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On Spectral Properties in Vector-Valued Beurling Algebra

H V Dedania[1], M K Kansagara[2]

[1] Department of Mathematics
Sardar Patel University, Vallabh Vidyanagar-388 120
Gujarat, India.
hvdedania@yahoo.com

[2] H & H B Kotak Institute of Science
Rajkot-360001, Gujarat, India.
milankansagara2011@gmail.com

Abstract

Let G be an LCA group, ω be a weight function on G, and $\mathcal{A}$ be a semisimple, commutative Banach algebra. We characterize some Banach algebra properties of vector-valued Beurling algebra $L^1(G,\omega,\mathcal{A})$ in terms of $L^1(G,\omega)$ and $\mathcal{A}$ using the Bochner integration theory. These properties are unique uniform norm property (UUNP), unique C^*-norm property (UC*NP), quasi divisor of zero property (QDZP), weak regularity (WR), regularity, and complete regularity (CR).

Keywords: Locally compact abelian group, vector-valued Beurling algebra, UUNP, UC*NP, regularity.

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1 Introduction and preliminaries

Let G be an LCA group, μ be the Haar measure on G, ω be a weight on G and $\mathcal{A}$ be a semisimple, commutative Banach algebra. Then the vector-valued, weighted, Banach space $L^1(G,\omega,\mathcal{A})$ of $\mathcal{A}$-valued Bochner integrable functions is a semisimple, commutative Banach algebra with respect to the convolution product; it is called *the vector-valued Beurling algebra* and its Gelfand theory is developed in [6]. In this paper, we characterize some Banach algebra properties of $L^1(G,\omega,\mathcal{A})$ in terms of $L^1(G,\omega)$ and $\mathcal{A}$ using the Bochner integration theory which is relatively simpler than the tensor product theory.

Most of the notations and terminologies used in this paper are same as in [6] except a few ones. For example, we use $\|\cdot\|_{\omega,\mathcal{A}}$ as the Banach algebra norm on $L^1(G,\omega,\mathcal{A})$ instead of $\|\cdot\|_{1,\omega}$. The elements of $L^1(G,\omega)$ are denoted by f,g,h, while the elements of $L^1(G,\omega,\mathcal{A})$ are denoted by $\widetilde{f},\widetilde{g},\widetilde{h}$, etc. Throughout this paper we assume that $\omega(s)\geq 1$ $(s\in G)$.

We start an introduction to the Bochner integration theory and our main reference for the Bochner integration theory is [8].

Definition 1.1. A function $f : G \longrightarrow \mathcal{A}$ is *strongly measurable* if $f^{-1}(U)$ is Borel measurable for every open set U of $\mathcal{A}$ and $f(G) \subset \mathcal{A}$ is separable.

Definition 1.2. A function $\widetilde{s} : G \longrightarrow \mathcal{A}$ is a *simple function* on G if it is of the form $\widetilde{s} = \sum_{i=1}^{n} \chi_{E_i, a_i}$, where $n \in \mathbb{N}, a_i \in \mathcal{A} \setminus \{0\}, E_i$ is a Borel measurable subset of G, and $E_i \cap E_j = \emptyset \ (i \neq j)$. Let $S(G, \mathcal{A})$ be the set of all simple functions on G.

Definition 1.3. A simple function $\widetilde{s} = \sum_{i=1}^{n} \chi_{E_i, a_i} \in S(G, \mathcal{A})$ is *Bochner integrable* if $\sum_{i=1}^{n} \|a_i\| \mu(E_i) < \infty$. In this case, define

$$\|\widetilde{s}\|_{1,\mathcal{A}} = \sum_{i=1}^{n} \|a_i\| \mu(E_i) \quad \text{and} \quad \int_G \widetilde{s}(t)\, d\mu(t) = \sum_{i=1}^{n} a_i \mu(E_i).$$

Let $L^1_s(G, \mathcal{A})$ be the set of all Bochner integrable simple functions on G. Usually we verify whether $(L^1_s(G, \mathcal{A}), \|\cdot\|_{1,\mathcal{A}})$ is a normed algebra with pointwise linear operations and convolution product $\star$.

Definition 1.4. Let ω be a weight on G such that $\omega(s) \geq 1 \ (s \in G)$. A simple function $\widetilde{s} = \sum_{i=1}^{n} \chi_{E_i, a_i}$ is *Bochner ω-integrable* on G if $\int_{E_i} \omega(t)\, d\mu(t) < \infty$ for each $1 \leq i \leq n$. In this case, define

$$\|\widetilde{s}\|_{\omega,\mathcal{A}} = \sum_{i=1}^{n} \Big(\int_{E_i} \omega(t)\, d\mu(t) \Big) \|a_i\| \quad \text{and} \quad \int_G \widetilde{s}(t)\, d\mu(t) = \sum_{i=1}^{n} a_i \mu(E_i).$$

Let $L^1_s(G, \omega, \mathcal{A})$ be the set of all Bochner ω-integrable simple functions on G. Again we verify whether $(L^1_s(G, \omega, \mathcal{A}), \|\cdot\|_{\omega,\mathcal{A}})$ is a normed algebra with pointwise linear operations and convolution product. Since $\omega(s) \geq 1 \ (s \in G)$, we have $L^1_s(G, \omega, \mathcal{A}) \subseteq L^1_s(G, \mathcal{A})$.

Definition 1.5. A strongly measurable function $f : G \longrightarrow \mathcal{A}$ is *Bochner ω-integrable* if there exists a sequence $\{\widetilde{s}_n\} \subset L^1_s(G, \omega, \mathcal{A})$ such that

1. $\widetilde{s}_n \longrightarrow f$ pointwise almost everywhere;
2. $\int_G \|\widetilde{s}_n(t) - \widetilde{s}_m(t)\| \omega(t)\, d\mu(t) \longrightarrow 0$ as $m, n \longrightarrow \infty$.

In this case, define

$$\|f\|_{\omega,\mathcal{A}} = \int_G \|f(t)\| \omega(t)\, d\mu(t) \quad \text{and} \quad \int_G f(t)\, d\mu(t) = \lim_{n \to \infty} \int_G \widetilde{s}_n(t)\, d\mu(t).$$

Let $L^1(G, \omega, \mathcal{A})$ be the set of all Bochner ω-integrable functions $f : G \longrightarrow \mathcal{A}$. Then $\Big(L^1(G, \omega, \mathcal{A}), \|\cdot\|_{\omega,\mathcal{A}}\Big)$ is a Banach algebra with pointwise linear operations and convolution product $\star$.

For reference, we state some useful lemmas without proofs and some results from [6, 7]. We start with a definition.

Definition 1.6. [7] An *ω-bounded generalised character on G* is a continuous homomorphism $\alpha : G \longrightarrow (\mathbb{C} \setminus \{0\}, \times)$ satisfying $|\alpha(s)| \leq \omega(s)$ for all $s \in G$. Let $\widehat{G}(\omega)$ be the set of all ω-bounded generalised characters on G equipped with the compact-open topology.

Theorem 1.7. [7, Th.2.8.2] Let G and ω be as above. Then the mapping $\widehat{G}(\omega) \longrightarrow \Delta(L^1(G, \omega))$; $\alpha \longmapsto \psi_\alpha$ is a homeomorphism, where ψ_α is defined as $\psi_\alpha(f) = \int_G \overline{\alpha(t)} f(t)\, d\mu(t)$.

Here we note that the second statement of the next result is not proved in [6]. However, it can be proved using [7, Cor.3.3.4].

Theorem 1.8. [6] Let $G, \omega, \mathcal{A}$ be as above.

1. The mapping $\widehat{G}(\omega) \times \Delta(\mathcal{A}) \longrightarrow \Delta(L^1(G, \omega, \mathcal{A}))$; $(\alpha, \varphi) \longmapsto \psi_{\alpha,\varphi}$ is a homeomorphism, where $\psi_{\alpha,\varphi}(\widetilde{f}) = \int_G \overline{\alpha(t)} \varphi(\widetilde{f}(t))\, d\mu(t)$.
2. The mapping $\partial L^1(G, \omega) \times \partial \mathcal{A} \longrightarrow \partial L^1(G, \omega, \mathcal{A})$; $(\alpha, \varphi) \longmapsto \psi_{\alpha,\varphi}$ is a homeomorphism, where $\partial \mathcal{B}$ is the Shilov boundary of the commutative Banach algebra $\mathcal{B}$.
3. Let ω be symmetric and $\mathcal{A}$ having a continuous algebra involution $*$. Then the mapping $\widehat{G} \times \widetilde{\Delta}(\mathcal{A}) \longrightarrow \widetilde{\Delta}(L^1(G, \omega, \mathcal{A}))$; $(\alpha, \varphi) \longmapsto \psi_{\alpha,\varphi}$ is a homeomorphism, where $\widetilde{\Delta}(\mathcal{B})$ is the hermitian Gelfand space of the commutative Banach $*$-algebra $\mathcal{B}$.

Lemma 1.9. For each $\varphi \in \Delta(\mathcal{A})$, there exists a unique continuous homomorphism $\eta_\varphi : L^1(G, \omega, \mathcal{A}) \longrightarrow L^1(G, \omega)$ satisfying

$$\eta_\varphi(\widetilde{s}) = \sum_{i=1}^{n} \varphi(a_i) \chi_{E_i} \quad \Big(\widetilde{s} = \sum_{i=1}^{n} \chi_{E_i, a_i} \in L^1_s(G, \omega, \mathcal{A})\Big).$$

Lemma 1.10. For each $\alpha \in \widehat{G}(\omega)$, there exists a unique continuous homomorphism $\eta_\alpha : L^1(G, \omega, \mathcal{A}) \longrightarrow \mathcal{A}$ satisfying

$$\eta_\alpha(\widetilde{s}) = \sum_{i=1}^{n} \psi_\alpha(\chi_{E_i}) a_i \quad \Big(\widetilde{s} = \sum_{i=1}^{n} \chi_{E_i, a_i} \in L^1_s(G, \omega, \mathcal{A})\Big).$$

Lemma 1.11. Let $f \in L^1(G, \omega)$, and $a \in \mathcal{A}$. Then $fa \in L^1(G, \omega, \mathcal{A})$ and $\psi_{\alpha,\varphi}(fa) = \widehat{a}(\varphi) \widehat{f}(\alpha)$ $(\alpha \in \widehat{G}(\omega);\ \varphi \in \Delta(\mathcal{A}))$.

Definition 1.12. Let $\pi_1 : \Delta(L^1(G, \omega, \mathcal{A})) \cong \widehat{G}(\omega) \times \Delta(\mathcal{A}) \longrightarrow \widehat{G}(\omega)$ be defined as $\pi_1(\psi_{\alpha,\varphi}) = \psi_\alpha$; and $\pi_2 : \Delta(L^1(G, \omega, \mathcal{A})) \cong \widehat{G}(\omega) \times \Delta(\mathcal{A}) \longrightarrow \Delta(\mathcal{A})$ be defined as $\pi_2(\psi_{\alpha,\varphi}) = \varphi$. The maps π_1 and π_2 are called the *projection maps*. Both of them are continuous as well as open.

2 UUNP, UC*NP and QDZP

A *uniform norm* on an algebra $\mathcal{B}$ is a (not necessarily complete) submultiplicative norm $|\cdot|$ on $\mathcal{B}$ satisfying the square property $|b^2| = |b|^2$ $(b \in \mathcal{B})$. In fact, H. V. Dedania has proved in [5] that the submultiplicativity of $|\cdot|$ is automatic in the presence of the square property. If $\mathcal{B}$ admits at least one uniform norm, then $\mathcal{B}$ is necessarily semi simple and commutative. The converse holds if $\mathcal{B}$ is a Banach algebra. Note that any two equivalent uniform norms must be identical. A semisimple, commutative Banach algebra $\mathcal{B}$ has *unique uniform norm property* (UUNP) if $\mathcal{B}$ admits exactly one uniform norm. The UUNP was introduced and extensively studied by Bhatt and Dedania in [3, 4].

It is proved in [7, Th.4.6.13] that if $\mathcal{A}$ and $\mathcal{B}$ are semi simple, commutative Banach algebras such that $\mathcal{A}\widehat{\otimes}_\pi\mathcal{B}$ is also semisimple, then $\mathcal{A}\widehat{\otimes}_\pi\mathcal{B}$ has UUNP if and only if both $\mathcal{A}$ and $\mathcal{B}$ have UUNP. Its proof uses the tensor product theory. Here we prove the similar result for $L^1(G,\omega,\mathcal{A})$ using Bochner integral theory.

Theorem 2.1. [3, Th.2.3] The following are equivalent.

1. $\mathcal{B}$ has UUNP.
2. If $F \subset \Delta(\mathcal{B})$ is closed and not containing $\partial\mathcal{B}$, then there exists $b \in \mathcal{B}$ such that $r(b) > 0$ (equivalently, $b \neq 0$) and $\widehat{b}(F) = \{0\}$.

Theorem 2.2. $L^1(G,\omega,\mathcal{A})$ has UUNP if and only if both $L^1(G,\omega)$ and $\mathcal{A}$ have UUNP.

Proof: Let $L^1(G,\omega,\mathcal{A})$ have UUNP. First, let F be a closed subset of $\widehat{G}(\omega)$ which does not contain $\partial L^1(G,\omega)$. Set $\widetilde{F} = F \times \Delta(\mathcal{A})$. Then $\partial L^1(G,\omega,\mathcal{A}) \cong \partial L^1(G,\omega) \times \partial\mathcal{A} \not\subset \widetilde{F}$. So by Theorem 2.1, there exists $\widetilde{f} \in L^1(G,\omega,\mathcal{A}) \setminus \{0\}$ such that $\widehat{\widetilde{f}}\,(\widetilde{F}) = \{0\}$. Choose $\alpha_0 \in \widehat{G}(\omega)$ and $\varphi_0 \in \Delta(\mathcal{A})$ such that $\psi_{\alpha_0,\varphi_0}(\widetilde{f}) \neq 0$, and set $f = \eta_{\varphi_0}(\widetilde{f}) \in L^1(G,\omega) \setminus \{0\}$, where η_{φ_0} is as in Lemma 1.9. Then for every $\alpha \in \widehat{G}(\omega)$, we have $\psi_\alpha(f) = \psi_\alpha(\eta_{\varphi_0}(\widetilde{f})) = \psi_{\alpha,\varphi_0}(\widetilde{f})$. Now if $\alpha \in F$, then $(\alpha,\varphi_0) \in \widetilde{F}$ and so we have $\psi_\alpha(f) = \psi_{\alpha,\varphi_0}(\widetilde{f}) = 0$. Thus $L^1(G,\omega)$ satisfies Theorem 2.1(2). Hence $L^1(G,\omega)$ has UUNP. Secondly, let E be a closed subset of $\Delta(\mathcal{A})$ which does not contain $\partial\mathcal{A}$. Set $\widetilde{E} = \widehat{G}(\omega) \times E$. Then $\partial L^1(G,\omega,\mathcal{A}) \cong \partial L^1(G,\omega) \times \partial\mathcal{A} \not\subset \widetilde{E}$. Therefore, by Theorem 2.1, there exists $\widetilde{f} \in L^1(G,\omega,\mathcal{A}) \setminus \{0\}$ such that $\widehat{\widetilde{f}}\,(\widetilde{E}) = \{0\}$. Choose $\alpha_0 \in \widehat{G}(\omega)$ and $\varphi_0 \in \Delta(\mathcal{A})$ such that $\psi_{\alpha_0,\varphi_0}(\widetilde{f}) \neq 0$, and set $a = \eta_{\alpha_0}(\widetilde{f}) \in \mathcal{A} \setminus \{0\}$, where η_{α_0} is as in Lemma 1.10. Then for every $\varphi \in \Delta(\mathcal{A})$, we have $\varphi(a) = \varphi(\eta_{\alpha_0}(\widetilde{f})) = \psi_{\alpha_0,\varphi}(\widetilde{f})$. Now if $\varphi \in E$, then $(\alpha_0,\varphi) \in \widetilde{E}$ and so we have $\varphi(a) = \psi_{\alpha_0,\varphi}(\widetilde{f}) = 0$. Thus $\mathcal{A}$ satisfies Theorem 2.1(2). Hence $\mathcal{A}$ has UUNP.

Conversely, let $L^1(G,\omega)$ and $\mathcal{A}$ have UUNP. Suppose, if possible, $L^1(G,\omega,\mathcal{A})$ does not have UUNP. Then by Theorem 2.1, there exists a closed subset $\widetilde{F}$ of $\Delta(L^1(G,\omega,\mathcal{A}))$ such that $\widetilde{F}$ does not contain $\partial L^1(G,\omega,\mathcal{A})$ and

$$|\widetilde{f}|_{\widetilde{F}} := \sup\{|\psi_{\alpha,\varphi}(\widetilde{f})| : \psi_{\alpha,\varphi} \in \widetilde{F}\} \quad (\widetilde{f} \in L^1(G,\omega,\mathcal{A}))$$

is a uniform norm on $L^1(G,\omega,\mathcal{A})$. So choose $\psi_{\alpha_0,\varphi_0} \sim (\alpha_0,\varphi_0) \in \partial L^1(G,\omega) \times \partial\mathcal{A}$ such that $\psi_{\alpha_0,\varphi_0} \notin \widetilde{F}$. Since $\widetilde{F}$ is a closed, so $\widetilde{F}^c$ is open, so by the definition of the product topology, there exist open sets U and V in $\widehat{G}(\omega)$ and $\Delta(\mathcal{A})$, respectively, such that $\alpha_0 \in U$ and $\varphi_0 \in V$ and $(U \times V) \cap \widetilde{F} = \emptyset$. Since $L^1(G,\omega)$ has UUNP and since U^c does not contain $\partial L^1(G,\omega)$, by Theorem 2.1, there exists $f \in L^1(G,\omega) \setminus \{0\}$ such that $\widehat{f}(U^c) = \{0\}$. Similarly, there exists $a \in \mathcal{A} \setminus \{0\}$ such that $\widehat{a}(V^c) = \{0\}$. Set $\widetilde{f} = fa$. Then $\widetilde{f} \in L^1(G,\omega,\mathcal{A}) \setminus \{0\}$. Now let $\psi_{\alpha,\varphi} \in \widetilde{F}$. Because $(U \times V) \cap \widetilde{F} = \emptyset$, either $\alpha \notin U$ or $\varphi \notin V$. If $\alpha \notin U$, then $\widehat{f}(\psi_\alpha) = 0$, and if $\varphi \notin V$, then $\widehat{a}(\varphi) = 0$. Hence by Lemma 1.11,

$$\psi_{\alpha,\varphi}(\widetilde{f}) = \psi_{\alpha,\varphi}(fa) = \widehat{a}(\varphi)\, \widehat{f}(\psi_\alpha) = 0.$$

Thus $|\widetilde{f}|_{\widetilde{F}} = 0$ which is a contradiction because $\widetilde{f} \neq 0$ and $|\cdot|_{\widetilde{F}}$ is a norm. Hence $L^1(G,\omega,\mathcal{A})$ has UUNP. ■

A C^**-norm* on a $*$-algebra $\mathcal{B}$ is a (not necessarily complete) submultiplicative norm $|\cdot|$ on $\mathcal{B}$ satisfying the C^*-property $|b^*b| = |b|^2$ $(b \in \mathcal{B})$. A Banach $*$-algebra $\mathcal{B}$ has *unique* C^**-norm property* (UC^*NP) if $\mathcal{B}$ admits exactly one C^*-norm. In fact, Z. Sebestyen has proved in [12] that the submultiplicativity of $|\cdot|$ is automatic in the presence of the C^*-property. Note that any two equivalent C^*-norms must be identical. So there is a natural comparision between the square property and the C^*-property of norms [2, 12]. Probably, B. A. Barnes was the first to study the UC^*NP in detail [1].

Next we characterize the UC^*NP of $L^1(G,\omega,\mathcal{A})$ in terms of $L^1(G,\omega)$ and $\mathcal{A}$. Note that if ω is symmetric on G (i.e., $\omega(-s) = \omega(s)$ $(s \in G)$), then $L^1(G,\omega)$ is $*$-semi simple. Further, if $\mathcal{A}$ is a $*$-algebra, then $L^1(G,\omega,\mathcal{A})$ is $*$-semi simple if and only if $\mathcal{A}$ is $*$-semi simple [6, Th.4.2].

Note that if $\mathcal{B}$ is a $*$-semisimple, commutative Banach $*$-algebra, then $\prod_{\mathcal{B}}$ used in [1, Prop.1.3], is exactly the hermitian Gelfand space of $\mathcal{B}$ (i.e. $\widetilde{\Delta}(\mathcal{B})$) and irreducible $*$-representations in [1] are exactly self adjoint complex homomorphisms (that is, $\varphi^* = \varphi$). So Theorem [1, Prop.1.3] is exactly as following.

Proposition 2.3. [1] Let $\mathcal{B}$ be a $*$-semisimple, commutative Banach $*$-algebra. Then $\mathcal{B}$ has unique C*-norm if and only if for every proper closed set $F \subset \widetilde{\Delta}(\mathcal{B})$ there exists $b \in \mathcal{B} \setminus \{0\}$, such that $\widehat{b}(F) = \{0\}$.

Theorem 2.4. Let ω be symmetric and $\mathcal{A}$ have a continuous algebra involution $*$. Then $L^1(G,\omega,\mathcal{A})$ has UC*NP if and only if both $L^1(G,\omega)$ and $\mathcal{A}$ have UC*NP.

Proof: Let $L^1(G,\omega,\mathcal{A})$ have UC*NP. Let F be a closed subset of $\widehat{G}$ and let $\widetilde{F} = F \times \widetilde{\Delta}(\mathcal{A})$. Then $\widetilde{F}$ is a closed subset of $\widetilde{\Delta}(L^1(G,\omega,\mathcal{A}))$. Since $L^1(G,\omega,\mathcal{A})$ has UC*NP, by Proposition 2.3, there exists $\widetilde{f} \in L^1(G,\omega,\mathcal{A}) \setminus \{0\}$ such that $\widehat{\widetilde{f}}\,(\widetilde{F}) = \{0\}$. Choose $\alpha_0 \in \widehat{G}$ and $\varphi_0 \in \widetilde{\Delta}(\mathcal{A})$ such that $\psi_{\alpha_0,\varphi_0}(\widetilde{f}) \neq 0$, and set $f = \eta_{\varphi_0}(\widetilde{f}) \in L^1(G,\omega) \setminus \{0\}$, where η_{φ_0} is as in Lemma 1.9. Then, for every $\alpha \in \widehat{G}$, we have, $\psi_\alpha(f) = \psi_\alpha(\eta_{\varphi_0}(\widetilde{f})) = \psi_{\alpha,\varphi_0}(\widetilde{f})$. Now if $\alpha \in F$, then $(\alpha, \varphi_0) \in \widetilde{F}$ and so we have $\psi_\alpha(f) = \psi_{\alpha,\varphi_0}(\widetilde{f}) = 0$. Hence, by Proposition 2.3, $L^1(G,\omega)$ has UC*NP. Secondly, let E be a closed subset of $\widetilde{\Delta}(\mathcal{A})$. Then $\widetilde{E} = \widehat{G} \times E$ is a closed subset of $\widetilde{\Delta}(L^1(G,\omega,\mathcal{A}))$. Since $L^1(G,\omega,\mathcal{A})$ has UC*NP by Proposition 2.3, there exists $\widetilde{f} \in L^1(G,\omega,\mathcal{A})\setminus\{0\}$ such that $\widehat{\widetilde{f}}\,(\widetilde{E}) = \{0\}$. Choose $\alpha_0 \in \widehat{G}$ and $\varphi_0 \in \widetilde{\Delta}(\mathcal{A})$ such that $\psi_{\alpha_0,\varphi_0}(\widetilde{f}) \neq 0$ and set $a = \eta_{\alpha_0}(\widetilde{f}) \in \mathcal{A}\setminus\{0\}$, where η_{α_0} is as in Lemma 1.10. Then, for every $\varphi \in \widetilde{\Delta}(\mathcal{A})$, we have $\varphi(a) = \varphi(\eta_{\alpha_0}(\widetilde{f})) = \psi_{\alpha_0,\varphi}(\widetilde{f})$. Now if $\varphi \in E$, then $(\alpha_0, \varphi) \in \widetilde{E}$ and so we have $\varphi(a) = \psi_{\alpha_0,\varphi}(\widetilde{f}) = 0$. Hence, by Proposition 2.3, $\mathcal{A}$ has UC*NP.

Conversely, let $L^1(G,\omega)$ and $\mathcal{A}$ have UC*NP. Suppose, if possible, $L^1(G,\omega,\mathcal{A})$ does not have UC*NP. Then, by Proposition 2.3, there exists a proper closed subset $\widetilde{E}$ of $\widetilde{\Delta}(L^1(G,\omega,\mathcal{A}))$ such that

$$|\widetilde{f}|_{\widetilde{E}} := \sup\{|\psi_{\alpha,\varphi}(\widetilde{f})| : \psi_{\alpha,\varphi} \in \widetilde{E}\} \quad (\widetilde{f} \in L^1(G,\omega,\mathcal{A}))$$

is a C*-norm on $L^1(G,\omega,\mathcal{A})$. Choose $\psi_{\alpha_0,\varphi_0} \sim (\alpha_0,\varphi_0) \in \widehat{G} \times \widetilde{\Delta}(\mathcal{A})$ such that $\psi_{\alpha_0,\varphi_0} \notin \widetilde{E}$. Since $\widetilde{E}$ is a closed, so $\widetilde{E}^c$ is open, so by the definition of the product topology, there exist open sets U and V in $\widehat{G}(\omega)$ and $\Delta(\mathcal{A})$, respectively, such that $\alpha_0 \in U$ and $\varphi_0 \in V$ and $(U \times V) \cap \widetilde{E} = \emptyset$. Define $\widetilde{F_1} = U^c \times \widetilde{\Delta}(\mathcal{A})$, $\widetilde{F_2} = \widehat{G} \times V^c$ and $\widetilde{F} = \widetilde{F_1} \cup \widetilde{F_2}$. Then $\widetilde{F}$ is a proper closed subset of $\widehat{G} \times \widetilde{\Delta}(\mathcal{A})$ such that $\widetilde{E} \subset \widetilde{F}$. Note that $L^1(G,\omega)$ is $*$- semisimple. Since $L^1(G,\omega)$ has UC*NP and U^c does not contain $\widehat{G}$, by Proposition 2.3, there exists $f \in L^1(G,\omega) \setminus \{0\}$ such that $\widehat{f}(U^c) = \{0\}$. Similarly, there exists $a \in \mathcal{A} \setminus \{0\}$ such that $\widehat{a}(V^c) = \{0\}$. Set $\widetilde{f} = fa \in L^1(G,\omega,\mathcal{A}) \setminus \{0\}$. Now let $\psi_{\alpha,\varphi} \in \widetilde{E}$. Since $(\alpha,\varphi) \in \widetilde{E} \subset \widetilde{F}$, either $(\alpha,\varphi) \in \widetilde{F_1}$ or $(\alpha,\varphi) \in \widetilde{F_2}$. Therefore either $\alpha \notin U$ or $\varphi \notin V$. If $\alpha \notin U$, then $\widehat{f}(\psi_\alpha) = 0$, and if $\varphi \notin V$, then $\widehat{a}(\varphi) = 0$. Hence, by Lemma 1.11,

$$\psi_{\alpha,\varphi}(\widetilde{f}) = \psi_{\alpha,\varphi}(fa) = \widehat{a}(\varphi)\, \widehat{f}(\psi_\alpha) = 0.$$

Thus $|\widetilde{f}|_{\widetilde{F}} = 0$ which is a contradiction because $\widetilde{f} \neq 0$ and $|\cdot|_{\widetilde{E}}$ is a norm. Hence $L^1(G,\omega,\mathcal{A})$ has UC*NP. ∎

Next we characterize the quasi divisor of zero property (QDZP) of $L^1(G,\omega,\mathcal{A})$ in terms of $L^1(G,\omega)$ and $\mathcal{A}$, which was introduced by M. J. Meyer [9].

Definition 2.5. [9, Definition 4, p-71] Let $\mathcal{B}$ be a semisimple, commutative, Banach algebra. Then $\mathcal{B}$ has *quasi divisor of zero property (QDZP)* if there exists an open set $U \subset \Delta(\mathcal{B})$ such that: (i) $\partial\mathcal{B} \subset \overline{U}$; (ii) For every open set $V \subset U$, there exist $b \in \mathcal{B}$ and a non-empty open set $W \subset V$ such that $\widehat{b}(V^c) = \{0\}$ and $\widehat{b}(W) = \{1\}$.

Theorem 2.6. $L^1(G,\omega,\mathcal{A})$ has QDZP if and only if $L^1(G,\omega)$ and $\mathcal{A}$ have QDZP.

Proof: Let $L^1(G,\omega,\mathcal{A})$ have QDZP. Then there is an open set $\widetilde{U} \subset \Delta(L^1(G,\omega,\mathcal{A}))$ which satisfies the properties stated in the definition of QDZP. Let $U_1 = \pi_1(\widetilde{U})$. Then U_1 is an open subset of $\widehat{G}(\omega)$, and

$$\partial L^1(G,\omega) = \pi_1(\partial L^1(G,\omega,\mathcal{A})) \subset \pi_1(\overline{\widetilde{U}}) \subset \overline{\pi_1(\widetilde{U})} = \overline{U}_1.$$

Let V_1 be any open subset of U_1. Then $\widetilde{V} = [V_1 \times \pi_2(\widetilde{U})] \cap \widetilde{U}$ is an open subset of $\widetilde{U}$. Since $L^1(G,\omega,\mathcal{A})$ has QDZP, there exist $\widetilde{f} \in L^1(G,\omega,\mathcal{A})$ and a non-empty open subset $\widetilde{W}$ of $\widetilde{V}$ such that $\widehat{\widetilde{f}}\,(\widetilde{V}^c) = \{0\}$ and $\widehat{\widetilde{f}}\,(\widetilde{W}) = \{1\}$. Choose $\psi_{\alpha_0,\varphi_0} \in \widetilde{W}$. Then there exists a basic open set $M \times N$ such that $\psi_{\alpha_0,\varphi_0} \sim (\alpha_0,\varphi_0) \in M \times N \subset \widetilde{W}$. Set $f = \eta_{\varphi_0}(\widetilde{f}) \in L^1(G,\omega)$, where η_{φ_0} is as in Lemma 1.9. Let $W_1 = \pi_1(M \times N) = M$ be a non-empty open subset of V_1. Then, for every $\alpha \in V_1^c$, $\psi_{\alpha,\varphi_0} \sim (\alpha,\varphi_0) \notin \widetilde{V}$, and so $\psi_\alpha(f) = \psi_\alpha(\eta_{\varphi_0}(\widetilde{f})) = \psi_{\alpha,\varphi_0}(\widetilde{f}) = 0$. Thus $\widehat{f}(V_1^c) = \{0\}$. Now, for every, $\alpha \in W_1$, $\psi_{\alpha,\varphi_0} \sim (\alpha,\varphi_0) \in M \times N \subset \widetilde{W}$, and so $\psi_\alpha(f) = \psi_\alpha(\eta_{\varphi_0}(\widetilde{f})) = \psi_{\alpha,\varphi_0}(\widetilde{f}) = 1$. Thus $\widehat{f}(W_1) = \{1\}$. Thus $L^1(G,\omega)$ has QDZP. By similar arguments, we can show that $\mathcal{A}$ has QDZP.

Conversely, let $L^1(G,\omega)$ and $\mathcal{A}$ have QDZP. Then there exist open subsets, $U_1 \subset \widehat{G}(\omega)$ and $U_2 \subset \Delta(\mathcal{A})$, which satisfies the properties in the definition of QDZP. Set $\widetilde{U} = U_1 \times U_2$. Then $\widetilde{U}$ is an open subset of $\Delta(L^1(G,\omega,\mathcal{A}))$, and

$$\partial L^1(G,\omega,\mathcal{A}) = \partial L^1(G,\omega) \times \partial\mathcal{A} \subset \overline{U}_1 \times \overline{U}_2 = \overline{U_1 \times U_2} = \overline{\widetilde{U}}.$$

Let $\widetilde{V}$ be any open subset of $\widetilde{U}$. We assume that $\widetilde{V} = V_1 \times V_2$ where V_1 and V_2 are open in $\widehat{G}(\omega)$ and $\Delta(\mathcal{A})$, respectively. Then $V_1 \subset U_1$ and $V_2 \subset U_2$. Since $L^1(G,\omega)$ and $\mathcal{A}$ have QDZP, there exist $f \in L^1(G,\omega), a \in \mathcal{A}$, and non-empty open subsets, $W_1 \subset V_1, W_2 \subset V_2$ such that

$$\widehat{f}(V_1^c) = \{0\}, \;\; \widehat{a}(V_2^c) = \{0\}, \;\; \widehat{f}(W_1) = \{1\} \text{ and } \widehat{a}(W_2) = \{1\}.$$

Set $\widetilde{f} = fa \in L^1(G,\omega,\mathcal{A})$ and $\widetilde{W} = W_1 \times W_2$. Then $\widetilde{W}$ is a non-empty open, subset of $\widetilde{V}$. Let $\psi_{\alpha,\varphi} \in \widetilde{V}^c$. Then either $\alpha \notin V_1$ or $\varphi \notin V_2$ and so $\psi_\alpha(f) = 0$ or $\varphi(a) = 0$. Hence, by Lemma 1.11, $\psi_{\alpha,\varphi}(\widetilde{f}) = \psi_\alpha(f)\varphi(a) = 0$. Thus $\widehat{\widetilde{f}}\,(\widetilde{V}^c) = \{0\}$. Now, for every $\psi_{\alpha,\varphi} \in \widetilde{W}$, $\alpha \in W_1$ and $\varphi \in W_2$, and so $\psi_\alpha(f) = 1$ and $\varphi(a) = 1$. Hence, by Lemma 1.11, $\psi_{\alpha,\varphi}(\widetilde{f}) = \psi_\alpha(f)\varphi(a) = 1 \cdot 1 = 1$. Thus $\widehat{\widetilde{f}}\,(\widetilde{W}) = \{1\}$. Thus $L^1(G,\omega,\mathcal{A})$ has QDZP. ■

3 Three Regularity Properties

The property "regularity" is one of the most important Banach algebra properties. There are various types of regularities studied in the literature. In this section, we are going to study three of them in $L^1(G,\omega,\mathcal{A})$; namely, weak regularity, regularity, and complete regularity.

Definition 3.1. [4, 9] A semi simple, commutative, Banach algebra $\mathcal{B}$ is

1. *weakly regular (WR)* if for each proper closed set $F \subset \Delta(\mathcal{B})$, there exists $b \in \mathcal{B} \setminus \{0\}$ such that $\widehat{b}(F) = \{0\}$.

2. *regular* if for each closed set $F \subset \Delta(\mathcal{B})$ and an element $\varphi \in \Delta(\mathcal{B}) \setminus F$, there exists an element $b \in \mathcal{B}$ such that $\widehat{b}(F) = \{0\}$ and $\widehat{b}(\varphi) = 1$.

3. *completely regular (CR)* if for every closed subset F and any compact subset K of $\Delta(\mathcal{B})$ with $F \cap K = \emptyset$, there exists $b \in \mathcal{B}$ such that $\widehat{b}(F) = \{0\}$ and $\widehat{b}(K) = \{1\}$.

Theorem 3.2. $L^1(G, \omega, \mathcal{A})$ is WR if and only if both $L^1(G, \omega)$ and $\mathcal{A}$ are WR.

Proof: Let $L^1(G, \omega, \mathcal{A})$ be WR. Let F be any proper closed subset of $\widehat{G}(\omega)$ and $\widetilde{F} = F_1 \times \Delta(\mathcal{A})$. Then $\widetilde{F}$ is a proper closed subset of $\Delta(L^1(G, \omega, \mathcal{A}))$. Since $L^1(G, \omega, \mathcal{A})$ is WR, there exists $\widetilde{f} \in L^1(G, \omega, \mathcal{A}) \setminus \{0\}$ such that $\widehat{\widetilde{f}}\,(\widetilde{F}) = \{0\}$. Choose $\psi_{\alpha_0,\varphi_0} \in \Delta(L^1(G, \omega, \mathcal{A}))$ such that $\psi_{\alpha_0,\varphi_0}(\widetilde{f}) \neq 0$. Set $f = \eta_{\varphi_0}(\widetilde{f}) \in L^1(G, \omega) \setminus \{0\}$, where η_{φ_0} is as in Lemma 1.9. Now if $\alpha \in F$, then $\psi_{\alpha,\varphi_0} \in \widetilde{F}$ and hence $\psi_\alpha(f) = \psi_\alpha(\eta_{\varphi_0}(\widetilde{f})) = \psi_{\alpha,\varphi_0}(\widetilde{f}) = 0$. Hence $L^1(G, \omega)$ is WR. Similarly, we prove that $\mathcal{A}$ is WR.

Conversely, let $L^1(G, \omega)$ and $\mathcal{A}$ be WR. Let $\widetilde{F}$ be any proper closed subset of $\Delta(L^1(G, \omega, \mathcal{A}))$. Then $\widetilde{F}^c$ is open in $\Delta(L^1(G, \omega, \mathcal{A}))$. Let $(\alpha_0, \varphi_0) \in \widetilde{F}^c$. Then there exists a basic open set $U \times V$ in $\widehat{G}(\omega) \times \Delta(\mathcal{A})$ such that $(\alpha_0, \varphi_0) \in U \times V \subset \widetilde{F}^c$. Set $F_U = \widehat{G}(\omega) \setminus U$ and $F_V = \Delta(\mathcal{A}) \setminus V$. Then both F_U and F_V are proper closed subsets of $\widehat{G}(\omega)$ and $\Delta(\mathcal{A})$ respectively. Since $L^1(G, \omega)$ and $\mathcal{A}$ are WR, there exist $f \in L^1(G, \omega) \setminus \{0\}$ and $a \in \mathcal{A} \setminus \{0\}$ such that $\widehat{f}(F_U) = \{0\}$ and $\widehat{a}(F_V) = \{0\}$. Set $\widetilde{f} = fa \in L^1(G, \omega, \mathcal{A}) \setminus \{0\}$. Let $\psi_{\alpha,\varphi} \sim (\alpha, \varphi) \in \widetilde{F}$. Then either $\alpha \in F_U$ or $\varphi \in F_V$, and so either $\widehat{f}(\psi_\alpha) = 0$ or $\widehat{a}(\varphi) = 0$. Hence, by Lemma 1.11, $\widehat{\widetilde{f}}\,(\psi_{\alpha,\varphi}) = \psi_{\alpha,\varphi}(\widetilde{f}) = \widehat{a}(\varphi)\widehat{f}(\psi_\alpha) = 0$. Since $\psi_{\alpha,\varphi} \in \widetilde{F}$ is arbitrary, $L^1(G, \omega, \mathcal{A})$ is WR. ∎

Theorem 3.3. $L^1(G, \omega, \mathcal{A})$ is regular if and only if both $L^1(G, \omega)$ and $\mathcal{A}$ are regular.

Proof: Let $L^1(G, \omega, \mathcal{A})$ be regular. Let F be any closed subset of $\Delta(\mathcal{A})$ and $\varphi_0 \in \Delta(\mathcal{A}) \setminus F$. Then $\widetilde{F} = \widehat{G}(\omega) \times F$ is a closed subset of $\Delta(L^1(G, \omega, \mathcal{A}))$. Let $\psi_{\alpha_0,\varphi_0} \in \Delta(L^1(G, \omega, \mathcal{A})) \setminus \widetilde{F}$. Since $L^1(G, \omega, \mathcal{A})$ is regular, there exists $\widetilde{f} \in L^1(G, \omega, \mathcal{A}) \setminus \{0\}$ such that $\widehat{\widetilde{f}}\,(\widetilde{F}) = \{0\}$ and $\psi_{\alpha_0,\varphi_0}(\widetilde{f}) = 1$. Take $a = \eta_{\alpha_0}(\widetilde{f}) \in \mathcal{A} \setminus \{0\}$, where η_{α_0} is as in Lemma 1.10. Then, for every $\varphi \in F$, $(\alpha_0, \varphi) \in \widetilde{F}$ and so $\varphi(a) = \varphi(\eta_{\alpha_0}(\widetilde{f})) = \psi_{\alpha_0,\varphi}(\widetilde{f}) = 0$ and $\varphi_0(a) = \varphi_0(\eta_{\alpha_0}(\widetilde{f})) = \psi_{\alpha_0,\varphi_0}(\widetilde{f}) = 1$. Hence $\mathcal{A}$ is regular. Similarly, using the homomorphism defined in Lemma 1.9, we prove that $L^1(G, \omega)$ is regular.

Conversely, let $L^1(G, \omega)$ and $\mathcal{A}$ be regular. Let $\widetilde{F}$ be any closed subset of $\Delta(L^1(G, \omega, \mathcal{A}))$ and $(\alpha_0, \varphi_0) \in \widetilde{F}^c$. Then there exists a basic open set $U \times V$ in $\widehat{G}(\omega) \times \Delta(\mathcal{A})$ such that $(\alpha_0, \varphi_0) \in U \times V \subset \widetilde{F}^c$. Set $F_U = \widehat{G}(\omega) \setminus U$ and $F_V = \Delta(\mathcal{A}) \setminus V$. Then $\alpha_0 \notin F_U$, $\varphi_0 \notin F_V$, and both F_U and F_V are closed subsets of $\widehat{G}(\omega)$ and $\Delta(\mathcal{A})$, respectively. Since $L^1(G, \omega)$ and $\mathcal{A}$ are regular, there exist $f \in L^1(G, \omega)$ and $a \in \mathcal{A}$ such that $\widehat{f}(F_U) = \{0\}, \widehat{a}(F_V) = \{0\}, \widehat{f}(\psi_{\alpha_0}) = 1$ and $\widehat{a}(\varphi_0) = 1$. Set $\widetilde{f} = fa \in L^1(G, \omega, \mathcal{A})$. Let $\psi_{\alpha,\varphi} \sim (\alpha, \varphi) \in \widetilde{F}$. Then either $\alpha \in F_U$ or $\varphi \in F_V$, and so either $\widehat{f}(\psi_\alpha) = 0$ or $\widehat{a}(\varphi) = 0$. Hence, by Lemma 1.11, $\widehat{\widetilde{f}}\,(\psi_{\alpha,\varphi}) = \psi_{\alpha,\varphi}(\widetilde{f}) = \widehat{a}(\varphi)\widehat{f}(\psi_\alpha) = 0$. Moreover, $\widehat{\widetilde{f}}\,(\psi_{\alpha_0,\varphi_0}) = \widehat{a}(\varphi_0)\widehat{f}(\psi_{\alpha_0}) = 1$. Thus $L^1(G, \omega, \mathcal{A})$ is regular. ∎

We do not know the converse of the following result.

Theorem 3.4. If $L^1(G,\omega,\mathcal{A})$ is CR, then $L^1(G,\omega)$ and $\mathcal{A}$ are CR.

Proof: Let $L^1(G,\omega,\mathcal{A})$ be CR. Let F and K be closed and compact subsets of $\widehat{G}(\omega)$, respectively, with $F \cap K = \emptyset$. Fix $\varphi_0 \in \Delta(\mathcal{A}) \setminus \{0\}$. Then $\widetilde{F} = F \times \{\varphi_0\}$ and $\widetilde{K} = K \times \{\varphi_0\}$ are closed and compact subsets of $\Delta(L^1(G,\omega,\mathcal{A}))$, respectively, and $\widetilde{F} \cap \widetilde{K} = \emptyset$. Since $L^1(G,\omega,\mathcal{A})$ is CR, there exists $\widetilde{f} \in L^1(G,\omega,\mathcal{A})$ such that $\widehat{\widetilde{f}}\,(\widetilde{F}) = \{0\}$ and $\widehat{\widetilde{f}}\,(\widetilde{K}) = \{1\}$. Take $f = \eta_{\varphi_0}(\widetilde{f}) \in L^1(G,\omega)$, where η_{φ_0} is as in Lemma 1.9. Then, for every $\alpha \in F$, $\psi_{\alpha,\varphi_0} \sim (\alpha,\varphi_0) \in \widetilde{F}$, and so $\psi_\alpha(f) = \psi_\alpha(\eta_{\varphi_0}(\widetilde{f})) = \psi_{\alpha,\varphi_0}(\widetilde{f}) = 0$. Thus $\widehat{f}(F) = \{0\}$. Now, for every $\beta \in K$, $\psi_{\beta,\varphi_0} \sim (\beta,\varphi_0) \in \widetilde{K}$, and so $\psi_\beta(f) = \psi_\beta(\eta_{\varphi_0}(\widetilde{f})) = \psi_{\beta,\varphi_0}(\widetilde{f}) = 1$. Thus $\widehat{f}(K) = \{1\}$. Thus $L^1(G,\omega)$ is CR. By similar arguments, we can show that $\mathcal{A}$ is CR. ∎

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