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Upper vertex covering number and well covered semigraphs

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Abstract

In this paper, we define minimal vertex covering sets with maximum cardinality in the given semigraph and the upper vertex covering number of a semigraph. We prove that this number does not increase when a vertex is removed from a semigraph. We also introduce well-covered semigraphs, approximately well covered semigraphs and proved some related results.

Keywords: Semigraph, α_b-set , well covered semigraph, approximately well covered semigraph.
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1 Introduction

Semigraphs provide an important generalization of graphs. Several interesting theorems on semigraphs have been proved in [3], [5] and [6]. Domination related parameters have also been studied in [4]. In this study, we include one more parameter namely vertex covering number of a semigraph. We consider minimal vertex set with maximum cardinality (called $\alpha_b - sets$). We prove that the upper vertex covering number of a semigraph does not increase when a vertex is removed from the semigraph.

We consider semigraphs for which either the vertex covering number is same as the upper vertex covering number or their differences exactly one. These semigraphs are called well covered and approximately well covered semigraphs respectively. For terminology related to semigraphs, the reader can refer to [6].

2 Preliminaries

Definition 2.1. [6] A semigraph G is a pair (V, X) where V is a non-empty set whose elements are called vertices of G and X is a set of n-tuples called edges of G, of distinct vertices, for various $n \geq 2$, satisfying the following conditions:

(i) Any two edges have at most one vertex in common.

(ii) Two edges $(u_1, u_2, ..., u_n)$ and $(v_1, v_2, ..., v_m)$ are considered to be equal if and only if

(a) $m = n$ and

(b) either $u_i = v_i$ for $1 \leq i \leq n$, or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Thus, the edge $e = (u_1, u_2, \ldots, u_n)$ is the same as $(u_n, u_{n-1}, \ldots, u_1)$. u_1 and u_n are called the end vertices of an edge e and $u_2, u_3, \ldots, u_{n-1}$ are called the middle vertices of e.

Definition 2.2. [6] A subset S of a semigraph G is said to be a vertex covering set of G, if for every edge e there is a vertex v in the edge e such that $v \in S$. A vertex covering set with minimum cardinality is called $\alpha_0 - set$ of G. The cardinality of a minimum vertex covering set of G is called the vertex covering number of G and it is denoted as $\alpha_0(G)$. It is obvious to see that a subset S of $V(G)$ is a minimum vertex covering set if and only if $V(G) - S$ is a maximum independent set.

Definition 2.3. Let G be a semigraph, $S \subseteq V(G)$ then S is said to be a minimal vertex covering set if

(1) S is a vertex covering set of G and

(2) $S - \{v\}$ is not a vertex covering set of G for every vertex v in S.

Definition 2.4. Let G be a semigraph. A minimal vertex covering set with maximum cardinality is said to be an upper vertex covering set ($\alpha_b - set$) of G. The number of vertices of an $\alpha_b - set$ is called the upper vertex covering number of G and it is denoted as $\alpha_b(G)$.

Obviously $\alpha_0(G) \leq \alpha_b(G)$. The strict inequality also holds as illustrated in the following example.

Example 2.5. Consider the semigraph G whose vertex set $V(G) = \{0, 1, 2, 3, 4, 5, 6\}$ and the edge set $E(G) = \{(1, 0, 4), (2, 0, 5), (3, 0, 6)\}$. Note that the set $S = \{1, 2, 3\}$ is a minimal vertex covering set of G with maximum cardinality and thus $\alpha_b(G) = 3$ and $\alpha_0(G) = 1$.

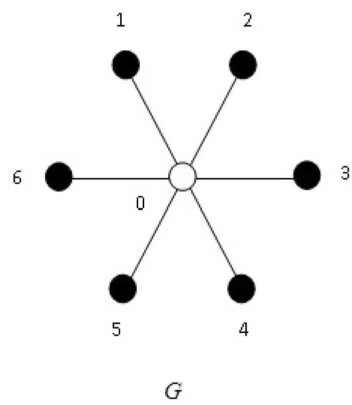

Figure 1

3 Main Results

Theorem 3.1. Let G be a semigraph and $S \subseteq V(G)$ be a vertex covering set of G then S is a minimal vertex covering set if and only if for every vertex v in S there is an edge f such that $S \cap f = \{v\}$.

Proof: Suppose S is minimal. Let $v \in S$ then there is an edge f such that $f \cap (S - \{v\}) = \phi$. However, $f \cap S \neq \phi$. Thus, it follows that $f \cap S = \{v\}$.
Conversely suppose the condition holds. Let $v \in S$ and f be an edge such that $f \cap S = \{v\}$ then $f \cap (S - \{v\}) = \phi$. Hence $S - \{v\}$ is not a vertex covering set of G. ∎

Subsemigraph of Type I: Let G be a semigraph and $v \in V(G)$. We consider the subsemigraph $G - \{v\}$ whose vertex set is $V(G) - \{v\}$ and the edge set is subedges obtained by removing the vertex v from every edge of G. We call this as subsemigraph of type I.

Now we consider minimal vertex covering sets of $G - \{v\}$ and G and prove that the upper vertex covering number does not increase when a vertex is removed from a semigraph.

Theorem 3.2. Let G be a semigraph and $v \in V(G)$ then

(1) Every minimal vertex covering set of $(G - \{v\})$ is a minimal vertex covering set of G and

(2) $\alpha_b(G - \{v\}) \leq \alpha_b(G)$.

Proof:
(1) Let S be a minimal vertex covering set of $G - \{v\}$, e be an edge of G and $e' = e - \{v\}$. Since $e' \cap S \neq \phi, e \cap s \neq \phi$. Thus, S is a vertex covering set of G.
Let $w \in S$. Since S is minimal in $G - \{v\}$, there is an edge f' such that $s \cap f' = \{w\}$. Let f be the edge of G such that $f - \{v\} = f'$ than $s \cap f' = \{w\}$. Hence S is a minimal vertex covering set of G.
(2) Let S be an $\alpha_b - set$ of $G - \{v\}$. Since S is a minimal vertex covering set of G, $\alpha_b(G) \geq |S| = \alpha_b(G - \{v\})$. Thus, $\alpha_b(G - \{v\}) \leq \alpha_b(G)$. ∎

In [7] we proved that for any semigraph G, $\alpha_0(G) \leq \alpha_0(G - \{v\})$. Thus, we have the following chain connecting $\alpha_0 - set$ and $\alpha_b - set$ of G and $(G - \{v\})$:

$$\alpha_0(G) \leq \alpha_0(G - \{v\}) \leq \alpha_b(G - \{v\}) \leq \alpha_b(G)$$

Now we give a condition under which the upper vertex covering number remains same when a vertex is removed.

Theorem 3.3. Let G be a semigraph and $v \in V(G)$. Then $\alpha_b(G - \{v\}) = \alpha_b(G)$ if and only if the set of all α_b-sets of G not containing v is equal to the set of all α_b-sets of $(G - \{v\})$.

Proof: Suppose $\alpha_b(G - \{v\}) = \alpha_b(G)$.
Let S be an $\alpha_b - set$ of $(G - \{v\})$. Then S is a minimal vertex covering set of G. Since $\alpha_b(G - \{v\}) = \alpha_b(G)$, S is an α_b-set of G not containing v. Let T be an α_b-set of G not containing v, then as proved earlier T is a minimal vertex covering set of $G - \{v\}$. Since $\alpha_b(G - \{v\}) = \alpha_b(G)$, T is an $\alpha_b - set$ in $G - \{v\}$. Hence both, set of all α_b-sets of $G - \{v\}$ and set of all α_b-sets of G not containing v are equal.

Conversely, Let T be an α_b-set in $G - \{v\}$, then T is also an $\alpha_b - set$ in G. Hence, $\alpha_b(G) = |T| = \alpha_b(G - \{v\})$. ∎

Theorem 3.4. Let G be a semigraph and $v \in V(G)$. If $\alpha_b(G - \{v\}) < \alpha_b(G)$ then there is a $\alpha_b - set$ of G say T such that $v \in T$.

Proof: Suppose $\alpha_b(G - \{v\}) < \alpha_b(G)$. Let S be an $\alpha_b - set$ in $G - \{v\}$. Then S is a minimal vertex covering set of G but it cannot be an $\alpha_b - set$ of G. Since S is a minimal vertex covering set of G, there is an $\alpha_b - set$ T of G such that $|S| < |T|$. If $v \notin T$ then T is a minimal vertex covering set of $G - \{v\}$. Hence $\alpha_b(G - \{v\}) \geq |T| > |S| = \alpha_b(G - \{v\})$, which is absurd. Thus, $v \in T$. ∎

Example 3.5. In Example 2.5, $\alpha_b(G - \{0\}) = \alpha_b(G) = 3$. Note that every $\alpha_b - set$ of $G - \{0\}$ is an $\alpha_b - set$ of G and conversely also.

Subsemigraph of Type II: Let G be a semigraph and $v \in V(G)$. We consider a semigraph $G - \{v\}$ in which the vertex set is $V(G) - \{v\}$ and the edge set is equal to the set of those edges of G which do not contain the vertex v. This is called the subsemigraph of type II.

Theorem 3.6. Let G be a semigraph and $v \in V(G)$. Let S be a minimal vertex covering set of $G - \{v\}$. Then either S is a minimal vertex covering set of G or $S \cup \{v\}$ is a minimal vertex covering set of G.

Proof: If S is a minimal vertex covering set of G then the theorem is proved. If not, then it implies that S is not a vertex covering set of G. Therefore there is an edge e of G such that $e \cap S = \phi$. Then it must be true that $v \in e$. Now consider the set $S \cup \{v\}$. It follows that $S \cup \{v\}$ is a vertex covering set of G which is also minimal. ∎

Theorem 3.7. Let G be a semigraph and $v \in V(G)$ then $\alpha_b(G - \{v\}) \leq \alpha_b(G)$.

Proof: Let S be an α_b-set of $G - \{v\}$. Then either S or $S \cup \{v\}$ is minimal vertex covering set of G. Therefore $\alpha_b(G) \geq |S| = \alpha_b(G - \{v\})$. Hence, $\alpha_b(G - \{v\}) \leq \alpha_b(G)$. ∎

Theorem 3.8. Let G be a semigraph and $v \in V(G)$. Then $\alpha_b(G - \{v\}) = \alpha_b(G)$ if and only if every $\alpha_b - set$ of $G - \{v\}$ is an $\alpha_b - set$ of G.

Proof: Suppose that $\alpha_b(G-\{v\}) = \alpha_b(G)$. Let S be an α_b-set of $G-\{v\}$. Then by Theorem 3.6, either S is minimal vertex covering set of G or $S \cup \{v\}$ is a minimal vertex covering set of G. If $S \cup \{v\}$ is a minimal vertex covering set of G then $\alpha_b(G) \geq |S \cup \{v\}| > |S| = \alpha_b(G-\{v\})$ and hence $\alpha_b(G-\{v\}) < \alpha_b(G)$, which is not true. Thus, S must be a minimal vertex covering set of G. Since $\alpha_b(G-\{v\}) = \alpha_b(G)$, S must be an $\alpha_b - set$ of G.

Conversely, suppose every $\alpha_b - set$ of $G-\{v\}$ is an $\alpha_b - set$ of G. Let S be an $\alpha_b - set$ of $G-\{v\}$. Then S is also an $\alpha_b - set$ of G. Thus $|S| = \alpha_b(G) = \alpha_b(G-\{v\})$. ∎

Corollary 3.9. If $\alpha_b(G-\{v\}) = \alpha_b(G)$ then there is an $\alpha_b - set$ of G, say S such that $v \notin S$.

Corollary 3.10. Let G be a semigraph and $v \in V(G)$. If v belongs to every $\alpha_b - set$ of G then $\alpha_b(G-\{v\}) < \alpha_b(G)$.

Theorem 3.11. Let G be a semigraph and $v \in V(G)$. Suppose there is an $\alpha_b - set$ T of G such that $v \in T$. If $\alpha_b(G-\{v\}) < \alpha_b(G)$ then $\alpha_b(G-\{v\}) = \alpha_b(G) - 1$.

Proof: Since T is a vertex covering set of G, $T-\{v\}$ is also a vertex covering set of $G-\{v\}$. If $T-\{v\}$ is a minimal vertex covering set of $G-\{v\}$ then since $\alpha_b(G-\{v\}) < \alpha_b(G)$ the set $T-\{v\}$ must be an $\alpha_b - set$ in $G-\{v\}$. Thus, $\alpha_b(G-\{v\}) = |T-\{v\}| = |T| - 1 = \alpha_b(G) - 1$.

Suppose $T-\{v\}$ is not minimal in $G-\{v\}$ then it contains a minimal vertex covering set T_1 of $G-\{v\}$. Since T_1 is a proper subset of T, T_1 cannot be a minimal vertex covering set of G. Hence by Theorem 3.6, $T_1 \cup \{v\}$ is a minimal vertex covering set of G. Now, $T_1 \cup \{v\} \subseteq T$ and both the sets are minimal in G. This implies that $T_1 \cup \{v\} = T$ and thus $T_1 = T - \{v\}$. This contradicts our assumption that $T-\{v\}$ is minimal in $G-\{v\}$. Thus, $T-\{v\}$ must be minimal in $G-\{v\}$. ∎

4 Well covered and Approximately well covered semigraphs

In this section we introduce two new concepts namely well covered semigraphs and approximately well covered semigraphs.

Definition 4.1. A semigraph G is said to be a well covered semigraph if $\alpha_0(G) = \alpha_b(G)$. Equivalently, all minimal vertex covering sets of G have the same cardinality.

Definition 4.2. Let G be a semigraph then G is said to be an approximately well covered semigraph if $\alpha_b(G) = \alpha_0(G) + 1$.

Example 4.3. Consider the semigraph G whose vertex set $V(G) = \{1, 2, 3, \ldots, 8, 9\}$ and edge set $E(G) = \{(1, 2, 3), (4, 1, 5), (6, 2, 7), (8, 3, 9)\}$. This semigraph is a well covered semigraph as $\alpha_0(G) = \alpha_b(G) = 3$.

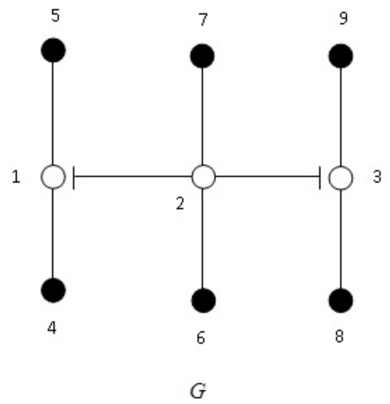

Figure 2

In fact the set $S = \{1, 2, 3\}$ is an $\alpha_0 - set$ of G. Any vertex covering set with more than three vertices will contain at least two vertices from the same edge of G and hence cannot be a minimal vertex covering set of G. Therefore $\alpha_b(G) = 3$.

Example 4.4. Consider the semigraph G whose vertex set $V(G) = \{0, 1, 2, 3, 4\}$ and edge set $E(G) = \{(1, 0, 2), (3, 0, 4), \}$. This semigraph is an approximately well covered semigraph of G because $\alpha_0(G) = 1$ and $\alpha_b(G) = 2$.

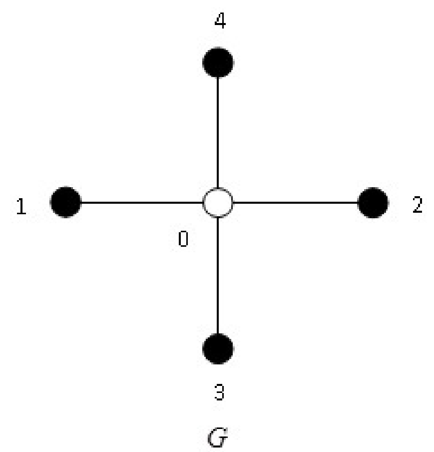

Figure 3

Theorem 4.5. Let G be a well covered semigraph. Then
(1) There is no vertex v such that $\alpha_0(G - \{v\}) > \alpha_0(G)$ and $\alpha_b(G - \{v\}) < \alpha_b(G)$.
(2) If $v \in V(G)$ then $G - \{v\}$ is also a well covered semigraph.
(3) If $v \in V(G)$ then $\alpha_0(G - \{v\}) = \alpha_0(G)$.
(4) If $v \in V(G)$ then $\alpha_b(G - \{v\}) = \alpha_b(G)$.

Proof: (1) For any vertex v of G ,$\alpha_0(G) \leq \alpha_0(G-\{v\}) \leq \alpha_b(G-\{v\}) \leq \alpha_b(G)$. Since $\alpha_0(G) = \alpha_b(G)$ we have $\alpha_0(G-\{v\}) = \alpha_b(G-\{v\}) = \alpha_0(G) = \alpha_b(G)$. Hence $\alpha_0(G-\{v\}) > \alpha_0(G)$ is not possible. Similarly, $\alpha_b(G-\{v\}) < \alpha_b(G)$ is also not possible.
(2) If $v \in V(G)$ then as proved in (1) $\alpha_0(G-\{v\}) = \alpha_b(G-\{v\})$. Hence $(G-\{v\})$ is well covered semigraph.
(3) and (4) are obvious. ∎

Theorem 4.6. Let G be an approximately well covered semigraph and $v \in V(G)$. Then

(1) Either $\alpha_0(G-\{v\}) = \alpha_b(G)$ or $\alpha_0(G-\{v\}) = \alpha_0(G) + 1$.

(2) If $\alpha_b(G-\{v\}) < \alpha_b(G)$ then $(G-\{v\})$ is a well covered semigraph.

(3) If $\alpha_b(G-\{v\}) = \alpha_b(G)$ then either $(G-\{v\})$ is a well covered or an approximately well covered semigraph.

Proof: (1) Since G is an approximately well covered semigraph, $\alpha_0(G) \leq \alpha_0(G-\{v\}) \leq \alpha_b(G-\{v\}) \leq \alpha_0(G)+1$. Hence either $\alpha_0(G-\{v\}) = \alpha_0(G)$ or $\alpha_0(G-\{v\}) = \alpha_0(G)+1$.
(2) Again $\alpha_0(G) \leq \alpha_b(G-\{v\}) \leq \alpha_0(G)+1$. Hence, if $\alpha_b(G-\{v\}) < \alpha_b(G)$ then it must be true that $\alpha_b(G-\{v\}) = \alpha_0(G)$. Also,$\alpha_0(G) < \alpha_0(G-\{v\}) \leq \alpha_b(G-\{v\})$. Hence $\alpha_0(G-\{v\}) = \alpha_b(G-\{v\})$. Thus, $(G-\{v\})$ is a well-covered semigraph.
(3) Suppose $\alpha_b(G-\{v\}) = \alpha_b(G)$. Then $\alpha_0(G-\{v\}) = \alpha_0(G)$ or $\alpha_0(G-\{v\}) = \alpha_0(G)+1 = \alpha_b(G) = \alpha_b(G-\{v\})$. In the first case it follows that $G-\{v\}$ is an approximately well covered and in the second case it follows that $G-\{v\}$ is well covered. ∎

Corollary 4.7. Let G be an approximately well covered semigraph and $v \in V(G)$. Then either $G-\{v\}$ is well covered or it is approximately well covered.

In all the following results we assume that the subsemigraph $G-\{v\}$ of G is of type II.

Theorem 4.8. Let G be a well covered semigraph and $v \in V(G)$. Then

(1) If $\alpha_0(G-\{v\}) = \alpha_0(G)$ then $G-\{v\}$ is well covered semigraph.

(2) If $\alpha_0(G-\{v\}) < \alpha_0(G)$ then either $G-\{v\}$ is well covered or approximately well covered semigraph.

Proof: (1) Suppose $\alpha_0(G-\{v\}) = \alpha_0(G)$. If $\alpha_b(G-\{v\}) > \alpha_0(G-\{v\})$ then it implies that $\alpha_b(G-\{v\}) > \alpha_b(G)$, which is not possible. Hence, $\alpha_b(G-\{v\}) = \alpha_0(G)$ and thus $G-\{v\}$ is a well covered semigraph.
(2) Suppose $\alpha_0(G-\{v\}) < \alpha_0(G)$ then $\alpha_0(G-\{v\}) = \alpha_0(G)-1$. If $\alpha_b(G-\{v\}) = \alpha_b(G) = \alpha_0(G)$ then $\alpha_b(G-\{v\}) = \alpha_0(G-\{v\})+1$. Hence $G-\{v\}$ is an approximately well covered semigraph. If $\alpha_b(G-\{v\}) < \alpha_b(G)$ then since $\alpha_0(G-\{v\}) \leq \alpha_b(G-\{v\}) < \alpha_0(G) = \alpha_b(G)$,0 it follows that $\alpha_b(G-\{v\}) = \alpha_0(G-\{v\})$. Hence $G-\{v\}$ is well covered semigraph of G. ∎

Corollary 4.9. Let G be a semigraph and $v \in V(G)$. Then either $G - \{v\}$ is an approximately well covered or a well covered semigraph.

Theorem 4.10. Let G be an approximately well covered semigraph and $v \in V(G)$ then,

(1) If $\alpha_0(G - \{v\}) = \alpha_0(G)$ then either $G - \{v\}$ is well covered or approximately well covered.

(2) If $\alpha_0(G - \{v\}) < \alpha_0(G)$ and $\alpha_b(G - \{v\}) < \alpha_b(G)$ then $G - \{v\}$ is either well covered or approximately well covered.

Proof: (1) Suppose $\alpha_0(G - \{v\}) = \alpha_0(G)$. If $\alpha_b(G - \{v\}) = \alpha_b(G)$ then $G - \{v\}$ is an approximately well covered semigraph. If $\alpha_b(G - \{v\}) < \alpha_b(G)$ then it follows that $\alpha_b(G - \{v\}) = \alpha_0(G - \{v\})$. Hence $G - \{v\}$ is well covered.
(2) Suppose $\alpha_0(G - \{v\}) < \alpha_0(G)$ and $\alpha_b(G - \{v\}) < \alpha_b(G)$. If $\alpha_b(G - \{v\}) = \alpha_0(G)$ then it follows that $\alpha_b(G - \{v\}) = \alpha_0(G - \{v\}) + 1$. Hence $G - \{v\}$ is an approximately well covered semigraph. Suppose $\alpha_b(G - \{v\}) = \alpha_0(G) - 1$ then it follows that $\alpha_b(G - \{v\}) = \alpha_0(G - \{v\})$. Hence, $G - \{v\}$ is well covered. ∎

Corollary 4.11. If G be an approximately well covered semigraph and $v \in V(G)$ then $G - \{v\}$ is either well covered or an approximately well covered.

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