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Recurrence relation on the number of spanning trees of generalized book graphs and related family of graphs

Nithya Sai Narayana

N.E.S. Ratnam College of Arts, Science and Commerce
Bhandup, Mumbai-400078, India.
narayana_nithya@yahoo.com

Abstract

The book graph denoted by $B_{n,2}$ is the cartesian product $S_{n+1} \times P_2$ where S_{n+1} is a star graph with n vertices of degree 1 and one vertex of degree n and P_2 is the path graph of 2 vertices. Let $\tau(B_{n,2})$ denote the number of spanning trees of $B_{n,2}$. Let $X_{n,p}$ denote the generalized form of book graph where a family of p-cycles which are n in number is merged at a common edge. In this paper, we discuss some recurrence relations satisfied by $X_{n,p}$ and spanning trees of these associated family of graphs.

Keywords: Book graph, spanning trees, recurrence relation.
AMS Subject Classification(2010): 05C05, 05C30, 05C85, 68R05.

1 Introduction and Preliminaries

Number of spanning trees of a graph representing a network represents the strength of the network and it is one of the important parameter associated with a graph. Cartesian product of two graphs G_1, G_2 denoted by $G_1 \times G_2$ is a graph with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices $(u_1, v_1), (u_2, v_2)$ of $G_1 \times G_2$ are adjacent if and only if either $u_1 = u_2$ and (v_1, v_2) is an edge in G_2 or $v_1 = v_2$ and (u_1, u_2) is an edge of G_1. The book graph denoted by $B_{n,2}$ is the cartesian product $S_{n+1} \times P_n$ where S_{n+1} is a star graph with n vertices of degree 1 and one vertex of degree n and P_2 is the path graph of n vertices. First observe that book graphs are planar graphs and examples of few book graphs and their planar representation are given below.

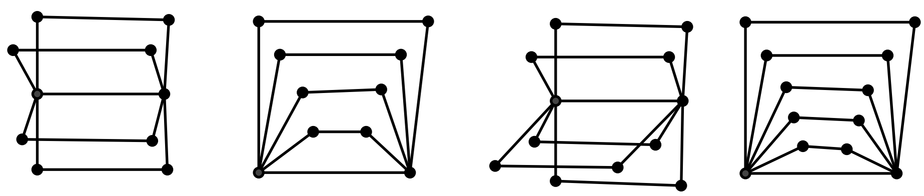

Figure 1: Book graphs $B_{4,2}$ and $B_{5,2}$ and their planar representation.

Definition 1.1. (i) Let $G = (V, E)$ be a graph. Let $e = xy \in E$ be an edge which is not a loop. The graph $G - e$ is obtained by removing the edge e from G and the graph $G.e$ is obtained by removing the edge e and merging the vertices x, y to a single vertex. Note that this new vertex is adjacent to all the vertices originally adjacent to the vertices x and y in G.
(ii) Suppose the vertices x, y are connected by the a simple path $P : x = v_0 v_1, v_2 \cdots v_k = y$. We assume that the vertices $v_1, v_2, \cdots v_{k-1}$ are not adjacent with any other vertices of G. We define $G - P$ is the graph obtained by removing the vertices $v_1, v_2, \cdots v_{k-1}$ from G and the graph $G.P$ is obtained by removing $v_1, v_2, \cdots v_{k-1}$ from G and merging x, y to a single vertex. Note that this new vertex is adjacent to all the vertices originally adjacent to the vertices x and y in G except the vertices v_1 and v_{k-1}.
(iii) Let $V_1 \subset V$ then the graph generated by V_1 denoted by $< V_1 >$ is a sub-graph of G whose vertex set is V_1 and edge set is the set of all edges of G having both the end vertices in V_1.

Theorem 1.2. (Fundamental recurrence relation of spanning trees of a graph)
Let $G = (V.E)$ be a graph and $e \in E(G)$ be an edge of G which is not a loop, then $\tau(G) = \tau(G - e) + \tau(G.e)$.

Theorem 1.3. If $G = (V, E)$ is a graph such that $V(G) = V_1 \cup V_2 \cup \cdots \cup V_n$ where $V_i \cap V_j = \{x\}$ for $i \neq j$. Let $G_i =< V_i >$ for $i = 1, 2 \cdots n$ and suppose the graph generated by $< Vi >$ does not have any edge common with $< V_j >$ for $i \neq j$ then $\tau(G) = \tau(G_1)\tau(G_2) \cdots \tau(G_n)$.

Theorem 1.4. If $G = (V, E)$ is a graph such that $V(G) = V_1 \cup V_2 \cup \cdots \cup V_n$ such that $V_i \cap V_{i+1}$ has exactly one vertex common and $< V_i >$ and $< V_j >$ has no edge common for $i \neq j$ then $\tau(G) = \tau(G_1)\tau(G_2) \cdots \tau(G_n)$.

Theorem 1.5. [2] Let $G = (V, E)$ be a planar graph. Let $V = V_1 \cup V_2$ be such that $V_1 \cap V_2 = \{x, y\}$. Let $e = xy \in E(G)$ and $E(G) =< V_1 > \cup < V_2 >$ be such that $< V_1 > \cap < V_2 >= \{e\}$ where e is the unique edge common to $< V_1 >$ and $< V_2 >$. Let $G_1 =< V_1 >$ and $G_2 =< V_2 >$. Then $\tau(G) = \tau(G_1)\tau(G_2) - \tau(G_1 - e)\tau(G_2 - e)$.

Proof: Number of spanning trees of G= number of spanning trees of G not containing e number of spanning trees of G containing e. Clearly number of spanning tree of $G_1 = \tau(G_1) = \tau(G_1 - e) + \tau(G_1.e)$ and number of spanning tree of $G_2 = \tau(G_2) = \tau(G_2 - e) + \tau(G_2.e)$.

$$\begin{aligned}\tau(G_1)\tau(G_2) &= [\tau(G_1 - e) + \tau(G_1.e)][\tau(G_2 - e) + \tau(G_2.e)] \\ &= \tau(G_1 - e)\tau(G_2 - e) + \tau(G_1.e)\tau(G_2 - e) + \tau(G_1 - e)\tau(G_2.e) + \tau(G_1.e)\tau(G_2.e).\end{aligned}$$

Thus,

$$\tau(G_1)\tau(G_2) - \tau(G_1 - e)\tau(G_2 - e) = \tau(G_1.e)\tau(G_2 - e) + \tau(G_1 - e)\tau(G_2.e) + \tau(G_1.e)\tau(G_2.e) \quad \cdots\cdots \text{(I)}$$

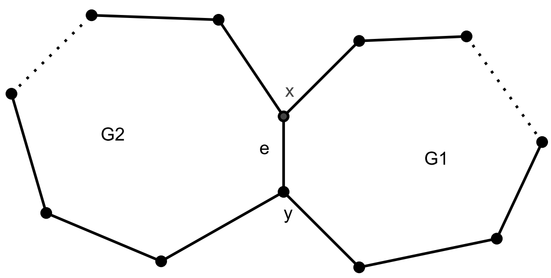

Figure 2

Consider a spanning tree T_1 of G_1 containing e and a spanning tree T_2 of G_2 containing e. From the two spanning trees T_1, T_2 we can construct a spanning of G containing e by merging the two spanning trees at e. Conversely consider a spanning tree of G containing e. By considering the induced sub-graph of T restricted to the vertices of G_1 and G_2 we get two spanning trees of G_1 and G_2 each of them containing e. Thus there is a bijective relation between the set of spanning trees of G containing e and the spanning trees of G_1 and G_2 each of them containing the edge e.

Note that the number of spanning trees of G_1 containing e is the same as the number of spanning trees of $G_1.e$ and the number of spanning trees of G_2 containing e is the same as the number of spanning trees of $G_2.e$ and the number of spanning trees of G containing e is the same as the number of spanning trees of $G.e$ and hence we have,

$$\tau(G.e) = \tau(G_1.e) \times \tau(G_2.e) \qquad \cdots\cdots \text{(II)}$$

Now consider a spanning tree T_1 of G_1 not containing e and a spanning tree T_2 of G_2 containing e. We construct a new graph G' by merging the two spanning trees. Note that in T_1 there is a unique path joining x and y and in T_2 the unique path joining x and y is the edge e. Thus G' contains a unique cycle containing e and is a spanning sub-graph of G and hence $G' - e$ is a spanning tree of G not containing e. Similarly by considering a spanning tree of G_2 not containing e and a spanning tree of G_1 containing e we can construct a spanning tree of G not containing e.

Conversely consider a spanning tree T of G not containing e. By considering the induced sub-graph of T containing the vertices of V_1 and V_2 we get two sub-graphs of G_1 and G_2 say G_1' and G_2'. First we prove that either there is a unique path in G_1' between x and y or there is a unique path in G_2' between x and y but not in both. Clearly if there is a unique path both in G_1' and in G_2' then $T_1 = G_1' \cup G_2'$ contains a cycle as there are two distinct paths in T between the vertices x and y and it is not possible as T is a spanning tree of G and it does not contain a cycle.

Suppose there is no path in G_1' between x and y then there must be a path between x and y in G_2' otherwise there is no path between x and y in T. If G_1' does not contain a path between

x and y then we add the edge e to G_1' to get a spanning tree of G_1 and in that case G_2' is a spanning tree of G_2. If G_2' does not contain a path between x and y, we add e to G_2' to get a spanning tree of G_2 and in that case G_1' is a spanning tree of G_1.

Note that there are exactly two possibilities for a spanning tree of G not containing e. The induced sub-graph containing the vertices of V_1 either contains a path between x and y or does not contain a path between x and y. In the first case we construct a spanning tree of G_1 not containing e and a spanning tree of G_2 containing e. In the second case we get a spanning tree of G_1 containing e and a spanning tree of G_2 not containing e. Thus we have,

$\tau(G - e) = \tau(G_1.e)\tau(G_2 - e) + \tau(G_1 - e)\tau(G_2.e).$ $\cdots\cdots$ (III)

Using Theorem 1.2 we get,

$$\begin{aligned}\tau(G) &= \tau(G - e) + \tau(G.e)\\ &= \tau(G_1.e)\tau(G_2 - e) + \tau(G_1 - e)\tau(G_2.e) + \tau(G_1.e)\tau(G_2.e) \text{ (from II and III)}\\ &= \tau(G_1)\tau(G_2) - \tau(G_1 - e)\tau(G_2 - e) \text{ (from I).}\end{aligned}$$

Thus the theorem is proved. ■

Theorem 1.6. [2] Let $G = (V, E)$ be a planar graph. Let $V(G) = V_1 \cup V_2$ be such that $V_1 \cap V_2 = \{x, y\}$. Let x and y be two vertices of G such that every path in G from $u_i \in V_1$ to $u_j \in V_2$ passes either through x or y and u and v are part of the same face of G. Let $< V_1 >= G_1$ and $< V_2 >= G_2$, then $\tau(G) = \tau(G_1)\tau(G_2\ .\ xy) + \tau(G_2)\tau(G_1\ .\ xy)$ where $G_1\ .\ xy, G_2\ .\ xy$ are obtained by merging the two vertices x, y into a single vertex so that the vertices adjacent to x, y would be adjacent to the new vertex.

Proof: Note that x, y may or may not be adjacent. Suppose x, y are adjacent vertices, then the edge $e = xy$ is included in exactly one of G_1 or G_2.

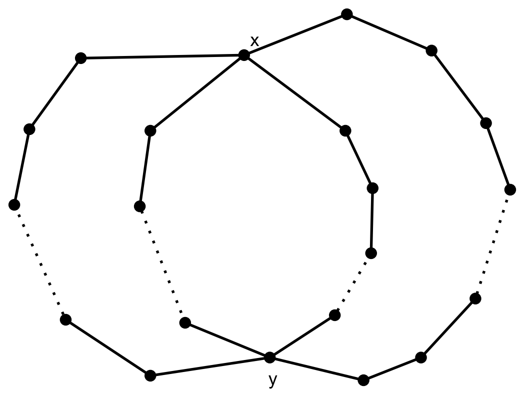

Figure 3

Consider a spanning tree T of G. We consider the sub-graph of T restricted to the vertices of V_1 and V_2. Let the sub-graph of T generated by V_1 be denoted by T_1' and the sub-graph of T generated by V_2 be denoted by T_2'. Note that there cannot be a path between x and y both in G_1' and G_2' as otherwise the union of two paths will give a cycle in T which is not possible. There are two possibilities. If there is a path in T_1' between x and y then there cannot be a path between x and y in T_2' and further if there is no path between x and y in T_1' then there must be a path between x and y in T_2' as T is connected.

Consider the first case(Type I) where T_1' does not have a path between x and y. Note that T_2' has a path between x and y. We prove that T_2' is a spanning tree of G_2 and $T_1'.xy$ is a spanning tree of $G_1.xy$.

Suppose T_2' is not a spanning tree of G_2. Let u, v be two vertices of G_2 which are not connected in G_2. Clearly in T, there exists a path consisting of vertices of G_2 between u and x or between u and y through which u is connected to a vertex of G_1. Similarly in T there exists a path consisting of vertices of G_2 between v and x or between v and y through which v is connected to a vertex of G_1. As per the assumption in T_2' there exists a path between x and y consisting of vertices of G_2 which implies that there exists a path between u and v consisting of vertices of G_2. It is a contradiction to our assumption and hence T_2' is a spanning tree of G_1.

Now we prove that $T_1'.xy$ is a spanning tree of $G_1.xy$. Let u, v be any two vertices in G_1. In T there exists a path from u and x or u and y, consisting of vertices of G_1 through which the vertex u is connected to a vertex of G_2 and similarly in T such path exists from v and x or v and y. In other words vertices of G_1 in T_1' are either connected to x or connected to y and hence in $T_1'.xy$ every pair of vertices of $G_2.xy$ are connected and is a spanning tree of $G_1.xy$.

Using similar argument it is clear that for the case(Type II) where T_1' have a path between x and y and there is no path between x and y in T_2', it can be proved that T_1' is a spanning tree of G_1 and in that case $T_2'.xy$ is a spanning tree of $G_2.xy$.

Thus every spanning tree T of G gives rise to either a spanning tree of G_1 and a spanning tree of $G_2.xy$ or a spanning tree of G_2 and a spanning tree of $G_1.xy$. Conversely with every spanning tree of G_1 and a spanning tree of $G_2.xy$ we get a spanning tree of G in which a path exists between x and y in G_1 and with every spanning tree of G_2 and a spanning tree of $G_1.xy$ we get a spanning tree of G in which a path exists between x and y in G_2.

Note that a spanning tree of G is either of Type I or of Type II and hence we get $\tau(G) = \tau(G_1)\tau(G_2\ .\ xy) + \tau(G_2)\tau(G_1\ .\ xy)$. ∎

2 Results on spanning trees of generalized book graph

Definition 2.1. Let $X_{n,p}$ denote a graph with n number of p-cycles with a common edge $e = xy$. We call this graph as generalized book graph as the graph becomes a book graph for $p = 4$

In this section we derive the recurrence relations satisfied by generalized book graphs and few more graphs obtained from the generalized books graphs.

Theorem 2.2. Let $X_{n,p}$ denote a graph with n number of p-cycles with a common edge $e = xy$ and let $Y_{n,p} = X_{n,p} - e$ then $X_{n,p}$ and $Y_{n,p}$ satisfy the following recurrence relations
(i) $\tau(X_{n,p}) = 2(p-1)\tau(X_{n-1,p}) - (p-1)^2\tau(X_{n-2,p})$
(ii) $\tau(Y_{n,p}) = (3p-4)\tau(Y_{n-1,p}) - (3p^2 - 8p + 5)\tau(Y_{n-2,p}) + (p^3 - 4p^2 + 5p - 2)\tau(Y_{n-3,p})$

Proof: Note that in $Y_{n,p}$ there exists p distinct paths between x and y of length $p-1$. Choosing any one such path and by removing each of $p-1$ edges between x and y and applying successively Theorem 1.2 we get $\tau(Y_{n,p}) = (p-2)\tau(Y_{n-1,p}) + \tau(X_{n-1,p})$
$\Rightarrow \tau(Y_{n-1,p}) = (p-2)\tau(Y_{n-2,p}) + \tau(X_{n-2,p})$ ......(*)

Further, $\tau(X_{n,p}) = \tau(G_1)\tau(G_2) - \tau(G_1 - e)\tau(G_2 - e)$ using Theorem 1.5, where G_1 is any p-cycle in $X_{n,p}$ containing e and G_2 is obtained from $X_{n,p}$ by removing the edges of G_1 other than the common edge e.
Thus, $\tau(X_{n,p}) = p\tau(X_{n-1,p}) - \tau(Y_{n-1,p})$
$\Rightarrow \tau(Y_{n-1,p}) = p\tau(X_{n-1,p}) - \tau(X_{n,p})$ and
$\tau(Y_{n-2,p}) = p\tau(X_{n-2,p}) - \tau(X_{n-1,p})$. ......(**)
Substituting in (*)
$p\tau(X_{n-1,p}) - \tau(X_{n,p}) = (p-2)[p\tau(X_{n-2,p}) - \tau(X_{n-1,p})] + \tau(X_{n-2,p})$
$\Rightarrow \tau(X_{n,p}) = p\tau(X_{n-1,p}) + (p-2)\tau(X_{n-1,p}) - \tau(X_{n-2-,p}) - p(p-2)\tau(X_{n-2,p})$,
Thus, $\tau(X_{n,p}) = 2(p-1)\tau(X_{n-1,p}) - (p-1)^2\tau(X_{n-2,p})$ which proves (i).
From (*), $\tau(X_{n-1,p}) = \tau(Y_{n,p}) - (p-2)\tau(Y_{n-1,p})$, $\tau(X_{n-2,p}) = \tau(Y_{n-1,p}) - (p-2)\tau(Y_{n-2,p})$ and $\tau(X_{n-3,p}) = \tau(Y_{n-2,p}) - (p-2)\tau(Y_{n-3,p})$.
Hence, $\tau(Y_{n,p}) - (p-2)\tau(Y_{n-1,p}) = 2(p-1)[\tau(Y_{n-1,p}) - (p-2)\tau(Y_{n-2,p})] - (p-1)^2[\tau(Y_{n-2,p}) - (p-2)\tau(Y_{n-3,p})]$.
Simplifying we get,

$$\begin{aligned}\tau(Y_{n,p}) =& [2(p-1) + (p-2)]\tau(Y_{n-1,p}) - [2(p-1)(p-2) + (p-1)^2]\tau(Y_{n-2,p}) \\ &+ (p-1)^2(p-2)\tau(Y_{n-3,p}) \\ =& (3p-4)\tau(Y_{n-1,p}) - (3p^2 - 8p + 5)\tau(Y_{n-2,p}) + (p^3 - 4p^2 + 5p - 2)\tau(Y_{n-3,p}).\end{aligned}$$

Hence (ii) is proved. ∎

The following well known result(which is actually a simple application of fundamental recurrence relation) is presented here. It is observed that it can also be arrived at by solving the recurrence relation mentioned above.

Corollary 2.3. (i) $\tau(X_{n,p}) = (p-1)^n + n(p-1)^{n-1}$ and (ii) $\tau(Y_{n,p}) = n(p-1)^{n-1}$.

Proof: From Theorem 2.2 (i), the characteristic equation of $\tau(X_{n,p})$ is $x^2-2(p-1)x+(p-1)^2=0$ and the solution to the recurrence relation is $\tau(X_{n,p})=c_1(p-1)^n+c_2n(p-1)^n$ with $\tau(X_{1,p})=p$ and $\tau(X_{2,p})=p^2-1$ and is given by $\tau(X_{n,p})=(p-1)^n+n(p-1)^{n-1}$.

From Theorem 2.2 (ii), the characteristic equation of $\tau(Y_{n,p})$ is $x^3-(3p-4)x^2+(3p^2-8p+5)x-(p^3-4p^2+5p-2)=0$ which implies $(x-(p-2))(x-(p-1))^2=0$ and the solution to the recurrence relation is $\tau(Y_{n,p})=c_1(p-2)^n+c_2(p-1)^n+c_3n(p-1)^n$ with $\tau(Y_{1,p})=1, \tau(Y_{2,p})=2(p-1)$ and $\tau(Y_{3,p})=3(p-1)^2$ is given by $\tau(Y_{n,p})=n(p-1)^{n-1}$. ■

Theorem 2.4. Suppose $G_{m,p:n,q}$ is a graph with m number of $p-$cycles and n number of $q-$cycles with a common edge $e=xy$, then
$\tau(G_{m,p:n,q})=(p-1)^{m-1}(q-1)^{n-1}[(p-1+m)(q-1+n)-mn]$.

Proof: Let $A_{m,p}$ denote m number of p cycles with common edge e and $B_{n,q}$ denote n number of q-cycles with the common edge e. Let $C_{m,p}=A_{m,p}-e$ and $D_{n,q}=B_{n,q}-e$. Then by Corollary 2.3, we have $\tau(A_{m,p})=(p-1)^m+m(p-1)^{m-1}$, $\tau(B_{n,q})=(q-1)^n+n(q-1)^{n-1}$, $\tau(C_{m,p})=m(p-1)^{m-1}$ and $\tau(D_{n,q})=n(q-1)^{n-1}$.
Using Theorem 1.5,

$$\begin{aligned}\tau(G_{m,p:n,q})&=\tau(A_{m,p})\tau(B_{n,q})-\tau(C_{m,p})\tau(D_{n,q})\\&=(p-1)^m+m(p-1)^{m-1}(q-1)^n+n(q-1)^{n-1}-m(p-1)^{m-1}n(q-1)^{n-1}\\&=(p-1)^{m-1}(q-1)^{n-1}[(p-1+m)(q-1+n)-mn].\end{aligned}$$ ■

Theorem 2.5. Let $E_1=X_{m,p}$ with a common base as e and $F_1=Y_{m,p}=X_{m,p}-e$. Let E_n and F_n be defined by joining n copies of E_1 and F_1 successively at an edge other than the base as given below. Then, $\tau(E_n), \tau(F_n)$ satisfy the following recurrence relations.

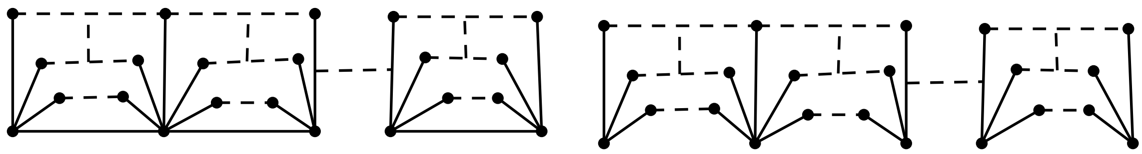

Figure 4: .Graph E_n and F_n.

(i) $\tau(E_n)=\alpha\tau(E_{n-1})-\alpha'^2\tau(E_{n-2})$ where $\alpha=(p-1)^m+m(p-1)^{m-1}$ and $\alpha'=(p-1)^{m-1}+m(p-1)^{m-2}$.
(ii) $\tau(F_n)=\beta\tau(F_{n-1})-\beta'^2\tau(F_{n-2})$ where $\beta=m(p-1)^{m-1}$ and $\beta'=(m-1)(p-1)^{m-2}$.

Proof: From Corollary 2.3, $\tau(E_1)=(p-1)^n+n(p-1)^{n-1}=\alpha(say)$ and $\tau(F_1)=n(p-1)^{n-1}=\beta(say)$ and by Theorem 1.5, $\tau(E_2)=\tau(E_1)\tau(E_1)-\tau(E_1-e)\tau(E_1-e)=\tau(X_{m,p})^2-\tau(X_{m-1,p})^2=\alpha^2-\beta^2$.
Using Theorem 1.5 we get,

$\tau(E_n) = \tau(E_1)\tau(E_{n-1}) - \tau(E_{n-2}).\tau(X_{m-1,p})\tau(X_{m-1,p}) = \alpha\tau(E_{n-1}) - (\alpha')^2\tau(E_{n-2})$.
(ii) Using similar argument we get,
$\tau(F_1) = \tau(Y_{m,p}) = m(p-1)^{m-1} = \beta(say)$ and $\tau(F_2) = Y_{m,p}^2 - Y_{m-1,p}^2 = \beta^2 - \beta'^2$ where $\beta' = (m-1)(p-1)^{m-2}$.

Using Theorem 1.5, we have $\tau(F_n) = \tau(F_{n-1})\tau(Y_{m,p}) - \tau(F_{n-2})\tau(Y_{m-1,p})^2 = \beta\tau(F_{n-1}) - (\beta')^2\tau(F_{n-2})$. ■

Theorem 2.6. Let $H_1 = G_{m,p:t,q}$ consisting of m number of p-cycles and t number of q-cycles with a common base e. Let H_n denote a graph containing n- copies of H_1 merged successively at edges other than the base as below. Then, $\tau(H_n)$ satisfies the recurrence relation given by
$\tau(H_n) = \lambda\tau(G_{n-1}) - \mu^2\tau(G_{n-2})$ where
$\lambda = (p-1)^{m-1}(q-1)^{t-1}[(p-1+m)(q-1+t) - mt]$ and
$\mu = (p-1)^{m-2}(q-1)^{t-2}[(p-2+m)(q-2+t) - (m-1)(t-1)]$ where $m \geq 2$ and $t \geq 2$.

Proof: Similar to the proof of Theorem 2.5 using Theorems 2.4 and 1.5. For $m = 1, t > 1$ and $m > 1, t = 1$ and $m = 1, t = 1$ similar results can be arrived. ■

Remark 2.7. The characteristic equation of $\tau(E_n)$ is given by $x^2 - ((p-1)^m + m(p-1)^{m-1})x + ((p-1)^{m-1} + (m-1)(p-1)^{m-2})^2 = 0$. Suppose θ_1, θ_2 are the roots of the characteristic equation then the general solution of $\tau(E_n)$ is given by $\tau(E_n) = c_1\theta_1^n + c_2\theta_2^n$ where c_1, c_2 are obtained by the solving the simultaneous equations $c_1\theta_1 + c_2\theta_2 = \tau(E_1)$ and $c_1\theta_1^2 + c_2\theta_2^2 = \tau(E_2)$. Similarly $\tau(F_n), \tau(H_n)$ can be obtained.

Example 2.8. We find the number of spanning trees of the following graph.

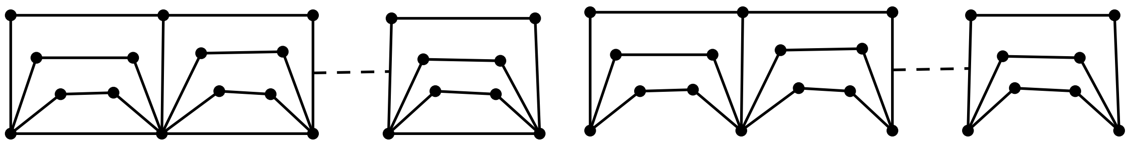

Figure 5: Graph E_n., F_n with $m = 3, p = 4$.

We find $\tau(E_n)$ with $m = 3, p = 4$. Applying Theorem 2.5 we get the recurrence relation satisfied by the graph is $\tau(E_n) = 54\tau(E_{n-1}) - 324\tau(E_{n-2})$ and the characteristic equation becomes $x^2 - 54x + 324 = 0$ whose roots are $\theta_1 = 27 + 9\sqrt{5}, \theta_2 = 27 - 9\sqrt{5}$ with $\tau(E_0) = 1$, $\tau(E_1) = 54$. Solving we get $\tau(E_n) = \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)(27 + 9\sqrt{5})^n + \left(\frac{-3+\sqrt{5}}{2\sqrt{5}}\right)(27 - 9\sqrt{5})^n$.

Example 2.9. We find the number of spanning trees of the following graphs.

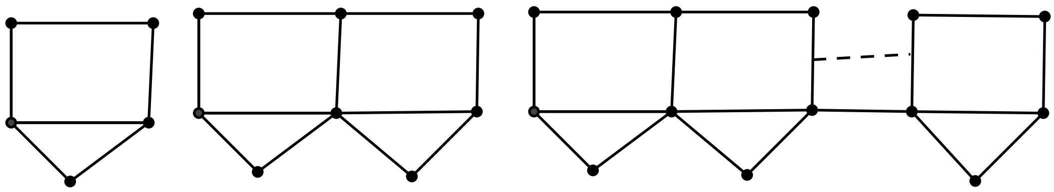

Figure 6: Graphs H_1 , H_2 and H_n.

We find $\tau(H_n)$ with $m = 1, p = 3, q = 4, t = 1$. Applying similar methods we get the recurrence relation satisfied by the graph is $\tau(H_n) = 11\tau(H_{n-1}) - 9\tau(H_{n-2})$ and the characteristic equation becomes $x^2 - 11x + 9 = 0$ whose roots are $\alpha = \frac{11+\sqrt{85}}{2}, \beta = \frac{11-\sqrt{85}}{2}$ with $\tau(H_1) = 11, \tau(H_2) = 112$. Solving we get $\tau(H_n) = \left(\frac{11+\sqrt{85}}{2\sqrt{85}}\right)(\frac{11+\sqrt{85}}{2})^n + \left(\frac{-11+\sqrt{85}}{2\sqrt{85}}\right)(\frac{11-\sqrt{85}}{2})^n$.

3 Number of spanning trees of some special family of book graphs

Theorem 3.1. Let $J_1 = X_{m,p}$ with common base and $Q_1 = Y_{m,p}$ with common base. Let J_n and Q_n be obtained by joining n copies of J_1 and Q_1 at a base vertex in circular form as below. Then
(i) $\tau(J_n) = n\alpha^{n-1}(p-1)^m$ where $\alpha = (p-1)^m + m(p-1)^{m-1}$.
(ii) $\tau(Q_n) = n\beta^{n-1}(p-1)^m$ where $\beta = m(p-1)^{m-1}$.

Proof: (i) Clearly, by Theorem 2.3 $\tau(J_1) = (p-1)^m + m(p-1)^{m-1} = \alpha(\text{say})$. Consider J_2 and

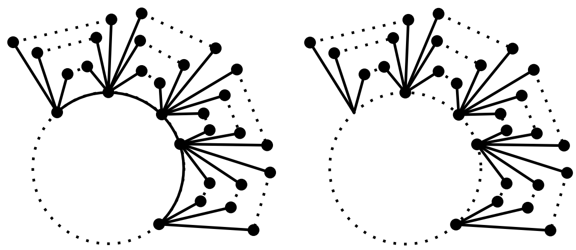

Figure 7: Graphs J_n and Q_n

we divide this graph into two parts with each of the two parts are J_1 with the common pair of vertices. We use Theorem 1.6 to get $\tau(J_2) = \tau(J_1)\tau(C_{p-1})^n + \tau(J_1)\tau(C_{p-1})^n = 2\tau(J_1)(p-1)^m = 2\alpha(p-1)^m$.
Considering J_n, we apply Theorem 1.6 taking $G_1 = J_1, G_2=$ the graph obtained by taking deleting J_1 from G which is the graph obtained by taking $n-1$ copies of J_1 and joining them in succession at a common base vertex $e = xy$ we get

$$\begin{aligned}\tau(J_n) &= \tau(J_1)\tau(G_2.xy) + \tau(G_2)\tau(J_1.xy)\\ &= \tau(J_1)\tau(J_{n-1}) + \tau(J_1)^{n-1}(p-1)^m\end{aligned}$$

$$
\begin{aligned}
&= \alpha\tau(J_{n-1}) + \alpha^{n-1}(p-1)^m \\
&= \alpha[\alpha\tau(J_{n-2}) + \alpha^{n-2}(p-1)^m] + \alpha^{n-1}(p-1)^m \\
&= \alpha^2\tau(J_{n-2}) + 2\alpha^{n-1}(p-1)^m \\
&= \alpha^3\tau(J_{n-3}) + 3\alpha^{n-1}(p-1)^m \\
&\vdots \\
&= \alpha^{n-2}\tau(J_2) + (n-2)\alpha^{n-1}(p-1)^m \\
&= \alpha^{n-2}2\alpha(p-1)^m + (n-2)\alpha^{n-1}(p-1)^m \\
&= \alpha^{n-1}(p-1)^m(2+n-2) = n\alpha^{n-1}(p-1)^m.
\end{aligned}
$$

(ii) Clearly, by Theorem 2.3 $\tau(Q_1) = m(p-1)^{m-1} = \beta(\text{say})$. Using Theorem 1.6 to get $\tau(Q_2) = 2\tau(Q_1)(p-1)^m = 2\beta(p-1)^m$, we have

$$
\begin{aligned}
\tau(Q_n) &= \tau(Q_1)\tau(Q_{n-1}) + \tau(Q_1)^{n-1}(p-1)^m \\
&= \beta(Q_{n-1}) + \beta^{n-1}(p-1)^m \\
&= \beta[\beta\tau(Q_{n-2}) + \beta^{n-2}(p-1)^m] + \beta^{n-1}(p-1)^m \\
&= \beta^2\tau(Q_{n-2}) + 2\beta^{n-1}(p-1)^m \\
&= \beta^3\tau(Q_{n-3}) + 3\beta^{n-1}(p-1)^m \\
&\vdots \\
&= \beta^{n-2}\tau(Q_2) + (n-2)\beta^{n-1}(p-1)^m \\
&= \beta^{n-2}2\beta(p-1)^m + (n-2)\beta^{n-1}(p-1)^m \\
&= n\beta^{n-1}(p-1)^m.
\end{aligned}
$$

∎

Example 3.2. We find the number of spanning trees of the following graph.

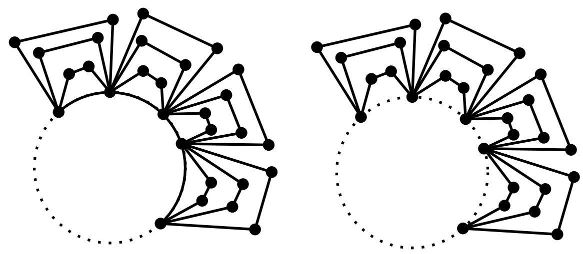

Figure 8

Here $m = 3$, $p = 4$ and $\alpha = 54$ and $\beta = 27$. Hence $\tau(J_n) = n54^{n-1} \times 3^3 = \frac{n}{2}54^n$ and $\tau(Q_n) = n27^{n-1} \times 3^3 = n27^n$.

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