + (t+1) = 4t+2$, we get a contradiction.

Consider (b) $v_f(1) = v_f(2) = t+1$, $v_f(0) = t$. Then spokes contribute $3t+1$ to $e_f(0)$ and from Theorem 2.4 case (i), it follows that the rim edges of $W_n$ contribute $t$ to $e_f(0)$. Thus $e_f(0) = (3t+1) + t = 4t+1$, we get a contradiction.

Consider (c) $v_f(0) = v_f(2) = t+1$, $v_f(1) = t$. Then spokes contribute $3t+1$ to $e_f(0)$ and from Theorem 2.4 case (i), it follows that the rim edges of $W_n$ contribute $t+1$ to $e_f(0)$. Thus

$e_f(0) = (3t + 1) + (t + 1) = 4t + 2$, we get a contradiction.

**Subcase(i):** $f(u) = 1$.

Consider (a) $v_f(0) = v_f(1) = t + 1$, $v_f(2) = t$. Then spokes contribute $t + 1$ to $e_f(0)$ and from Theorem 2.4 case (ii), it follows that the rim edges of $W_n$ contribute $t + 2$ to $e_f(0)$. Thus $e_f(0) = (t + 1) + (t + 2) = 2t + 3$, we get a contradiction.

Consider (b) $v_f(1) = v_f(2) = t + 1$, $v_f(0) = t$. Then spokes contribute $t$ to $e_f(0)$ and from Theorem 2.4 case (ii), it follows that the rim edges of $W_n$ contribute $t + 1$ to $e_f(0)$. Thus $e_f(0) = t + (t + 1) = 2t + 1$, we get a contradiction.

Consider (c) $v_f(0) = v_f(2) = t + 1$, $v_f(1) = t$. Then spokes contribute $t + 1$ to $e_f(0)$ and from Theorem 2.4 case (ii), it follows that the rim edges of $W_n$ contribute $t + 2$ to $e_f(0)$. Thus $e_f(0) = (t + 1) + (t + 2) = 2t + 3$, we get a contradiction.

**Subcase (iii):** $f(u) = 2$.

Consider (a) $v_f(0) = v_f(1) = t + 1$, $v_f(2) = t$. Then spokes contribute $t + 1$ to $e_f(0)$ and from Theorem 2.4 case (ii), it follows that the rim edges of $W_n$ contribute $t + 2$ to $e_f(0)$. Thus $e_f(0) = (t + 1) + (t + 2) = 2t + 3$, we get a contradiction.

Consider (b) $v_f(0) = t$, $v_f(1) = v_f(2) = t + 1$. Then spokes contribute $t$ to $e_f(0)$ and from Theorem 2.4 case (ii), it follows that the rim edges of $W_n$ contribute $t + 1$ to $e_f(0)$. Thus $e_f(0) = t + (t + 1) = 2t + 1$, we get a contradiction.

Consider (c) $v_f(0) = v_f(2) = t + 1$, $v_f(1) = t$. Then spokes contribute $t + 1$ to $e_f(0)$ and from Theorem 2.4 case (ii), it follows that the rim edges of $W_n$ contribute $t + 2$ to $e_f(0)$. Thus $e_f(0) = (t + 1) + (t + 2) = 2t + 3$, we get a contradiction. In all the above cases, we see that $W_n$ is not geometric mean cordial. ∎

**Theorem 2.15.** $K_{n,n}$ is not geometric mean cordial for $n \geq 3$.

**Proof:** Let $V(K_{n,n}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, \cdots, u_n\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$. Then we see that $K_{n,n}$ has $2n$ vertices and $n^2$ edges.

**Case (i):** $n \equiv 0 \pmod 3$.

Let $n = 3t$, where $t \geq 1$. Then $K_{n,n}$ has $6t$ vertices and $9t^2$ edges. If $K_{n,n}$ admits a geometric mean cordial labeling, then we must have

$$v_f(0) = v_f(1) = v_f(2) = 2t \qquad \cdots (1)$$

$$e_f(0) = e_f(1) = e_f(2) = 3t^2 \qquad \cdots (2)$$

Suppose (1) holds. Since $v_f(0) = 2t$, $2t$ vertices of $V(K_{n,n})$ are labeled with 0. If these $2t$ vertices are in $V_1$, then they are adjacent with $3t$ vertices in $V_2$ and hence $2t \times 3t = 6t^2$ edges have the label 0. Similar case arises when these $2t$ vertices are in $V_2$.

If $t$ vertices are in $V_1$ and the remaining $t$ vertices are in $V_2$, then $t \times 3t + t \times 2t = 5t^2$ edges have the label 0.

In general, if $2t-i$ vertices are in $V_1$ and $i$ vertices are in $V_2$, where $0 \leq i \leq 2t$, then we have $e_f(0) = (2t-i)3t + i\{3t-(2t-i)\} = 6t^2 - 2it + i^2$. The maximum value for $e_f(0) = 6t^2$ is obtained by putting $i = 0$ or $2t$ and the minimum value for $e_f(0) = 5t^2$ is obtained by putting $i = t$. Thus $5t^2 \leq e_f(0) \leq 6t^2$. This is a contradiction to $e_f(0) = 3t^2$ from (2). Thus $K_{n,n}$ is not a geometric mean cordial graph.

**Case(ii):** $n \equiv 1 \pmod 3$.

Let $n = 3t+1$, where $t \geq 1$. Then $K_{n,n}$ has $6t+2$ vertices and $9t^2+6t+1$ edges. If $K_{n,n}$ admits a geometric mean cordial labeling, then we must have

(i) $v_f(0) = 2t$, $v_f(1) = v_f(2) = 2t+1$ and $e_f(0) = e_f(1) = 3t^2+2t$, $e_f(2) = 3t^2+2t+1$

(ii) $v_f(0) = v_f(2) = 2t+1$, $v_f(2) = 2t$ and $e_f(0) = 3t^2+2t+1$, $e_f(1) = e_f(2) = 3t^2+2t$

(iii) $v_f(0) = v_f(1) = 2t+1$, $v_f(2) = 2t$ and $e_f(0) = e_f(2) = 3t^2+2t$, $e_f(1) = 3t^2+2t+1$.

Suppose $(i)$ holds. Since $v_f(0) = 2t$, $2t$ vertices of $V(K_{n,n})$ are labeled with 0.

If these $2t$ vertices are in $V_1$, then they are adjacent with $3t+1$ vertices in $V_2$ and hence $2t \times (3t+1) = 6t^2+2t$ edges have the label 0. Similar case arises when these $2t$ vertices are in $V_2$.

If $t$ vertices are in $V_1$ and the remaining $t$ vertices are in $V_2$, then $t\times(3t+1)+t\times(2t+1) = 5t^2+2t$ edges have the label 0.

In general, if $2t-i$ vertices are in $V_1$ and $i$ vertices are in $V_2$, where $0 \leq i \leq 2t$, then we have $e_f(0) = (2t-i)(3t+1) + i\{3t+1-(2t-i)\} = 6t^2+2t-2it+i^2$. The maximum value for $e_f(0) = 6t^2$ +2t is obtained by putting $i = 0$ or $2t$ and the minimum value for $e_f(0) = 5t^2+2t$ is obtained by putting $i = t$. Thus $5t^2+2t \leq e_f(0) \leq 6t^2+2t$. This is a contradiction to $e_f(0) = 3t^2+2t$, from (i).

Suppose $(ii)$ holds. Since $v_f(0) = 2t+1$, $2t+1$ vertices of $V(K_{n,n})$ are labeled with 0.

If these $2t+1$ vertices are in $V_1$, then they are adjacent with $3t+1$ vertices in $V_2$ and hence $(2t+1)\times(3t+1) = 6t^2+5t+1$ edges have the label 0. Similar case arises when these $2t+1$ vertices are in $V_2$.

If $2t$ vertices are in $V_1$ and the remaining 1 vertex is in $V_2$, then $2t\times(3t+1)+1\times(t+1) = 6t^2+2t+t+1 = 6t^2+3t+1$ edges have the label 0.

In general, if $2t+1-i$ vertices are in $V_1$ and $i$ vertices are in $V_2$, where $0 \leq i \leq 2t+1$, then we have $e_f(0) = (2t+1-i)(3t+1)+i\{(3t+1)-((2t+1)-i)\} = 6t^2+5t+1-2it-i+i^2$. The maximum value for $e_f(0) = 6t^2+5t+1$ is obtained by putting $i = 0$ or $2t+1$ and the minimum value for $e_f(0) = 6t^2+3t+1$ is obtained by putting $i = 1$. Thus $6t^2+3t+1 \leq e_f(0) \leq 6t^2+5t+1$. This is a contradiction to $e_f(0) = 3t^2+2t+1$ from $(ii)$.

Suppose $(iii)$ holds. Since $v_f(1) = 2t+1$, $2t+1$ vertices of $V(K_{n,n})$ are labeled with 1.

If $t$ vertices are in $V_1$ and the remaining $t+1$ vertices are in $V_2$, then $t\times(t+1) = t^2+t$ edges having the label 1.

If $t-1$ vertices are in $V_1$ and the remaining $t+2$ vertices are in $V_2$, then $(t-1)\times(t+2) = t^2+t$

- 2 edges have the label 1.

In general, if $2t + 1 - i$ vertices are in $V_1$ and $i$ vertices are in $V_2$, where $0 \leq i \leq t + 2$, then we have $e_f(0) = (2t+1-i) \times i = 2it + i - i^2$. The maximum value for $e_f(0) = t^2 + t$ is obtained by putting $i = 0$ or $t + 2$ and the minimum value for $e_f(0) = t^2 + t - 2$ is obtained by putting $i = t - 1$. Thus $t^2 + t - 2 \leq e_f(0) \leq t^2 + t$. This is a contradiction to $e_f(0) = 3t^2 + 2t + 1$ from (i). Thus $K_{n,n}$ is not a geometric mean cordial labeling.

**Case(iii):** $n \equiv 2 \pmod 3$.

Let $n = 3t + 2$, where $t \geq 1$. Then $K_{n,n}$ has $6t + 4$ vertices and $9t^2 + 12t + 4$ edges. If $K_{n,n}$ admits a geometric mean cordial labeling, then it should be

(i) $v_f(0) = v_f(1) = 2t + 1$, $v_f(2) = 2t + 2$ and $e_f(0) = e_f(1) = 3t^2 + 4t + 1$, $e_f(2) = 3t^2 + 4t + 2$

(ii) $v_f(0) = v_f(2) = 2t + 1$, $v_f(2) = 2t + 2$ and $e_f(0) = 3t^2 + 4t + 2$, $e_f(1) = e_f(2) = 3t^2 + 4t + 1$

(iii) $v_f(0) = 2t + 2$, $v_f(1) = v_f(2) = 2t + 1$ and $e_f(0) = e_f(2) = 3t^2 + 4t + 1$, $e_f(1) = 3t^2 + 4t + 2$

Suppose $(i)$ holds. Since $v_f(0) = 2t + 1$, $2t + 1$ vertices of $V(K_{n,n})$ are labeled with 0.

If these $2t + 1$ vertices are in $V_1$, then they are adjacent with $3t + 2$ vertices in $V_2$ and hence $(2t + 1) \times (3t + 2) = 6t^2 + 7t + 2$ edges have the label 0. Similar case arises when these $2t + 1$ vertices are in $V_2$.

If $2t$ vertices are in $V_1$ and the remaining one vertex is in $V_2$, then $2t \times (3t + 2) + 1 \times (t + 2) = 6t^2 + 4t + t + 2 = 6t^2 + 5t + 2$ edges have the label 0.

In general, if $2t + 1 - i$ vertices are in $V_1$ and $i$ vertices are in $V_2$, where $0 \leq i \leq 2t + 1$, then we have $e_f(0) = ((2t+1)-i)(3t+2) + i\{(3t+2) - ((2t+1)-i)\} = 6t^2 + 7t + 2 - 2it - i + i^2$. The maximum value for $e_f(0) = 6t^{2+}7t + 2$ is obtained by putting $i = 0$ or $2t + 1$ and the minimum value for $e_f(0) = 6t^2{+}5t{+}2$ is obtained by putting $i = 1$. Thus $6t^2{+}5t{+}2 \leq e_f(0) \leq 6t^2{+}7\text{t}{+}2$. This is a contradiction to $e_f(0) = 3t^2 + 4t + 1$ from (i).

Suppose $(ii)$ holds. Since $v_f(1) = 2t + 2$, $2t + 2$ vertices of $V(K_{n,n})$ are labeled with 1.

If these $2t{+}2$ vertices are in $V_1$, then there is no vertex labeled 1 in $V_2$ and hence $(2t{+}1) \times 0 = 0$ edges have the label 1. Similar case arises when these $2t + 2$ vertices are in $V_2$.

If $t + 2$ vertices are in $V_1$ and the remaining $t$ vertices in $V_2$, then $(t + 2) \times t = t^2 + 2t$ edges have the label 1.

In general, if $2t + 2 - i$ vertices are in $V_1$ and $i$ vertices are in $V_2$, where $0 \leq i \leq 2t + 2$, then we have $e_f(1) = (2t + 2 - i) \times i = 2it + 2i - i^2$. The maximum value for $e_f(1) = t^2 + 2t$ is obtained by putting $i = 0$ or $2t + 2$ and the minimum value for $e_f(1) = 0$ is obtained by putting $i = t$. Thus $0 \leq e_f(1) \leq t^2{+}2\text{t}$. This is a contradiction to $e_f(1) = 3t^2 + 4t + 1$ from (i).

Suppose $(iii)$ holds. Since $v_f(0) = 2t + 2$, $2t + 2$ vertices of $V(K_{n,n})$ are labeled with 0.

If these $2t + 2$ vertices are in $V_1$, then they are adjacent with $3t + 2$ vertices in $V_2$ and hence $(2t + 2) \times (3t + 2) = 6t^2 + 10t + 4$ edges have the label 0. Similar case arises when these $2t + 2$

vertices are in $V_2$.

If $t$ vertices are in $V_1$ and the remaining $t+2$ vertices in $V_2$, then $t\times(3t+2)+(t+2)(2t+2) = 5t^2+8t+4$ edges have the label 0.

In general, if $2t-i$ vertices are in $V_1$ and $i$ vertices are in $V_2$, where $0 \leq i \leq 2t+1$, then we have $e_f(0) = (2t+2-i)(3t+2)+i\{(3t+2)-((2t+2)-i)\} = 6t^2+10t+4-2it-2i+i^2$. The maximum value for $e_f(0) = 6t^{2+}10t+4$ is obtained by putting $i = 0$ or $2t+2$ and the minimum value for $e_f(0) = 5t^2+8t+4$ is obtained by putting $i = t$. Thus $5t^2+8t+4 \leq\ e_f(0) \leq\ 6t^2$+10t+4. This is a contradiction to $e_f(0) = 3t^2+4t+1$, from (i). Thus $K_{n,n}$ is not a geometric mean cordial graph. ■

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