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Odd-even sum labeling of some graphs

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Abstract

A (p, q) graph $G = (V, E)$ is said to be an odd-even sum graph if there exists an injective function $f : V(G) \to \{\pm 1, \pm 3 \pm 5, ..., \pm(2p-1)\}$ such that the induced mapping $f^* : E(G) \to \{2, 4, 6, ..., 2q\}$ defined by $f^*(uv) = f(u) + f(v)\ \forall\ uv \in E(G)$ is bijective. The function f is called an odd-even sum labeling of G. In this paper we study odd-even sum labeling of path $P_n (n \geq 2)$, star $K_{1,n} (n \geq 1)$, bistar $B_{m,n}$,$S(K_{1,n})$, $B(m, n, k)$ and some standard graphs.

Keywords: odd-even sum graph, odd-even sum labeling.
AMS Subject Classification(2010): 05C78.

1 Introduction

Graphs considered in this paper are finite, undirected and without loops or multiple edges. Let $G = (V, E)$ be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary[3]. For number theoretic terminology we follow [1]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions.If the domain of the mapping is the set of vertices(edges/both) then the labeling is called a vertex(edge/total) labeling. There are several types of graph labeling and a detailed survey is found in [6].

Harary [4] introduced the notion of a sum graph. A graph $G = (V, E)$ is called a sum graph if there is an bijection f from V to a set of $+ve$ integers S such that $xy \in E$ if and only if $(f(x) + f(y)) \in S$. In 1991 Harary [5] defined a real sum graph. An injective function $f : V(G) \to \{0, 1, 2, ..., q\}$ is an odd sum labeling [2] if the induced edge labeling f^* defined by $f^*(uv) = f(u) + f(v)\ \forall\ uv \in E(G)$ is bijective and $f^* : E(G) \to \{1, 3, 5, ..., 2q - 1\}$. A graph is said to be an odd sum graph if it admits an odd sum labeling. Ramya et al. introduced skolem even-vertex-odd difference mean labeling in [10]. Ponraj et al. [9] defined pair sum labeling.

Motivated by these, we introduce a new concept called odd-even sum labeling.A (p, q) graph $G = (V, E)$ is said to be an odd-even sum graph if there exists an injective function $f : V(G) \to \{\pm 1, \pm 3 \pm 5, ..., \pm(2p - 1)\}$ such that the induced mapping $f^* : E(G) \to \{2, 4, 6, ..., 2q\}$ defined by $f^*(uv) = f(u) + f(v)\ \forall\ uv \in E(G)$ is bijective. The function f is called an odd-even sum labeling of G. A graph which admits odd-even sum labeling is called an odd-even sum graph. We use the following definitions in the subsequent section.

Definition 1.1. [3] A complete bipartite graph $K_{1,n}(n \geq 1)$ is called a star and it has $n+1$ vertices and n edges.

Definition 1.2. [3] The bistar graph $B_{m,n}$ is obtained from a copy of star $K_{1,m}$ and a copy of star $K_{1,n}$ by joining the vertices of maximum degree by an edge.

Definition 1.3. For each vertex v of a graph G, take a new vertex v'. Join v' to all the vertices of G adjacent to v. The graph $S(G)$ thus obtained is called splitting graph of G.

Definition 1.4. [9] The graph $B(m, n, k)$ is obtained from a path of length k by attaching the star $K_{1,m}$ and $K_{1,n}$ with its pendent vertices.

Definition 1.5. [3] The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has p vertices) and p copies of G_2 and joining the i^{th} vertices of G_1 to every vertex in the i^{th} copy of G_2.

Definition 1.6. The comb $P_n \odot K_1$ is obtained from a path $P_n = u_1u_2...u_n$ by joining a vertex v_i to $u_i(1 \leq i \leq n)$.

Definition 1.7. A coconut tree $CT(m, n)$ is the graph obtained from the path P_m by appending n new pendent edges at an end vertex of P_m.

Definition 1.8. Let $X_i \in N$. Then the cater pillar $S(X_1, X_2, ..., X_n)$ is obtained from the path P_n by joining X_i vertices to each of i^{th} vertex of $P_n(1 \leq i \leq n)$.

2 Main Results

Theorem 2.1. Path P_n, $(n \geq 2)$ is an odd-even sum graph.

Proof: Let V(P_n)= $\{v_i/1 \leq i \leq n\}$ and E(P_n)= $\{v_iv_{i+1}/1 \leq i \leq n-1\}$. Then $|V(P_n)| = n$ and $|E(P_n)| = n - 1$.
Let $f : V(P_n) \to \{\pm1, \pm3 \pm 5, ..., \pm(2n-1)\}$ be defined as follows.
Case(i): n is odd.

$$f(v_{2i+1}) = n + 2i; \qquad 0 \leq i < \frac{n+1}{2},$$
$$f(v_{2i}) = -n + 2i; \quad 1 \leq i < \frac{n+1}{2}.$$

Case(ii): n is even.

$$f(v_{2i+1}) = -(n-1) + 2i; \quad 0 \leq i < \frac{n}{2},$$
$$f(v_{2i}) = n - 1 + 2i; \quad 1 \leq i \leq \frac{n}{2}.$$

Let f^* be the induced edge labeling of f. Then $f^*(v_iv_{i+1}) = 2i$, $1 \leq i \leq n-1$. The induced edge labels are $2, 4, 6, ..., 2n-2$ which are all distinct. Hence path P_n is an odd-even sum graph. ■

Theorem 2.2. The star $K_{1,n}(n \geq 1)$ is an odd-even sum graph.

Proof: Let $v, v_1, v_2, ..., v_n$ be the vertices of $K_{1,n}$. Let $vv_i; \quad 1 \leq i \leq n$ be the edges of $K_{1,n}$. Then $|V(K_{1,n})| = n + 1$ and $|E(K_{1,n})| = n$.
Let $f : V(K_{1,n}) \to \{\pm1, \pm3 \pm 5, ..., \pm(2n + 1)\}$be defined as follows.

$f(v) = 2n + 1,$

$f(v_i) = -[2n + 1 - 2i]; \quad 1 \leq i \leq n.$

Let f^* be the induced edge labeling of f. Then $f^*(vv_i) = 2i$, $1 \leq i \leq n$. The induced edge labels are $2, 4, 6, ..., 2n$ which are all distinct. Hence the star $K_{1,n}$ is an odd-even sum graph. ■

Theorem 2.3. The cycle C_n is not an odd-even sum graph when $n \equiv 2 \pmod 4$ (or) $n \equiv 3 \pmod 4$.

Proof: Let V(C_n)= $\{v_i/1 \leq i \leq n\}$ and E(C_n)= $\{v_iv_{i+1}, v_nv_1/1 \leq i \leq n-1\}$. Then $|V(C_n)| = n$ and $|E(C_n)| = n$.
Suppose C_n is an odd-even sum graph.
Let $f : V(C_n) \to \{\pm1, \pm3 \pm 5, ..., \pm(2n - 1)\}$ be an odd-even sum labeling of C_n.

Case(i): $n \equiv 2 \pmod 4$.

$f(v_1) + f(v_2) + f(v_2) + f(v_3) + ... + f(v_n) + f(v_1) = 2 + 4 + 6 + ... + 2n$

$2(f(v_1) + f(v_2) + f(v_3) + ... + f(v_n)) = n(n + 1)$

$f(v_1) + f(v_2) + f(v_3) + ... + f(v_n) = \frac{n(n+1)}{2}$ which is odd.

This contradicts the choice of n.

Case(ii): $n \equiv 3 \pmod 4$.

$f(v_1) + f(v_2) + f(v_2) + f(v_3) + ... + f(v_n) + f(v_1) = 2 + 4 + 6 + ... + 2n$

$2(f(v_1) + f(v_2) + f(v_3) + ... + f(v_n)) = n(n + 1)$

$f(v_1) + f(v_2) + f(v_3) + ... + f(v_n) = \frac{n(n+1)}{2}$ which is even.

This contradicts the choice of n and the theorem is proved. ■

Theorem 2.4. Bistar $B_{m,n}$ is an odd-even sum graph.

Proof: Let $v, w, v_1, v_2, ..., v_n, w_1, w_2, ..., w_n$ be the vertices of $B_{m,n}$. Let $vv_i; \quad ww_j; \quad 1 \leq i \leq m, 1 \leq j \leq n$ and vw be the edges of $B_{m,n}$. Then $|V(B_{m,n})| = m+n+2, |E(B_{m,n})| = m+n+1$.
Let $f : V(B_{m,n}) \to \{\pm1, \pm3 \pm 5, ..., \pm2(m + n) + 3\}$be defined as follows.

$f(v) = 2(m + n) + 3,\ f(w) = -1$

$f(v_i) = -[2m + 2n + 3 - 2i]; \quad 1 \leq i \leq m,$

$f(w_j) = 2m + 2n + 3 - 2j; \quad 1 \leq j \leq n.$

Let f^* be the induced edge labeling of f. Then

$f^*(vv_i) = 2i; \quad 1 \leq i \leq m,$

$f^*(vw) = 2m + 2n + 2,$

$f^*(ww_j) = 2m + 2n + 2 - 2j, 1 \leq j \leq n.$

The induced edge labels are $2, 4, 6, ..., 2m + 2n + 2$ which are all distinct. Hence bistar $B_{m,n}$ is an odd-even sum graph. ■

Theorem 2.5. $S(K_{1,n})$ is an odd-even sum graph.

Proof: Let $v_1, v_2, ..., v_n$ be the pendant vertices, v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, ..., u_n$ be the added vertices corresponding to $v, v_1, v_2, ..., v_n$ to obtain $S(K_{1,n})$. Then $|V(S(K_{1,n}))| = 2n + 2$ and $|E(S(K_{1,n}))| = 3n$.
Let $f : V(S(K_{1,n})) \to \{\pm 1, \pm 3 \pm 5, ..., \pm 4n + 3\}$be defined as follows.

$f(u) = 4n + 3,$

$f(v) = 2n + 3,$

$f(u_i) = -(2i + 1); \quad 1 \leq i \leq n,$

$f(v_i) = 2i - 1; \quad 0 \leq i < n.$

Let f^* be the induced edge labeling of f. Then

$f^*(vu_i) = 2n + 2 - 2i; \quad 1 \leq i \leq n,$

$f^*(vv_i) = 2n + 2i; 1 \leq i \leq n,$

$f^*(uv_i) = 4n + 2i; 1 \leq i \leq n.$

The induced edge labels are $2, 4, 6, ..., 6n$ which are all distinct. Hence $S(K_{1,n})$ is an odd-even sum graph. ■

Theorem 2.6. The graph $B(m, n, k)$ is an odd-even sum graph.

Proof: Let $V(B(m, n, k)) = \{v_i, v_j', u_l : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq k\}$ and $E(B(m, n, k)) = \{u_1v_i, u_kv_j', u_lu_{l+1} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq k - 1\}$. Then $|V(B(m, n, k))| = m + n + k$ and $|E(B(m, n, k))| = (m + n + k) - 1$.
Let $f : V(B(m, n, k)) \to \{\pm 1, \pm 3, ..., \pm(2m + 2n + 2k - 1)\}$ be defined as follows.

Case(i): k is odd.

$f(v_{i+1}) = -1 - 2i; \quad 0 \leq i \leq m - 1,$

$f(u_{2i+1}) = 2m + 2n + 2k - 1 - 2i; \quad 0 \leq i < \frac{k+1}{2},$

$f(u_{2i}) = f(v_m) - 2i; \quad 1 \leq i \leq \frac{k-1}{2},$

$f(v_j') = f(u_{k-1}) - 2j; \quad 1 \leq j \leq n.$

Case(ii): k is even.

$f(v_{i+1}) = -1 - 2i; \quad 0 \leq i \leq m - 1,$

$f(u_{2i+1}) = 2m + 2n + 2k - 1 - 2i; \quad 0 \leq i < \frac{k}{2},$

$f(u_{2i}) = f(v_m) - 2i; \quad 1 \leq i \leq \frac{k}{2},$

$f(v_j') = f(u_{k-1}) - 2j; \quad 1 \leq j \leq n.$

In both cases, let f^* be the induced edge labeling of f. Then

$f^*(u_1v_i) = 2m + 2n + 2k - 2i; \quad 1 \leq i \leq m,$

$f^*(u_iu_{i+1}) = 2n + 2k - 2i; \quad 1 \le i < k,$

$f^*(u_kv_j^{'}) = 2n + 2 - 2j; \quad 1 \le j \le n.$

The induced edge labels are $2, 4, 6, ..., (2m + 2n + 2k - 2)$ which are all distinct. Hence $B(m, n, k)$ is an odd-even sum graph. ■

Theorem 2.7. Coconut tree is an odd-even sum graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of path P_n. Let G be the coconut tree $CT(m, n)$. Let $V(G) = \{v_i, v_j^{'}/1 \le i \le m, 1 \le j \le n\}$ and $E(G) = \{v_iv_{i+1}, v_j^{'}v_m/1 \le i < m, 1 \le j \le n\}$. Then $|V(G)| = m + n \quad and \quad |E(G)| = m + n - 1$.
Let $f : V(G) \to \{\pm 1, \pm 3, ..., \pm(2m + 2n - 1)\}$ be defined as follows.

Case(i): m is odd.

$f(v_{2i+1}) = -1 - 2i; \quad 0 \le i \le \frac{m-1}{2},$

$f(v_{2i+2}) = 2m + 2n - 1 - 2i; \quad 0 \le i < \frac{m-1}{2},$

$f(v_j^{'}) = f(v_{m-1}) - 2j; \quad 1 \le j \le n.$

Case(ii): m is even.

$f(v_{2i+1}) = -1 - 2i; \quad 0 \le i < \frac{m}{2},$

$f(v_{2i+2}) = 2m + 2n - 1 - 2i; \quad 0 \le i < \frac{m}{2},$

$f(v_j^{'}) = f(v_{m-1}) - 2j; \quad 1 \le j \le n.$

In both the cases, let f^* be the induced edge labeling of f. Then

$f^*(v_iv_{i+1}) = 2m + 2n - 2i; \quad 1 \le i < m,$

$f^*(v_mv_j) = f^*(v_{m-1}v_m) - 2j; \quad 1 \le j \le n.$

The induced edge labels are $2, 4, 6, ..., (2m + 2n - 2)$ which are all distinct. Hence the graph G is an odd-even sum graph. ■

Theorem 2.8. Caterpillar $S(X_1, X_2, ..., X_n)$ is an odd-even sum graph for all $n > 1$.

Proof: Let $V(S(X_1, X_2, ..., X_n)) = \{v_i : \quad 1 \le i \le n\} \cup \{v_{ij} : \quad 1 \le i \le n, \ 1 \le j \le X_i\}$ and $E(S(X_1, X_2, ..., X_n)) = \{v_iv_{i+1} : \quad 1 \le i \le n - 1\} \cup \{v_iv_{ij} : \quad 1 \le i \le n, \ 1 \le j \le X_i\}$.
Then $|V(S(X_1, X_2, ..., X_n))| = X_1 + X_2 + ... + X_n + n$ and $|E(S(X_1, X_2, ..., X_n))| = X_1 + X_2 + ... + X_n + n - 1$.
Let $f : V(S(X_1, X_2, ..., X_n)) \to \{\pm 1, \pm 3, ..., \pm 2(X_1 + X_2 + ... + X_n + n) - 1\}$ be defined as follows.

Case(i): n is odd.

$f(v_{2i+1}) = 2(X_1 + X_2 + ... + X_n + n) - 1 - 2i; \quad 0 \le i < \frac{n+1}{2},$

$f(v_{2i+2}) = -1 - 2i; \quad 0 \le i < \frac{n-1}{2},$

$f(v_{1(i+1)}) = -(2n - 1) - 2i; \quad 0 \le i \le X_1 - 1,$

$f(v_{2(i+1)}) = 2(X_1 + X_2 + ... + X_n + n) - 1 - 2n - 2(X_1 - 1) - 2i; \quad 0 \le i \le X_2 - 1,$

$f(v_{(2j+3)(i+1)}) = f(v_{(2j+1)(X_{2j+1})}) - 2X_{2(j+1)} - 2i; \quad 0 \le i < X_{2j+3}, 0 \le j < \frac{n-1}{2},$

$f(v_{(2j+4)(i+1)}) = f(v_{(2j+2)(X_{2j+2})}) - 2X_{2j+3} - 2i; \quad 0 \le i < X_{2j+4}, 0 \le j < \frac{n-3}{2}.$

Case(ii): n is even.

$f(v_{2i+1}) = 2(X_1 + X_2 + ... + X_n + n) - 1 - 2i; \quad 0 \le i < \frac{n}{2},$

$f(v_{2i+2}) = -1 - 2i; \quad 0 \le i < \frac{n}{2},$

$f(v_{1(i+1)}) = -(2n-1) - 2i; \quad 0 \le i \le X_1 - 1,$

$f(v_{2(i+1)}) = 2(X_1 + X_2 + ... + X_n + n) - 1 - 2n - 2(X_1 - 1) - 2i; \quad 0 \le i \le X_2 - 1,$

$f(v_{(2j+3)(i+1)}) = f(v_{(2j+1)(X_{2j+1})}) - 2X_{2(j+1)} - 2i; \quad 0 \le i < X_{2j+3}, 0 \le j < \frac{n-2}{2},$

$f(v_{(2j+4)(i+1)}) = f(v_{(2j+2)(X_{2j+2})}) - 2X_{2j+3} - 2i; \quad 0 \le i < X_{2j+4}, 0 \le j < \frac{n-2}{2}.$

In both cases, let f^* be the induced edge labeling of f. Then

$f^*(v_i v_{i+1}) = 2(X_1 + X_2 + ... + X_n + n) - 2i; \quad 1 \le i \le n-1,$

$f^*(v_1 v_{1j}) = f^*(v_{n-1} v_n) - 2j; \quad 1 \le j \le X_1,$

$f^*(v_i v_{ij}) = f^*(v_{i-1} v_{(i-1)(X_{i-1})}) - 2j; \quad 2 \le i \le n, 1 \le j \le X_i.$

The induced edge labels are $2, 4, 6, ..., 2(X_1 + X_2 + ... + X_n + n - 1)$ which are all distinct. Hence caterpillar is an odd-even sum graph. ■

Illustration 2.9. Odd-even sum labeling of $S(3, 5, 2, 7, 4)$ is given in Figure 1.

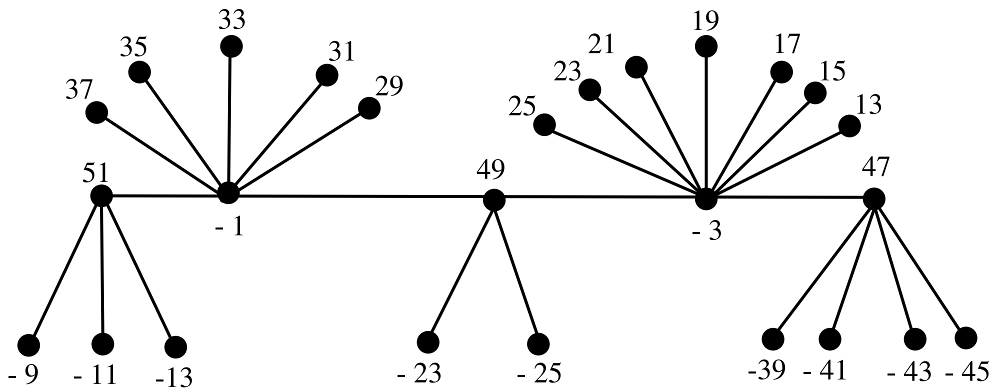

Figure 1: Odd-even sum labeling of $S(3, 5, 2, 7, 4)$.

Corollary 2.10. The graph $P_n \odot nK_1 (n \ge 2)$ admits an odd-even sum labeling.

Proof: Let $X_1 = X_2 = ... = X_n = K$ in Theorem 2.8. Then the result follows. ■

Corollary 2.11. The graph $P_{n-1}(1, 2, 3, ..., n)$ admits an odd-even sum labeling.

Proof: Let $X_i = i$ in Theorem 2.8. Then the result follows. ■

Corollary 2.12. Comb is odd-even sum labeling.

Proof: Let $X_1 = X_2 = ... = X_n = 1$ in Theorem 2.8. Then the result follows. ∎

Corollary 2.13. The graph obtained by deleting a pendent edge $P_n^+ - e_0$ admits an odd-even sum labeling.

Proof: Let $X_1 = X_2 = ... = X_{n-1} = 1, X_n = 0$ in Theorem 2.8. Then the result follows. ∎

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