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Divisor cordial labeling in context of ring sum of graphs

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Abstract

A graph $G = (V, E)$ is said to have a divisor cordial labeling if there is a bijection $f : V(G) \to \{1, 2, \dots |V(G)|\}$ such that if each edge $e = uv$ is assigned the label 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a divisor cordial labeling, then it is called divisor cordial graph. In this paper we derive divisor cordial labeling of ring sum of different graphs.

Keywords: Divisor cordial labeling, ring sum of two graphs.
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1 Introduction

By a graph, we mean a simple, finite, undirected graph. For terms not defined here, we refer to Gross and Yellen [3]. For standard terminology and notations related to number theory we refer to Burton [4]. Varatharajan et al.[7] introduced the concept of divisor cordial labeling of a graph. The divisor cordial labeling of various types of graphs are presented in [6, 8]. The brief summary of definitions which are necessary for the present investigation are provided below.

Definition 1.1. A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.

Notation 1.2. For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Then
$v_f(i) :=$ number of vertices of G having label i under f.
$e_f(i) :=$ number of edges of G having label i under f^*.

Definition 1.3. A binary vertex labeling f of a graph G is called cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[1]. The concept was generalized and extended to $k-$equitable labeling[2]. There are other labeling schemes with minor variations in cordial theme such as the product cordial labeling, total product cordial labeling, prime cordial labeling and divisor cordial labeling. The present work is focused on divisor cordial labeling.

Definition 1.4. Let $G = (V, E)$ be a simple, finite, connected and undirected graph. A bijection $f : V \to \{1, 2, \ldots |V|\}$ is said to be divisor cordial labeling if the induced function $f^* : E \to \{0, 1\}$ defined by

$$f^*(e = uv) = \begin{cases} 1; & \text{if } f(u)|f(v) \text{ or } f(v)|f(u), \\ 0; & \text{otherwise,} \end{cases}$$

satisfies the condition $|e_f(0) - e_f(1)| \leq 1$. A graph that admits a divisor cordial labeling is called a divisor cordial graph.

Definition 1.5. A chord of a cycle C_n is an edge joining two non-adjacent vertices.

Definition 1.6. Two chords of a cycle are said to be twin chords if they form a triangle with an edge of the cycle C_n.
For positive integers n and p with $5 \leq p+2 \leq n, C_{n,p}$ is the graph consisting of a cycle C_n with twin chords with which the edges of C_n form cycle C_p,C_3 and C_{n+1-p} without chords.

Definition 1.7. A cycle with triangle is a cycle with three chords which by themselves form a triangle.
For positive integers p, q, r and $n \geq 6$ with $p + q + r + 3 = n$, $C_n(p, q, r)$ denotes a cycle with triangle whose edges form the edges of cycles C_{p+2}, C_{q+2}, C_{r+2} without chords.

Definition 1.8. Ring sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denoted by $G_1 \oplus G_2$, is the graph $G_1 \oplus G_2 = (V_1 \cup V_2, (E_1 \cup E_2) - (E_1 \cap E_2))$.

Remark 1.9. Throughout this paper we consider the ring sum of a graph G with $K_{1,n}$ by considering any one vertex of G and the apex vertex of $K_{1,n}$ as a common vertex.

2 Main Results

Theorem 2.1. $C_n \oplus K_{1,n}$ is a divisor cordial graph for all $n \in \mathbb{N}$.

Proof: Let $V(C_n \oplus K_{1,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of C_n and $V_2 = \{v = u_1, v_1, v_2, \ldots, v_n\}$ be the vertex set of $K_{1,n}$. Here $v_1, v_2, \ldots, v_n$ are pendant vertices and v is the apex vertex of $K_{1,n}$. Also $|V(C_n \oplus K_{1,n})| = |E(C_n \oplus K_{1,n})| = 2n$.

We define a labeling $f : V(C_n \oplus K_{1,n}) \to \{1, 2, 3, \ldots, 2n\}$ as follows:

$f(u_1) = f(v) = 2, f(v_1) = 1,$

$f(u_i) = 2i - 1; \quad 2 \leq i \leq n,$
$f(v_j) = 2j; \quad 2 \leq j \leq n.$

According to this pattern the vertices are labeled such that for any edge $e = u_iu_{i+1}$ in C_n, $f(u_i) \nmid f(u_{i+1})$, $1 \leq i \leq n-1$.

Also $f(v_1) \mid f(v)$ and $f(v) \mid f(v_j)$ for each j, $2 \leq j \leq n$.
Hence $e_f(1) = e_f(0) = n$.

Thus the graph $C_n \oplus K_{1,n}$ admits a divisor cordial labeling and hence $C_n \oplus K_{1,n}$ is a divisor cordial graph. ■

Example 2.2. A divisor cordial labeling of $C_5 \oplus K_{1,5}$ is shown in Figure 1.

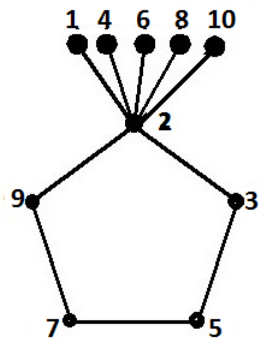

Figure 1: A divisor cordial labeling of $C_5 \oplus K_{1,5}$.

Theorem 2.3. The graph $G \oplus K_{1,n}$ is a divisor cordial graph for all $n \geq 4, n \in \mathbb{N}$, where G is the cycle C_n with one chord forming a triangle with two edges of C_n.

Proof: Let G be the cycle C_n with one chord. Let $V(G \oplus K_{1,n}) = V_1 \cup V_2$, where V_1 is the vertex set of G and V_2 is the vertex set of $K_{1,n}$. Let $u_1, u_2, \ldots, u_n$ be the successive vertices of C_n and $e = u_2u_n$ be the chord of C_n.

The vertices u_1, u_2, u_n form a triangle with the chord e. Let $v_1, v_2, \ldots, v_n$ be the pendant vertices, v be the apex vertex of $K_{1,n}$. Take $v = u_1$. Also $|V(G \oplus K_{1,n})| = 2n$ and $|E(G \oplus K_{1,n})| = 2n + 1$.

We define a labeling $f : V(G \oplus K_{1,n}) \to \{1, 2, 3, \ldots, 2n\}$ as follows:
$f(u_1) = f(v) = 2, f(v_1) = 1,$
$f(u_i) = 2i - 1; \quad 2 \leq i \leq n,$
$f(v_j) = 2j; \quad 2 \leq j \leq n.$

According to this pattern the vertices are labeled such that for any edge $e = u_iu_{i+1} \in G$, $f(u_i) \nmid f(u_{i+1})$, $1 \leq i \leq n-1$.

Also $f(v_1) \mid f(v)$ and $f(v) \mid f(v_j)$ for each j, $2 \leq j \leq n$.

Hence $e_f(1) = n, e_f(0) = n + 1$.

Thus $|e_f(0) - e_f(1)| \leq 1$ and the graph $G \oplus K_{1,n}$ admits divisor cordial labeling. Therefore, $G \oplus K_{1,n}$ is a divisor cordial graph. ■

Example 2.4. A divisor cordial labeling of ring sum of C_6 with one chord and $K_{1,6}$ is shown in Figure 2.

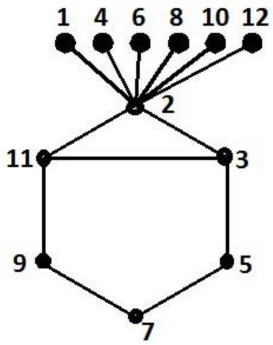

Figure 2: A divisor cordial labeling of ring sum of C_6 with one chord and $K_{1,6}$.

Theorem 2.5. The graph $G \oplus K_{1,n}$ is a divisor cordial graph for all $n \geq 5, n \in \mathbb{N}$, where G is the cycle with twin chords forming two triangles and another cycle C_{n-2} with the edges of C_n.

Proof: Let G be the cycle C_n with twin chords, where chords form two triangles and one cycle C_{n-2}. Let $V(G \oplus K_{1,n}) = V_1 \cup V_2$.

$V_1 = \{u_1, u_2, \ldots, u_n\}$ is the vertex set of C_n, $e_1 = u_n u_2$ and $e_2 = u_n u_3$ are the chords of C_n.

$V_2 = \{v = u_1, v_1, v_2, \ldots, v_n\}$ is the vertex set of $K_{1,n}$, where $v_1, v_2, \ldots, v_n$ are pendant vertices and $v = u_1$ is the apex vertex.

Also $|V(G \oplus K_{1,n})| = 2n$ and $|E(G \oplus K_{1,n})| = 2n + 2$.

We define a labeling $f : V(G \oplus K_{1,n}) \to \{1, 2, 3, \ldots, 2n\}$ as follows:

$$f(u_1) = f(v) = 2, f(v_1) = 3, f(u_n) = 2n - 1, f(u_{n-1}) = 1,$$
$$f(u_i) = 2i + 1; \quad 2 \leq i \leq n - 2,$$
$$f(v_j) = 2j; \quad 2 \leq j \leq n.$$

According to this pattern the vertices are labeled such that for any edge $e = u_i u_{i+1} \in G$, $f(u_i) \nmid f(u_{i+1})$, $1 \leq i \leq n - 1$.

Also $f(v_1) \mid f(v)$ and $f(v) \mid f(v_j)$ for each j, $2 \leq j \leq n$. and $f(u_{n-1}) \mid f(u_{n-2})$. Hence $e_f(0) = n + 1 = e_f(1)$.

Thus the graph admits a divisor cordial labeling and hence $G \oplus K_{1,n}$ is a divisor cordial graph. ■

Example 2.6. A divisor cordial labeling of ring sum of cycle C_7 with twin chords and $K_{1,7}$ is shown in Figure 3.

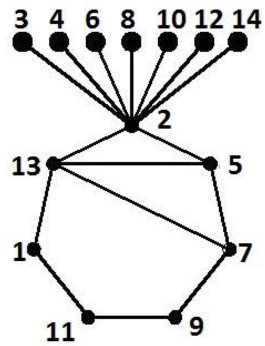

Figure 3: A divisor cordial labeling of ring sum of C_7 with twin chords and $K_{1,7}$.

Theorem 2.7. The graph $G \oplus K_{1,n}$ is a divisor cordial graph for all $n \geq 6$, $n \in \mathbb{N}$, where G is a cycle with triangle $C_n(1, 1, n-5)$.

Proof: Let G be cycle with triangle $C_n(1, 1, n-5)$. Let $V(G \oplus K_{1,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots, u_n\}$ is the vertex set of G and $V_2 = \{v = u_1, v_1, v_2, \ldots, v_n\}$ is the vertex set of $K_{1,n}$. Here $v_1, v_2, \ldots, v_n$ are the pendant vertices and v is the apex vertex of $K_{1,n}$.

Let u_1, u_3 and u_{n-1} be the vertices of triangle formed by edges $e_1 = u_1u_3, e_2 = u_3u_{n-1}$ and $e_3 = u_1u_{n-1}$. Also $|V(G \oplus K_{1,n})| = 2n$ and $|E(G \oplus K_{1,n})| = 2n + 3$.

We define a labeling $f : V(G \oplus K_{1,n}) \to \{1, 2, 3, \ldots, 2n\}$ as follows:

$f(u_1 = v) = 2, f(v_1) = 3, f(u_2) = 1,$
$f(u_i) = 2i - 1; \quad 3 \leq i \leq n,$
$f(v_j) = 2j; \quad 2 \leq j \leq n.$

According to this pattern the vertices are labeled such that for any edge $e = u_iu_{i+1} \in G$, $f(u_i) \nmid f(u_{i+1})$, $1 \leq i \leq n-1$.

Also $f(v_1) \nmid f(v)$ and $f(v) \mid f(v_j)$ for each j, $2 \leq j \leq n$. $f(v) \mid f(u_2), f(u_2) \mid f(u_3)$.

Then $|e_f(0) - e_f(1)| \leq 1$. Hence $G \oplus K_{1,n}$ is a divisor cordial graph. ■

Example 2.8. A divisor cordial labeling of ring sum of cycle C_8 with triangle and $K_{1,8}$ is shown in Figure 4.

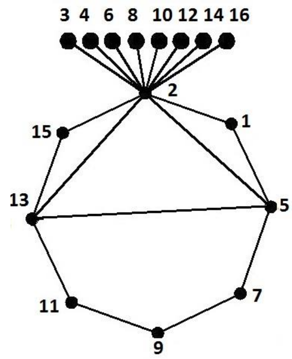

Figure 4: A divisor cordial labeling of ring sum of $C_8(1,1,3)$ and $K_{1,8}$.

Theorem 2.9. The graph $P_n \oplus K_{1,n}$ is a divisor cordial graph for all $n \in \mathbb{N}$.

Proof: Let $V(P_n \oplus K_{1,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots, u_n\}$ is the vertex set of P_n and $V_2 = \{v = u_1, v_1, v_2, \ldots, v_n\}$ is the vertex set of $K_{1,n}$. Here $v_1, v_2, \ldots, v_n$ are the pendant vertices and v is the apex vertex. Also $|V(G \oplus K_{1,n})| = 2n, |E(G \oplus K_{1,n})| = 2n - 1$.

We define a labeling $f : V(G \oplus K_{1,n}) \to \{1, 2, 3, \ldots, 2n\}$ as follows:

$f(u_1) = f(v) = 2, f(v_1) = 1,$
$f(u_i) = 2i - 1; \quad 2 \le i \le n,$
$f(v_j) = 2j; \quad 2 \le j \le n.$

According to this pattern the vertices are labeled such that for any edge $e = u_i u_{i+1} \in G$, $f(u_i) \nmid f(u_{i+1})$, $1 \le i \le n-1$.Also $f(v) \mid f(v_j)$ for each j, $1 \le j \le n$.

Then we have $e_f(1) = n, e_f(0) = n - 1$. Hence the graph admits a divisor cordial labeling. Theredore, $P_n \oplus K_{1,n}$ is a divisor cordial graph. ■

Example 2.10. A divisor cordial labeling of $P_5 \oplus K_{1,5}$ is shown in Figure 5.

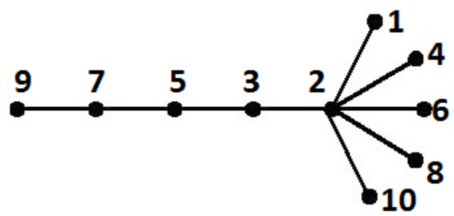

Figure 5: A divisor cordial labeling of $G \oplus K_{1,5}$.

Definition 2.11. The double fan graph DF_n is defined as $DF_n = P_n + 2K_1$.

Theorem 2.12. The graph $DF_n \oplus K_{1,n}$ is a divisor cordial graph for every $n \in \mathbb{N}$.

Proof: Let $V(DF_n \oplus K_{1,n}) = V_1 \cup V_2$, where $V_1 = \{u, w, u_1, u_2, \ldots, u_n\}$ be the vertex set of DF_n and $V_2 = \{v = w, v_1, v_2, \ldots, v_n\}$ be the vertex set of $K_{1,n}$. Here $v_1, v_2, \ldots, v_n$ are pendant vertices and v be the apex vertex of $K_{1,n}$. Also $|V(G \oplus K_{1,n})| = 2n+2, |E(G \oplus K_{1,n})| = 4n-1$. We define a labeling $f : V(G \oplus K_{1,n}) \to \{1, 2, 3, \ldots, 2n+2\}$ as follows:

$f(w) = f(v) = 1, f(u) = p$, where p is largest prime number.

$f(v_1) = 2, f(u_1) = 3,$

$f(u_i) = 2i;\ 1 \leq i \leq n.$

Assign the remaining labels to the remaining vertices $v_1, v_2, \ldots, v_n$ in any order.

According to this pattern the vertices are labeled such that for any edge $e = u_i u_{i+1}$, $f(u_i) \nmid f(u_{i+1})$, $1 \leq i \leq n-1$. Also $f(v) \mid f(v_j)$ for each j, $1 \leq j \leq n$. and $f(v) \mid f(u_i)$ for each i, $1 \leq i \leq n$. and $f(u) \nmid f(u_i)$ for each i, $1 \leq i \leq n$.

Then we have $e_f(1) = 2n, e_f(0) = 2n-1$. Hence the graph admits divisor cordial labeling. Therefore, $DF_n \oplus K_{1,n}$ is a divisor cordial graph. ■

Example 2.13. A divisor cordial labeling of $DF_5 \oplus K_{1,5}$ is shown in Figure 6.

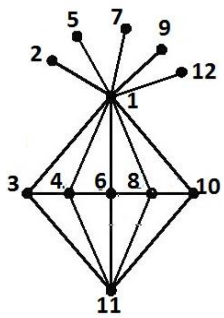

Figure 6: A divisor cordial labeling of $DF_5 \oplus K_{1,5}$.

Definition 2.14. The flower fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm. It contains three types of vertices: an apex of degree $2n$, n vertices of degree 4 and n vertices of degree 2.

Theorem 2.15. The graph $fl_n \oplus K_{1,n}$ is a divisor cordial graph for every $n \in \mathbb{N}$.

Proof: Let $V(fl_n \oplus K_{1,n}) = V_1 \cup V_2$, $V_1 = \{u, u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\}$ be the vertex set of fl_n, where u is the apex vertex, $u_1, u_2, \ldots, u_n$ are the internal vertices and $w_1, w_2, \ldots, w_n$ are the external vertices. Let $V_2 = \{v = w_1, v_1, v_2, \ldots, v_n\}$ be the vertex set of $K_{1,n}$, where $v_1, v_2, \ldots, v_n$ are pendant vertices and v is the apex vertex of $K_{1,n}$. Also note that $|V(fl_n \oplus K_{1,n})| = 3n+1, |E(G \oplus K_{1,n})| = 5n$.

We define a labeling $f : V(fl_n \oplus K_{1,n}) \to \{1, 2, 3, \ldots, 3n+1\}$ as follows:

$f(u) = 1,$
$f(u_i) = 2i + 1; \quad 1 \leq i \leq n,$
$f(w_i) = 2i; \quad 1 \leq i \leq n.$

Assign the remaining labels to the remaining vertices $v_1, v_2, \ldots, v_n$ in any order. According to this pattern the vertices are labeled such that for any edge $e = u_i u_{i+1} \in G, f(u_i) \nmid f(u_{i+1})$, $1 \leq i \leq n-1$. Further $f(u) \mid f(u_i)$, $f(u) \mid f(w_i)$ for each i, $1 \leq i \leq n$ and $f(v) \mid f(v_i)$ if i is odd, $1 \leq i \leq n$.

Then $|e_f(0) - e_f(1)| \leq 1$. Hence the graph admits divisor cordial labeling and $fl_n \oplus K_{1,n}$ is a divisor cordial graph. ■

Example 2.16. A divisor cordial labeling of the graph $fl_4 \oplus K_{1,4}$ is shown in Figure 7.

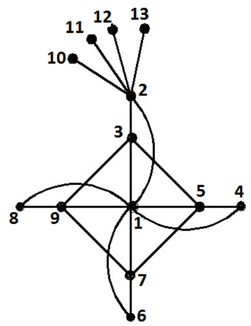

Figure 7: A divisor cordial labeling of $fl_4 \oplus K_{1,4}$.

Remark 2.17. In all the above theorems, for the ring sum operation one can consider any arbitrary vertex of G and by different permutations of the vertex labels provided in the above defined labeling pattern one can easily check that the resultant graph is divisor cordial.

Concluing Remarks: The divisor cordial labeling is an invariant of cordial labeling by minor variation in the definition using divisor of a number. Here we have derived divisor cordial graphs in context of the operation ring sum of graphs. It is interesting to see whether divisor cordial graphs are invariant under ring sum or any other graph operation or not.

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