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Some extremal problems in $(d, d+1)-$regular graphs

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Abstract

By $(d, d+1)$ - regular graph, we mean a graph whose vertices are all having degree either d or d+1. Two graphs are said to be sequentially equivalent (SE) if they have the same degree sequence. In this paper, we discuss some extremal problems of general $(d, d+1)$ -regular graphs, and SE $(d, d+1)$ - regular graphs.

Keywords: degree sequence, $(d, \mathrm{d}+1)$ – regular graphs, semiregular graph**.**

AMS Subject Classification (2010): 05C35.

1 Introduction

In general by an extremal problem, we mean a problem that searches boundaries of the collection of objects under consideration. The objects may be elements, sets, functions, structures, algorithms and so on. In graph theory, there are a lot of extremal problems that provide a base for the development of the subject in the desired direction. The extremal problems of enquiring minimum or maximum number of edges or vertices of the given kind of graph so as to get the desired structure, the minimum number of edges required to ensure the connectedness of the graph containing n vertices, the optimum size of the graph that guarantees the presence or absence of the well-known structure in it are some of the interesting fundamental extremal problems in graph theory [2].

Recently, Xiao-Dong Zhang [6] studied on the extremal problems in a given set consisting of all simple connected graphs with the same graphical sequence. In this paper, we discuss some extremal problems of $(d, d+1)$– regular graphs. By $(d, d+1)$ - regular graph we mean a graph whose vertices all have degrees in the set $\{d, d+1\}$. Such graphs are a class of semiregular graphs [3]. In general, the graphs whose vertices all have degrees in the set $\{r, s\}$ are called semiregular graphs. Allison Northup [1] defined the graphs in which every vertex has exactly n number of vertices at distance two as n-semiregular graphs. The basic properties of these graphs can be seen in [4]. The n-semiregularity of certain special graphs has been discussed in [5]. The factorization of $(d, d+1)$ - regular graphs have been studied in [3].

Throughout this paper, by a graph we mean the simple connected graph. In this paper, we particularly discuss the minimum number of edges and the maximum number of possible $(d, d+1)$ - graphs that have same graphical sequence of the graph containing n-vertices.

2 Observations

The following are some of the observations that we can see from the $(d, d+1)$ – regular graphs, for d = 1,2,3,4.

i. There is only one (1,2) - regular graph with 3 vertices
ii. There is only one (2,3) - regular graph with 4 vertices
iii. A (2,3) - regular graph has at least four vertices and five edges
iv. Every (3,4) - regular graph has at least five vertices and eight edges
v. Every (4,5) - regular graph has at least six vertices and thirteen edges

3 Extremal Problems

Lemma 3.1. Every $(d, d+1)$ - regular graph has atleast $d+2$ vertices.

Proof: Let G be a $(d, d+1)$ - regular graph. Since, there exists a vertex v in $V(G)$ such that $(v) = v+1$, G must have at least $d+2$ vertices.

Theorem 3.2. The minimum number of edges in a $(d, d+1)$ -regular graph is $a_d = \left\lceil \frac{(d+1)^2}{2} \right\rceil$, where $a_0 = 1$ and $a_1 = 2$.

Proof: It can be seen that the $(d, d+1)$- regular graph of order n with minimum number of edges would have $n-2$ vertices of degree d, and 2 vertices of degree $d+1$ when d is even. Hence, by the property that the number of edges in a graph G is $\frac{\sum d}{2}$, where d is degree of a vertex in a given graph G, and also by the fact that every $(d, d+1)$-regular graph has at least $d+2$ vertices. Thus the minimum number of edges in the $(d, d+1)$- regular graph when d is even is given by $a_d = \frac{(n-2)d+2(d+1)}{2} = \frac{(d+1)^2}{2} + \frac{1}{2} = \left\lceil \frac{(d+1)^2}{2} \right\rceil$.

Similarly, if d is odd, the $(d, d+1)$- regular graph of order n with minimum number of edges would have $(n-1)$ vertices of degree d and there is only one vertex of degree $d+1$. Hence the minimum number of edges in the $(d, d+1)$- regular graph when d is odd is given by

$a_d = \frac{(n-1)d+d+1}{2} = \frac{(d+1)d+d+1}{2} = \left\lceil \frac{(d+1)^2}{2} \right\rceil$. ■

Theorem 3.3. The number of SE $(2,3)$- regular graphs with n vertices is $\left\lfloor \frac{n}{2} \right\rfloor$ when n is odd and $\frac{n-2}{2}$ when n is even.

Proof: A $(2,3)$-regular graph should have at least 4 vertices. Since the number of vertices with odd degree is always even, the $(2,3)$-regular graph will have exactly 2 vertices with degree 3. Only one such graph is possible. Hence the theorem is true when $n = 4$. A $(2,3)$- regular graph can be identified with the number of vertices with degree 3, because any two $(2,3)$- regular graphs with same number of vertices with degree 3 are isomorphic to each other. Hence the number $(2,3)$-regular graphs with n vertices is equal to the number of even numbers $\leq n$ which is clearly $\left\lfloor \frac{n}{2} \right\rfloor$ when n is odd and $\frac{n-2}{2}$ when n is even. In the later case, the number of 0 and n are excluded, otherwise the graph becomes either 2-regular or 3-regular respectively instead of $(2,3)$-regular. ■

Theorem 3.4. For every $(2,3)$-regular graph with n-vertices, there is a $(3,4)$ - regular graph with *n* vertices, where $n \geq 4$.

Proof: Let *G* be a $(2,3)$ - regular graph with *n* vertices. First let us take *n* be odd. Let $d_1, \ldots\ldots, d_n$ be the degree sequence of G such that $d_i \geq d_j$,$1 \leq i,j \leq n$ and $d_i = 2 \; or \; 3$. Since *n* is odd and there is always even number of vertices of odd degree in a graph G, there are $\left\lfloor \frac{n}{2} \right\rfloor$ number of $(2,3)$ graphs and hence $\left\lfloor \frac{n}{2} \right\rfloor$ number of degree sequences so that each sequence contains even number of 3's. The two extreme possibilities are

i. The sequence containing only two 3's.

ii. The sequence containing $n-1$ number of 3's.

For the both cases, the sequence $d_1, d_2, (d_3+1), \ldots\ldots (d_{n-1}+1), (d_n+1)$ is the degree sequence of the corresponding $(3,4)$ $-$graphs. For the other cases, let $d_1 = d_2 = \cdots = d_i = 3$ and $d_{i+1} = d_{i+2} = \cdots = d_n = 2$. Then $d_1, d_2, \ldots\ldots, d_i$, $(d_{i+1}+2)$,$(d_{i+2}+2)$,…,(d_n+2) is a sequence of corresponding $(3,4)$ - regular graph. Suppose if n is even, there are also $\frac{n}{2}$ number of $(3,4)$ - regular graphs and hence $\frac{n}{2}$ degree sequences containing even number of 3's. The two extreme possibilities are:

i. The sequence containing two 3's and

ii. The sequence containing $(n-2)$ number of 3's.

For both the cases, the sequence $(d_1+1), (d_2+1), (d_3+1), \ldots\ldots, (d_n+1)$ is the degree sequence of the corresponding $(3,4)$ - regular graph.

For all the other cases, if $d_1 = d_2 = \cdots = d_i = 3$ and $d_{i+1} = d_{i+2} = \cdots = d_n = 2$. Then $d_1 d_2, \ldots\ldots d_i \; (d_{i+1}+2) \; (d_{i+2}+2) \ldots\ldots (d_n+2)$ is the corresponding graphical sequence of the required $(3,4)$ – graph. ■

Example 3.5. There are four SE $(2,3)-$ regular graphs with 10 vertices. The corresponding degree sequences are 3322222222, 3333222222, 3333332222 and 3333333322. Adding 1 to each integer in the sequence containing two 3's and eight 3's and adding 2 to two in the remaining sequences, we get four degree sequences 4433333333, 4444443333, 4444333333 and 4444444433. These four sequences represent the four $(3,4)$ - regular graphs. They are also SE graphs. The following theorem gives the more generalized form of Theorem 3.4.

Theorem 3.6. For every $(2,3)$ - regular graph with $n \geq 5$ vertices, there is a $(d, d+1)$- graph where $d = 3,4, \ldots\ldots n-2$ with the same number of vertices.

Proof: This theorem has already been proved for $d = 3$. We also know that every $(d, d+1)$ - regular graph should at least have $d+2$ vertices and therefore $(n-2, n-1)$ - regular graph is the maximal (d, d+1) - regular graph with n vertices. Hence $d = n-2$ is an upper bound for d. Also, for $n = 4$, there is only one $(2,3)$ - regular graph and the construction of further (*d*, *d*+1) - regular graphs is not possible with 4 vertices. Hence n must be greater than or equal to 5. First let *n* be even and $d_1, d_{2,} d_3, d_4, \ldots\ldots, d_n$ be the degree sequence such that $d_1 = d_2 = d_3 = \cdots = d_i$ and $d_{i+1} = d_{i+2} = \cdots = d_n$ such that $d_i > d_{i+1}$, then there are $\frac{n-2}{2}$ number of $(2,3)$ - regular graphs with *n* vertices.

Hence $i = 2,4,6, \dots, \frac{n-2}{2}$. It can be found that the graphical sequence of $(d+1, d+2)$ - regular graph from $(d, d+1)$ - regular graph can be obtained as follows:

i. $(d_1+1), (d_2+1), \dots\dots, (d_n+1)$ for $i = 2, \frac{n-2}{2}$

ii. $d_1, \dots\dots, d_i, (d_{i+1}+2), (d_{i+2}+2), \dots, (d_n+2)$ for all other cases

Hence the theorem is proved when *n* is even.

When *n* is odd, $d_1, \dots\dots, d_n$ be the graphical sequence of $(d, d+1)$ - regular graph with *n* vertices. Let $d_1 = d_2 = d_3 = \cdots = d_i$ and $d_{i+1} = d_{i+2} = \cdots = d_n$. Let $d_i > d_{i+1}$. In the case of (2,3) - regular graph $d_i = 3$ and $d_{i+1} = 2$. It can be observed that there are $\frac{n-1}{2}$ number of $(d, d+1)$ - regular graphs when n is odd and hence correspondingly there are $\frac{n-1}{2}$ degree sequences. It can also be found that the following sequences of $(d+1, d+2)$ –regular graphs can be constructed from $(d, d+1)$ - regular graphs.

i. $d_1, d_2, (d_3+1), \dots\dots, d_{n-1}+1), (d_n+2)$ if d is even and $i = 2, \frac{n-2}{2}$

ii. $(d_1+1), (d_2+1), \dots\dots, (d_{n-1}+1), (d_n+2)$ if d is odd and $i = 2, \frac{n-2}{2}$

iii. $d_1, d_2, \dots\dots, d_i, (d_{i+1}+2), (d_{i+2}+2), \dots, (d_n+2)$ for all other cases. ■

Theorem 3.7. The number of SE $(d, d+1)-$ regular graphs with $n \geq d+2$ vertices is

$$\begin{cases} \frac{n-2}{2} & if\ n\ is\ even \\ \frac{n-1}{2} & if\ n\ is\ odd \end{cases}$$

Proof: The number of SE $(d, d+1)-$ regular graphs with $n \geq d+2$ vertices is the number of combinations of the integers d, $d+1$ among n places such that the odd integer occurs at even number of times. It can be found that such combinations are $\frac{n-2}{2}$ when n is even and $\frac{n-1}{2}$ when n is odd.

Theorem 3.8. For every $(d, d+1)$ - regular graph with n vertices there is a SE $(d+1, d+2)$- regular graph, except when $d+2 = n-1$

Proof: Let *G* be a $(d, d+1) -$ regular graph with *n* vertices. Then *n* must be greater than $d+1$. If $d+2 = n-1$, then $n = d+3$. So $(d+2, d+3)$- regular graph is not possible with $d+3$ vertices. Hence the theorem is not true when $d+2 = n-1$.

Let $d_1, \dots\dots, d_i, d_{i+1}, \dots, d_n$ be the degree sequence of $(d, d+1)$ - regular graph with *n* vertices. If *n* is even, then the number of SE $(d, d+1)-$ regular graphs with *n* vertices is $\frac{n-2}{2}$. The degree sequences of these $\frac{n-2}{2}$ number of graphs respectively contain $2,4, \dots\dots, (n-2)$ times of either d or $d+1$. Consider the degree sequences which contain either d or $d+1$ exactly two times. More precisely, there will be exactly one degree sequence each containing

i. d two times ; $d+1$, $(n-1)$ times and

ii. d, $(n-2)$ times; $(d+1)$, 2 times.

In each case, by adding 1 to each degree we get degree sequences of $(d+1, d+2)$ - regular graphs, respectively with

i. $d + 1$ two times, and $(d + 2)$, $(n - 2)$ times and

ii. $(d + 1)$, $(n - 2)$ times and $(d + 2)$, two times.

For all other degree sequences, by adding 2 to each d gives the degree sequence of a $(d + 1, d + 2)$ - regular graph. This proves the theorem when n is even. If n is odd, there will be $\frac{n-1}{2}$ number of $(d, d + 1)$ - regular graphs. The theorem completes, if we define a degree sequence for a $(d + 1, d + 2)$ - regular graph from the degree sequence of a $(d, d + 1)$ - regular graph. Let $d_1, \dots\dots, d_n$ be the degree sequence of $(d,\ d + 1)$ - regular graphs. The various possible cases are as follows:

Values of *d*	Degree sequence of $(d + 1, d + 2)$-regular graphs
<u>When *d* is even</u>	
i. d + 1 occurs two times	$d_1 d_2 (d_3 + 1) \dots\dots (d_{n-1} + 1)(d_n + 2)$
ii. d + 1 occurs (n-1) times	$d_1 d_2 (d_3 + 1) \dots\dots (d_{n-1} + 1)(d_n + 2)$
iii. For all other cases when $d_1 = d_2 = d_3 = \cdots \dots . d_i = d + 1$ $d_{i+1} = d_{i+2} = \cdots \dots . d_n = d$	$d_1 d_2 \dots\dots d_i (d_{i+1} + 2) \dots\dots (d_n + 2)$
<u>When *d* is odd</u>	
i. d occurs either 2 times or (*n*-1) times	$(d_1 + 1)(d_2 + 1) \dots\dots (d_{n-1} + 1)(d_n + 2)$
ii. $d_1 = d_2 = d_3 = \cdots \dots . = d_i = d + 1$ $d_{i+1} = d_{i+2} = \cdots \dots . = d_n = d$	$d_1 d_2 \dots\dots d_i (d_{i+1} + 2) \dots\dots (d_n + 2)$

Conclusion: In this paper, some minimal $(d, d + 1)$ - regular graphs have been observed. The minimum number of edges in the general $(d, d + 1)$ - regular graphs, the constructions of $(d, d + 1)$ - regular graph from (2,3)- regular graph and the number of SE $(d, d + 1)$ - regular graphs have been discussed in detail for the given order of the graph.

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