International Journal of Mathematics and Soft Computing
Vol.6, No.2. (2016), 13 - 19.

ISSN Print : 2249 – 3328
ISSN Online: 2319 – 5215

Graceful and odd graceful labeling of graphs

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Abstract

In this paper we prove gracefulness and odd gracefulness of arbitrary super -subdivision of triangular snake, where labeling of the vertices follows arithmetic progression.

Keywords: **S**uper- subdivisions of graph, arbitrary super subdivisions of graph, triangular snake.

AMS Subject Classification (2010): 0578C.

1 Introduction

The graceful labeling was introduced by Rosa [8] as β – valuation and later Golomb [4] named it as graceful labeling. Graph labelling is an active area of research in graph theory which has rigorous applications in coding theory, communication networks, optimal layouts and graphs decomposition problems. A lot of research has been done on graceful graphs. Kathiresan [6] proved that subdivisions of ladder are graceful. Sethuraman and Selvaraju [9] have introduced super subdivisions of graphs and proved that there exists a super subdivision of c_{n}, $n \geq 3$ with certain conditions. Kathiresan [5] also proved that arbitrary super subdivisions of stars are graceful. Ramchandran and Sekar [7] have given graceful labeling of super subdivision of ladder. Ujwala Deshmukh and Bhatavadekar [2] proved that arbitrary super subdivisions of cycle are graceful.

2 Definitions and Notations

Definition 2.1. The triangular snake T_n is a graph containing a path of length n with vertices $u_1, u_2, \ldots\ldots u_n, u_{n+1}$ and each pair of consecutive vertices u_i, u_{i+1} is joined to a common vertex $v_i, i = 1, 2 \ldots\ldots, n$. Thus it has $2n + 1$ vertices and $3n$ edges.

Definition 2.2. A quadrilateral snake Q_n is obtained from a path $u_1, u_2, \ldots\ldots u_n$ by joining u_i and u_{i+1} to two new vertices v_i and w_i respectively for $1 \leq i \leq n - 1$.

Definition 2.3. Let G be a graph with p vertices and q edges. A graphs H is said to be a subdivisions of G if H is obtained by subdivision of every edge of G exactly once. H is denoted by $S(G)$. Thus $|v| = p + q$ and $|E| = 2q$.

Definition 2.4. Let G be a graph with p vertices and q edges. A graphs H is said to be a super subdivisions of G if it is obtained from G by replacing every edge e of G by a complete bipartite $\mathrm{k}_{2,\mathrm{m}}$ is denoted by $SS(G)$ thus $|\mathrm{v}| = \mathrm{p} + \mathrm{mq}$ and $|\mathrm{E}| = 2\mathrm{mq}$.

Definition 2.5. Let G be a graph with p vertices and q edges. A graphs H is said to be a arbitrary super subdivision of G if it is obtained from G by replacing every edge $\mathrm{e}_\mathrm{i}$of G by a complete bipartite graph $\mathrm{k}_{2,\mathrm{m}_\mathrm{i}}$ $\mathrm{i} = 1,2 \ldots.\mathrm{q}$. H is denoted by ASS(G) Thus $|\mathrm{v}| = \mathrm{p} + \sum_{\mathrm{i}-1}^{\mathrm{q}} \mathrm{m}_\mathrm{i}$ and $|\mathrm{E}| = \sum_{\mathrm{i}=1}^{\mathrm{q}} 2\mathrm{m}_\mathrm{i}$.

Definition 2.6. Let $f : V \rightarrow \{ 0,1,2,\ldots,q\}$ be an injective function and for an edge $e = uv$, the induced edge labelling $\mathrm{f}^*\colon \mathrm{E(G)} \rightarrow \{ 1,2,...,\mathrm{q}\}$ is a bijective function such that $\mathrm{f}^*\,(\mathrm{uv}) = |\mathrm{f(u)} - \mathrm{f(v)}|$. Then function f is called graceful labeling. A graph which admits graceful labeling is called a graceful graph.

Definition 2.7. Let $f : V \rightarrow \{ 0,1,\ldots, 2q\text{-}1\}$ be an injective function and for an edge $\mathrm{e} = \mathrm{uv}$, the induced edge labeling $\mathrm{f}^*\colon \mathrm{E(G)} \rightarrow \{ 1,3,5, \ldots, 2\mathrm{q}\text{-}1\}$ is a bijective function such that $\mathrm{f}^*\,(\mathrm{uv}) = |\mathrm{f(u)} - \mathrm{f(v)}|$. A function f is called odd graceful labeling. A graph which admits odd graceful labeling is called an odd graceful graph.

Notations for Triangular snake:

Let $u_{i,} i = 1,2, \ldots., n+1$ be vertices of path of length n and $v_{i,} i = 1,2, \ldots., n$ be vertices adjacent to u_i and u_{i+1} respectively.

Let $x_i^k, k = 1,2,..,m_i^1$ be the vertices of the m_i^1 vertices part of k_{2,m_i^1} replacing the edge $u_i v_i$ $i = 1,2,\ldots,n$.

Let $y_i^k, k = 1,2,..,m_i^2$ be the vertices of the m_i^2 vertices part of k_{2,m_i^2} replacing the edge $u_{i+1}\, v_i$ for $i = 1,2,\ldots,n$.

Let $z_i^k, k = 1,2,..,m_i^3$ be the vertices of the m_i^3 vertices part of k_{2,m_i^3} replacing the edge $u_i\, u_{i+1}$ for $i = 1,2,\ldots,n$.

3 Main Results

Theorem 3.1. Arbitrary super subdivisions of triangular snake where $m_i^3 \equiv o \bmod 2, i = 1,2,\ldots,n$ is graceful.

Proof: Let $m = \sum_{i=1}^{n} m_i^1 + \sum_{i=1}^{n} m_i^2 + \sum_{i=1}^{n} m_i^3$.

Define a labeling f as follows:

$$f(u_i) = 2i - 2 \qquad , 1 \le i \le n+1;$$

$$f(v_i) = 2i - 1 \qquad , 1 \le i \le n.$$

Let $\propto_i = f(u_i) + 2m - 2(\sum_{r=1}^{i-1} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3)$, i =2,3,..., n and $\propto_1 = 2m$.

$$f\left(x_i^k\right) = \propto_i + 2 - 2k \quad , k = 1\ to\ m_i^1, i = 1 \text{ to } n.$$

$\beta_i = f\left(v_i\right) + 2m - 2(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3)$. i =2,3,…, n and $\beta_1 = 1+2m-2m_1^1$

$$f\left(y_i^k\right) = \beta_i + 2 - 2k \quad , k = 1\ to\ m_i^2, i = 1\ to\ n$$

$\gamma_i = f\left(u_i\right) + 2m - 2(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i} m_r^2 + \sum_{r=0}^{i-1} m_r^3$, i =2,3,…n and $\gamma_1 = 2m - 2m_1^1 - 2m_1^2$

$$f\left(z_i^k\right) = \gamma_i + 4 - 4l \quad , l = 1,2, \dots, \frac{m_i^3}{2} \qquad \text{for } k = 1,3 \dots\dots, m_i^3 - 1$$

$$= \gamma_i + 3 - 4l \quad , l = 1,2, \dots, \frac{m_i^3}{2} \qquad \text{for } k = 2,4 \dots\dots, m_i^3.$$

Table 1: Range of vertex labels.

Graphs	Range of vertex labels		
k_{2,m_1^1}	$\propto_1 = 2m$	to	$\propto_1 - 2m_1^1 + 2$
k_{2,m_1^2}	$\beta_1 = \propto_1 - 2m_1^1 + 1$	to	$\beta_1 - 2m_1^2 + 2$
k_{2,m_1^3}	$\gamma_1 = \beta_1 - 2m_1^2 - 1$	to	$\gamma_1 - 2m_1^3 + 2$
k_{2,m_i^1}	$\propto_i$	to	$\propto_i - 2m_1^1 + 2$
k_{2,m_i^2}	$\beta_i = \propto_1 - 2m_i^1 + 1$	to	$\beta_i - 2m_i^2 + 2$
k_{2,m_i^3}	$\gamma_i = \beta_1 - 2m_i^2 + 1$	to	$\gamma_i - 2m_i^3 + 2$

Table 2: Range of edge labels.

Graphs	Range of edge labels		
k_{2,m_1^1}	$2m$	to	$2\text{m} - 2m_1^1 + 1$
k_{2,m_1^2}	$2\text{m} - 2m_1^1$	to	$2\text{m} - 2m_1^1 - 2m_1^2 + 1$
k_{2,m_1^3}	$2\text{m} - 2m_1^1 - 2m_1^2$	to	$2\text{m} - 2m_1^1 - 2m_1^2 - 2m_1^3 + 1$
k_{2,m_i^1}	$2m - 2\ (\sum_{r=1}^{i-1} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3)$	to	$2m - 2\left(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right) + 1$
k_{2,m_i^2}	$2m - 2\left(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right)$	to	$2m - 2\left(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right) + 1$
k_{2,m_i^3}	$2m - 2\left(\sum_{r=0}^{i} m_r^1 + \sum_{r=0}^{i} m_r^2 + \sum_{r=0}^{i-1} m_r^3\right)$	to	$2m - 2\left(\sum_{r=0}^{i} m_r^1 + \sum_{r=0}^{i} m_r^2 + \sum_{r=0}^{i} m_r^3\right) + 1$
k_{2,m_n^1}	$2(m_n^3 + m_n^2) + 1$	to	$2(m_n^3 + m_n^2 + m_n^1)$
k_{2,m_n^2}	$2(m_n^3) + 1$	to	$2(m_n^3 + m_n^2)$
$k_{2,}m_n^3$	1	to	$2\ m_n^3$

From Table 1, labels of vertices of $m_i^1 -$ part of k_{2,m_i^1} and m_i^2- part of k_{2,m_i^2} are in arithmetic progression with common difference -2, Thus they are distinct.

Whereas, the labels of vertices of $m_i^3 -$ part of k_{2,m_i^3} are from γ_i to γ_i - $2m_i^3 + 2$ follows arithmetic progression A_l and B_l alternatively where

$$A_l = \gamma_i + 4 - 4l,$$

$$B_l = \gamma_i + 3 - 4l \text{ for } l = 1,2,\ldots,\frac{m_i^3}{2}.$$

As $A_l = B_l + 1$, $A_i \neq B_j$ for all i and j.

Thus vertex labels of $m_i^3 -$ parts of k_{2,m_i^3} are distinct. The labels of vertex on triangular snake are $f(u_i) = 2i - 2$ for , $1 \leq i \leq n + 1$ and $f(v_i) = 2i - 1$ for $1 \leq i \leq n$ which are distinct and are less than least label of $m_i^j -$ part of $k_{2,}m_i^j, j = 1,2,3$.

Thus vertex labels are distinct. From Table 2, it can be clearly seen that all edge weights are distinctly covered. ■

Theorem 3.2. Arbitrary super Subdivision of triangular snake where $m_i^3 \equiv o \; mod \; 2, i = 1$ to n is odd graceful.

Proof: Define $f: V \rightarrow \{0,1,2,\ldots,4m\text{-}1\}$ by

$$f(u_i) = 4i - 4 \qquad i = 1 \; to \; n + 1 \;,$$

$$f(u_i) = 4i - 2 \qquad i = 1 \; to \; n.$$

Define $\propto_i = f(u_i) + (4m - 1) - 4 \; \left(\sum_{r=1}^{i-1} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right)$, i=2,3,…, n and

$\alpha_1 = 4m - 1.$

$$f(x_i^k) = \propto_i - 4 - 4k, \qquad k = 1 \; to \; m_i^1, \; i = 1 \; to \; n\,.$$

$\beta_i = f(v_i) + (4m - 1) - 4(1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3)$, i =2,3,…, n and

$\beta_1 = 4m - 4m_1^1 + 1$

$$f(y_i^k) = \beta_i - 4 - 4k, \qquad k = 1 \; to \; m_i^2, i = 1 \; to \; n.$$

$\gamma_i = f(u_i) + (2m - 1) - 4(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i} m_r^2 + \sum_{r=1}^{i} m_r^3)$, i =2,3,…, n and

$\gamma_1 = 4m - 4m_1^1 - 4m_1^2 - 1.$

$$f(z_i^k) = \gamma_i + 6 - 6l, \qquad l = 1,2,\ldots,\frac{m_i^3}{2}, \qquad for \; k = 1,3,\ldots,m_i^3 - 1,$$

$$= \gamma_i + 4 - 6l, \qquad l = 1,2,\ldots,\frac{m_i^3}{2}, \qquad for \; k = 2,4,\ldots,m_i^3.$$

Table 3: Range of vertex labels.

Graphs	Range of vertex labels		
k_{2,m_1^1}	$\propto_1 = 4m - 1$	to	$\propto_1 - 4m_1^1 + 4$
k_{2,m_1^2}	$\beta_1 = \propto_1 - 4m_1^1 + 2$	to	$\beta_1 - 4m_1^2 + 4$
k_{2,m_1^3}	$\gamma_1 = \beta_1 - 4m_1^2 - 2$	to	$\gamma_1 - 4m_1^3 + 6$
k_{2,m_i^1}	$\propto_i$	to	$\propto_i - 4m_1^1 + 4$
k_{2,m_i^2}	β_i	to	$\beta_i - 4m_i^2 + 4$
k_{2,m_i^3}	γ_i	to	$\gamma_i - 4m_i^3 + 6$

Table 4: Range of edge labels.

Graph Parts	Range of edge labels		
k_{2,m_1^1}	$4m - 1$	to	$4\text{m} - 4m_1^1 + 1$
k_{2,m_1^2}	$4\text{m} - 4m_1^1 - 1$	to	$4\text{m} - 4m_1^1 - 4m_1^2 + 1$
k_{2,m_1^3}	$4\text{m} - 4m_1^1 - 4m_1^2 - 1$	to	$4\text{m} - 4m_1^1 - 4m_1^2 - 4m_1^3 + 1$
k_{2,m_i^1}	$4m - 4\left(\sum_{r=1}^{i-1} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right) - 1$	to	$4m - 4\left(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right) + 1$
k_{2,m_i^2}	$4m - 4\left(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i-1} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right) - 1$	to	$4m - 4\left(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right) + 1$
k_{2,m_i^3}	$4m - 4\left(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i} m_r^2 + \sum_{r=1}^{i-1} m_r^3\right) - 1$	to	$4m - 4\left(\sum_{r=1}^{i} m_r^1 + \sum_{r=1}^{i} m_r^2 + \sum_{r=1}^{i} m_r^3\right) + 1$
k_{2,m_n^1}	$4(m_n^3 + m_n^2) + 1$	to	$4(m_n^3 + m_n^2 + m_n^1) - 1$
k_{2,m_n^2}	$4(m_n^3) + 1$	to	$4(m_n^3 + m_n^2) - 1$
$k_{2,}m_n^3$	1	to	$4\,m_n^3 - 1$

From Table 3, the labels of vertices of $m_i^1 -$ part of k_{2,m_i^1} and m_i^2- part of k_{2,m_i^2} are in arithmetic progression with common difference -4, thus they are distinct. Whereas, the labels of vertices of $m_i^3 -$ part of k_{2,m_i^3} are from γ_i *to* γ_i - $4m_i^3 + 2$ follows arithmetic progression A_l and B_l alternatively where

$$A_l = \gamma_i + 6 - 6l \qquad l = 1,2,\dots,\frac{m_r^3}{2}$$

$$B_l = \gamma_i + 4 - 6l \qquad l = 1,2,\dots,\frac{m_r^3}{2}$$

As $A_l = B_l + 2$, we have $A_i \neq B_j$ for all i and j.

Thus the vertex labels of $m_i^3 -$ parts of k_{2,m_i^3} are distinct. The labels of the vertices on triangular snake are $f(u_i) = 4i - 4, 1 \leq i \leq n + 1$ and $f(v_i) = 4i - 2, 1 \leq i \leq n$, which are district. Also, the labels of the vertices on the triangular snake are even and the labels of the vertices of m_l^j part of k_{2,m_i^j}, j = 1,2,3 are odd .

Thus the vertex labels are distinct. From Table 4, it can be clearly seen that all edge weights are distinctly covered. ■

Corollary 3.3. If f is a graceful labelling of Ass(Tn) and $f^*$is a odd graceful labelling of Ass(Tn) then

$$f^*(u) = 2f(u) \qquad if \quad f(u) < p$$
$$f^*(u) = 2f(u) - 1 \qquad if \quad f(u) \geq P$$

Illustration 3.4. Graceful labeling of ASS(T4) is given in Figure 1.

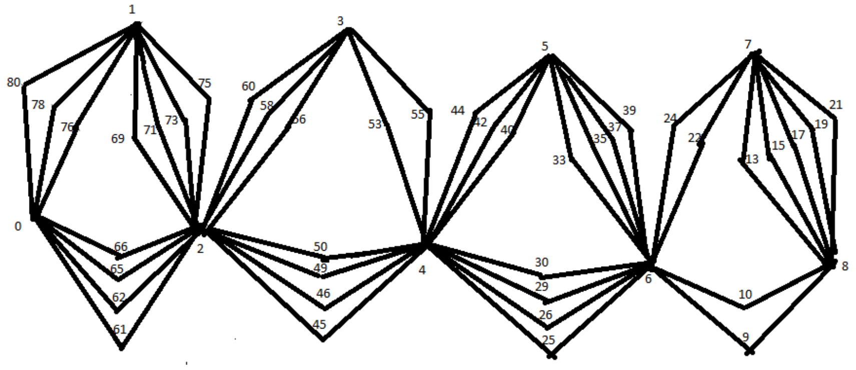

Figure1: Graceful labelling of ASS(T4).

Illustration 4.5. Odd graceful labeling of ASS(T4) is given in Figure 2.

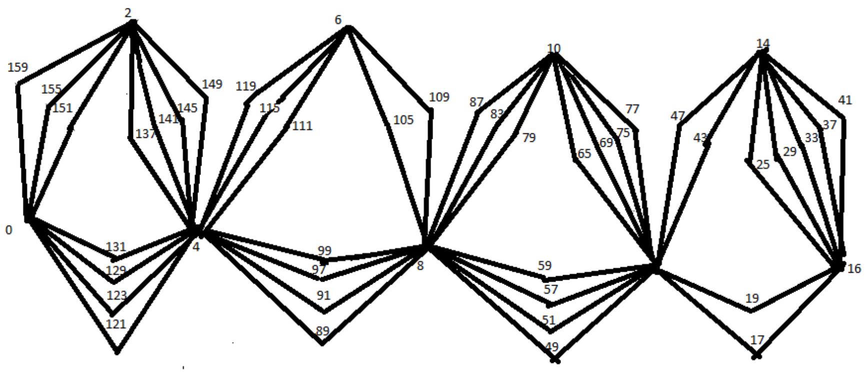

Figure 2: Odd Graceful labelling of ASS (T4).

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