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Neighborhood-prime labeling of some union graphs

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Abstract

In this paper, we investigate the neighborhood-prime labeling for union of two cycles, union of two wheels and union of a finite number of paths.

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1 Introduction

In the past thirty years several research papers investigating the primality of various graphs have been published. In [4], [6] and [7] fans, helms, flowers, stars, wheels W_{2n}, books, the (m,n)-gon star $S_n^{(m)}$ are shown as prime graphs. In [8], it is shown that the union of stars $S_m \cup S_n$, the union of cycles and stars $C_m \cup S_n$ are prime graphs. For a comprehensive list of results regarding prime graphs, readers may refer to [2]. It is a well-known result that the cycle C_n is a prime graph for all n. Further in [1], it is shown that the graph $C_n \cup C_m$ is prime if either n is even or m is even.

Motivated by the study of prime labeling of graphs, we introduced neighborhood-prime labeling for graphs in [5] and showed that the path P_n is a neighborhood-prime graph for all n and the cycle C_n is a neighborhood-prime graph if and only if $n \not\equiv 2 \pmod 4$. Also we showed that certain path and cycle related graphs are neighborhood-prime graphs.

In the present work, we derive necessary and sufficient conditions under which the graph $C_n \cup C_m$ is neighborhood-prime. We also show that union of two wheels and a union of a finite number of paths are neighborhood-prime graphs.

Note that all graphs considered in this paper are simple, finite and undirected. We follow Gross and Yellen [3] for graph theoretic terminology and notations.

Definition 1.1. Let $G = (V(G), E(G))$ be a graph with n vertices and for $v \in V(G)$, let $N(v)$ denote the open neighborhood of v. A bijective function $f : V(G) \rightarrow \{1, 2, 3, \ldots, n\}$ is said to be a neighborhood-prime labeling of G, if for every vertex $v \in V(G)$ with $deg(v) > 1$, $gcd\{f(u) : u \in N(v)\} = 1$. A graph which admits neighborhood-prime labeling is called a neighborhood-prime graph.

Remark 1.2. If in a graph G, every vertex is of degree at most 1, then such a graph is neighborhood-prime vacuously.

2 Some important lemmas

In this section we derive some important lemmas which are used to prove our main results.

Lemma 2.1. Let n be any integer of the form $4k+2$. Suppose $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the cycle C_n which are all labeled with 0 or 1 in such a way that the vertices labeled with 0 and the vertices labeled with 1 are equal in number. Then there exists at least one i, $1 \leq i \leq n$, such that v_{i-1} and v_{i+1} are labeled with 0 where the indices are taken modulo n.

For the proof of this lemma we refer to [5]. Note that this lemma immediately implies the following lemma.

Lemma 2.2. Let n be any integer of the form $4k+2$. Suppose $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the cycle C_n which are all labeled with 0 or 1 in such a way that the vertices labeled with 0 are greater than or equal to the number of vertices labeled with 1. Then there exists at least one i, $1 \leq i \leq n$, such that v_{i-1} and v_{i+1} are labeled with 0 where the indices are taken modulo n.

We use Lemma 2.2 to derive a similar result for the cycle C_{4k}.

Lemma 2.3. Let n be any integer of the form $4k$. Suppose $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the cycle C_n in which $2k+1$ or more vertices are labeled with 0 and the remaining vertices are labeled with 1. Then there exists at least one i, $1 \leq i \leq n$, such that v_{i-1} and v_{i+1} are labeled with 0 where the indices are taken modulo n.

Proof: Suppose the lemma fails for some cycle C_{4k_0} with consecutive vertices as $v_1, v_2, \ldots, v_{4k_0}$. This means that there exists a function $f : \{v_1, v_2, \ldots, v_{4k_0}\} \to \{0, 1\}$ such that:

(i) Cardinality of the set $\{v_i : f(v_i) = 0\}$ is at least $2k_0 + 1$;

(ii) $f(v_{i-1})$ and $f(v_{i+1})$ are simultaneously not equal to zero.

Now consider the cycle C_{4k_0+2} with consecutive vertices as $u_1, u_2, \ldots, u_{4k_0}, u_{4k_0+1}, u_{4k_0+2}$ and define $g : \{u_1, u_2, \ldots, u_{4k_0}, u_{4k_0+1}, u_{4k_0+2}\} \to \{0, 1\}$ by

$$g(u_i) = \begin{cases} f(v_i) & 1 \leq i \leq 4k_0 \\ 1 & i = 4k_0+1, 4k_0+2 \end{cases}$$

The definition of g clearly suggests that cardinality of the set $\{u_i : g(u_i) = 0\}$ is at least $2k_0 + 1$ and moreover, $g(u_{i-1})$ and $g(u_{i+1})$ cannot be simultaneously zero for $1 \leq i \leq 4k_0 + 2$. But this contradicts Lemma 2.2 and so our supposition is wrong. ■

Lemma 2.4. Let n be any integer of the form $2k+1$. Suppose $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the cycle C_n in which $k+1$ vertices are labeled with 0 and k vertices are labeled with 1. Then there exists at least one i, $1 \leq i \leq n$, such that v_{i-1} and v_{i+1} are labeled with 0 where the indices are taken modulo n.

Proof: We prove the lemma by induction on k. For $k = 1$ and $k = 2$, the lemma follows easily. Now assuming the lemma for all the cycles C_{2k+1} with $k \leq k_0$, we prove it for the cycle $C_{2(k_0+1)+1}$. The proof is by contradiction.

Let $u_1, u_2, \ldots, u_{2(k_0+1)+1}$ be the consecutive vertices of the cycle $C_{2(k_0+1)+1}$ and suppose there does not exist i ($1 \leq i \leq 2(k_0+1)+1$), such that u_{i-1} and u_{i+1} are labeled with 0. But if this happens then since there are k_0+2 vertices labeled with 0 and k_0+1 vertices labeled with 1 in the cycle $C_{2(k_0+1)+1}$, there must exist two consecutive vertices in $C_{2(k_0+1)+1}$ labeled with 0. So let u_j and u_{j+1} be some consecutive vertices labeled with 0. This in addition to our above supposition that no two alternate vertices are labeled with 0, implies that $u_{j-2}, u_{j-1}, u_{j+2}, u_{j+3}$ are labeled with 1. Now consider the cycle C with vertices $u_1, u_2, \ldots, u_{j-2}, u_{j+3}, u_{j+4} \ldots, u_{2(k_0+1)+1}$. Note that C is a cycle of length $2k_0 - 1$ in which u_{j-2} and u_{j+3} are labeled with 1. This along with our supposition suggests that C does not contain a pair of alternate vertices labeled with 0. But C is a cycle of length $2k_0 - 1$ and so this is a contradiction to our induction hypothesis. By the principle of mathematical induction the lemma follows for all k.

∎

We now observe that Lemma 2.4 immediately implies the following lemma.

Lemma 2.5. Let n be any integer of the form $2k+1$. Suppose $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the cycle C_n in which $k+1$ or more vertices are labeled with 0 and the remaining vertices are labeled with 1. Then there exists at least one i, $1 \leq i \leq n$, such that v_{i-1} and v_{i+1} are labeled with 0 where the indices are taken modulo n.

3 Main Results

We begin with the investigation about neighborhood-prime labeling for $C_n \cup C_m$ in all possible cases. Later we prove that union of two wheels and union of a finite number of paths are neighborhood-prime graphs.

Theorem 3.1. If n and m are odd integers, then the graph $G = C_n \cup C_m$ is not neighborhood-prime.

Proof: Let $f : V(G) \to \{1, 2, \ldots, n+m\}$ be any bijective function. Assuming that $n = 2k_1 + 1$ and $m = 2k_2+1$ for some positive integers k_1 and k_2, it follows that either the cycle C_n is labeled with $k_1 + 1$ or more even integers or the cycle C_m is labeled with $k_2 + 1$ or more even integers under f. Thus if we identify all even and odd integers of the set $\{1, 2, \ldots, n + m\}$ by 0 and 1 respectively, then in view of Lemma 2.5 it is quite clear that f cannot be a neighborhood-prime labeling for G. ∎

Theorem 3.2. Let m be an odd integer. Then the graph $G = C_n \cup C_m$ is neighborhood-prime if and only if $n \equiv 0 (\text{mod } 4)$.

Proof: First we show that G is a neighborhood-prime graph if $n \equiv 0 (\text{mod } 4)$. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of the cycle C_n and $u_1, u_2, \ldots, u_m$ be the consecutive vertices of the cycle C_m. Define $f : V(G) \to \{1, 2, \ldots, n+m\}$ by

$$\begin{aligned} f(v_{2j-1}) &= \frac{n}{2} + j, & 1 \le j \le \frac{n}{2}, \\ f(v_2) &= n+1, & \\ f(v_{2j}) &= j, & 2 \le j \le \frac{n}{2} \\ f(u_{2j-1}) &= n + \frac{m-1}{2} + j, & 1 \le j \le \frac{m+1}{2}, \\ f(u_2) &= 1, & \\ f(u_{2j}) &= n + j, & 2 \le j \le \frac{m-1}{2}. \end{aligned}$$

We claim that f is a neighborhood-prime labeling. For this we need to show that if w is an arbitrary vertex of G and $S = \{f(p) : p \in N(w)\}$, then the gcd of the numbers in the set S is 1. If w is a vertex different from v_1, v_3, v_n, u_1, u_3 and u_m then this follows because S consists of two consecutive integers in such cases. The remaining cases are discussed below.

Case 1: $w = v_1$.
In this case $S = \{f(v_n), f(v_2)\} = \{\frac{n}{2}, n+1)\}$. But $\frac{n}{2}$ and $n+1$ are relatively prime.

Case- 2: $w = v_3$.
Since n is even, here it suffices to observe that $S = \{n+1, 2\}$.

Case 3: $w = v_n$.
Note that here $S = \{n, \frac{n}{2} + 1\}$. Since $n \equiv 0 (\text{mod } 4)$, the gcd of the numbers in the set S is once again 1.

Case 4: $w = u_1, u_3$.
In this case $u_2 \in N(w)$ and so S contains the integer $f(u_2) = 1$ which gives gcd of the numbers in the set S to be 1.

Case 5: $w = u_m$.
In this case $S = \{\frac{m+1}{2} + n, \frac{m-1}{2} + n\}$, a set of consecutive integers and hence the gcd is 1.

Conversely, assume that $n \not\equiv 0$ (mod 4). Then n is odd or $n \equiv 2$ (mod 4). But if n is odd then since m is also odd, Theorem 3.1 implies that G is not neighborhood-prime. Now assume that $n = 4k_1 + 2$ and $m = 2k_2 + 1$ for some positive integers k_1, k_2 and consider a bijective function $f : V(G) \to \{1, 2, \ldots, n+m\}$. Then either the cycle C_n of graph G is labeled with $2k_1 + 1$ (or more) even integers, or the cycle C_m of graph G is labeled with $k_2 + 1$ (or more) even integers under f. In view of Lemma 2.2 and Lemma 2.5, it follows that f cannot be a neighborhood-prime labeling for the graph G. ∎

Example 3.3. The following figure shows the neighborhood-prime labeling of $C_{12} \cup C_9$.

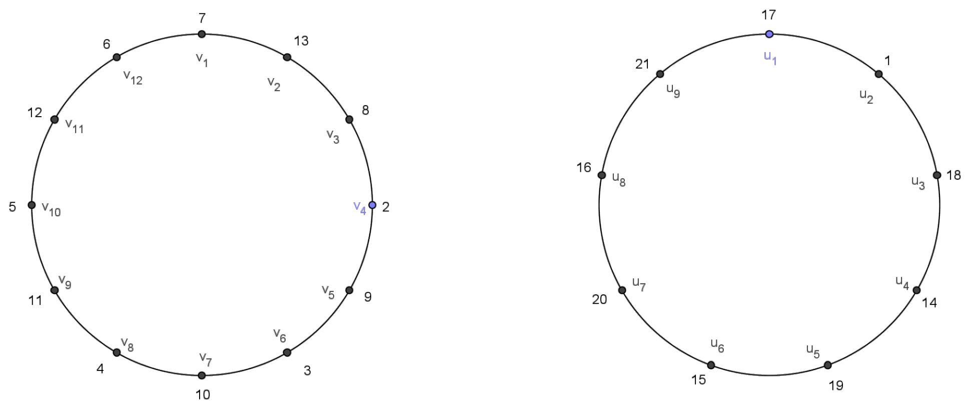

Figure 1: Neighborhood-prime labeling of $C_{12} \cup C_9$.

Theorem 3.4. The graph $G = C_n \cup C_m$ is not neighborhood-prime, if

(1) $n, m \equiv 2 \pmod 4$;

(2) $n \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$.

Proof: For (1), assume that $n = 4k_1 + 2$ and $m = 4k_2 + 2$ and consider an arbitrary bijective function $f : V(G) \to \{1, 2, \ldots, n + m\}$. Then either the cycle C_n of graph G is labeled with $2k_1 + 1$ (or more) even integers, or the cycle C_m of graph G is labeled with $2k_2 + 1$ (or more) even integers under f. In view of Lemma 2.2, it follows that f cannot be a neighborhood-prime labeling for the graph G.
For (2), assume that $n = 4k_1$ and $m = 4k_2 + 2$ and consider an arbitrary bijective function $f : V(G) \to \{1, 2, \ldots, n + m\}$. Then an argument similar to (1), along with Lemma 2.2 and Lemma 2.3 implies that f cannot be a neighborhood-prime labeling for the graph G. ∎

Theorem 3.5. Let n and m be multiples of 4. Then the graph $G = C_n \cup C_m$ is neighborhood-prime.

Proof: Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of the cycle C_n and $u_1, u_2, \ldots, u_m$ be the consecutive vertices of the cycle C_m. It is easy to prove the theorem if n or m is equal to 4. Without loss of generality, assume that $m = 4$ and define a bijection $f : V(G) \to \{1, 2, \ldots, n + 4\}$ by

$$f(v_{2j-1}) = \frac{n}{2} + j, \qquad 1 \le j \le \frac{n}{2},$$

$$\begin{aligned} f(v_{2j}) &= j, & 1 \le j \le \frac{n}{2} \\ f(u_{2j-1}) &= n+2+j, & j = 1,2, \\ f(u_{2j}) &= n+j, & j = 1,2. \end{aligned}$$

It is easy to verify that the function f is a neighborhood-prime labeling on G.
Now assume that $n, m > 4$. We define a bijection between $V(G)$ and $\{1, 2, \ldots, n+m\}$ as per the following two cases and show that it is a neighborhood-prime labeling on G.
Case 1: $n + 2 \not\equiv 0(\text{mod } 3)$.
Define a bijection $f : V(G) \to \{1, 2, \ldots, n+m\}$ by

$$\begin{aligned} f(v_{2j-1}) &= \frac{n}{2}+j, & 1 \le j \le \frac{n}{2}, \\ f(v_2) &= n+1, \\ f(v_4) &= n+2, \\ f(v_{2j}) &= j, & 3 \le j \le \frac{n}{2} \\ f(u_1) &= 1, \\ f(u_{2j-1}) &= n+\frac{m}{2}+j, & 2 \le j \le \frac{m}{2}, \\ f(u_2) &= n+\frac{m}{2}+1, \\ f(u_4) &= 2, \\ f(u_{2j}) &= n+j, & 3 \le j \le \frac{m}{2}. \end{aligned}$$

Here we need to show that if w is an arbitrary vertex of G and $S = \{f(p) : p \in N(w)\}$, then the gcd of the two numbers in the set S is 1. If $w \neq v_1, v_5, v_n, u_2, u_3, u_5, u_m$; then this follows because S consists of two consecutive integers in such cases. If $w = u_2, u_m$ then S contains 1 and so we are through. If $w = u_3, u_5$ then $S = \{n + \frac{m}{2} + 1, 2\}$ and $S = \{n + 3, 2\}$ respectively. But $n + \frac{m}{2} + 1$ and $n + 3$ are odd numbers and so their gcd with 2 is 1. Now if $w = v_5$, then $S = \{n + 2, 3\}$. But $n + 2$ and 3 are relatively prime due to our assumption that $n + 2 \not\equiv 0(\text{mod } 3)$. The remaining two cases, that is, $w = v_1, v_n$ follows from the following two results (under the assumption that n is a multiple of 4) respectively.

$$gcd\left(f(v_2), f(v_n)\right) = gcd\left(n+1, \frac{n}{2}\right) = 1;$$

$$gcd\left(f(v_1), f(v_{n-1})\right) = gcd\left(\frac{n}{2}+1, n\right) = 1.$$

Now if $n + 2 \equiv 0(\text{mod } 3)$ then we make a minor change in the definition of f and obtain a neighborhood-prime labeling on G.
Case 2: $n + 2 \equiv 0 \ (\text{mod } 3)$ (and so $n + 4 \not\equiv 0 \ (\text{mod } 3)$).

Define a bijection $g : V(G) \to \{1, 2, \ldots, n+m\}$ by

$$\begin{aligned} g(w) &= f(w),\ w \neq v_6, u_6 \\ g(v_6) &= f(u_6), \\ g(u_6) &= f(v_6). \end{aligned}$$

The reader may verify that g defines a neighborhood-prime labeling on G. ∎

Example 3.6. The following figures illustrate the two cases of Theorem 3.5.

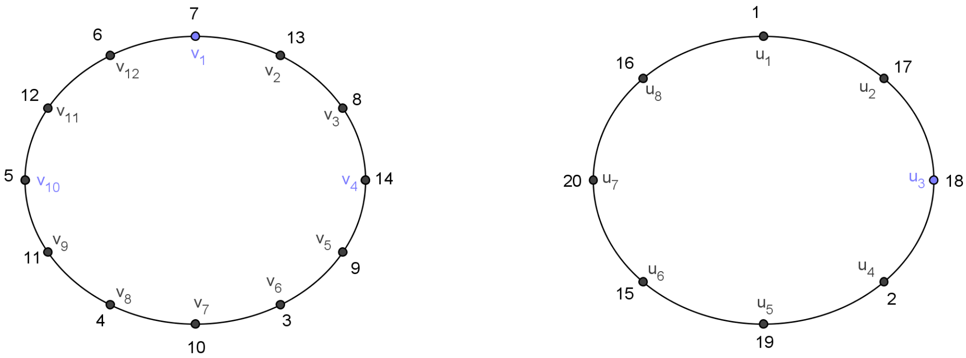

Figure 2: Neighborhood-prime labeling of $C_{12} \cup C_8$.

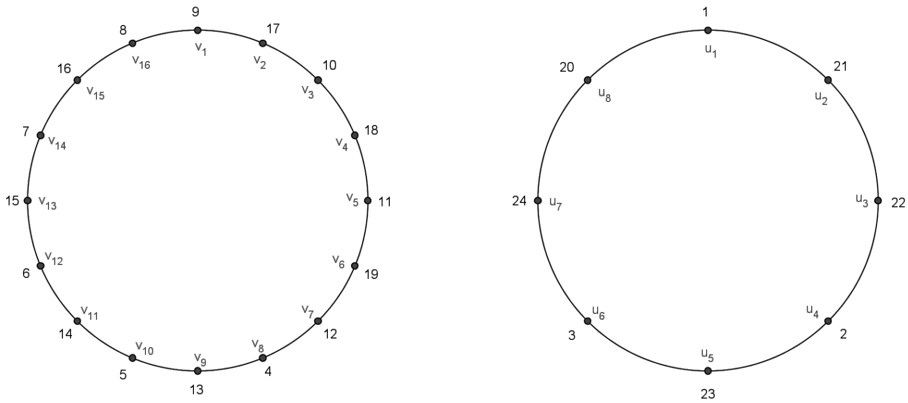

Figure 3: Neighborhood-prime labeling of $C_{16} \cup C_8$.

Observe that Theorem 3.1, 3.4 and 3.5 together give the following result for the special case $C_n \cup C_n$.

Theorem 3.7. The graph $C_n \cup C_n$ is neighborhood-prime if and only if $n \equiv 0 (\mathrm{mod}\ 4)$.

$C_n \cup C_m$ is a neighborhood-prime graph only for some restricted values of m and n where as in case of wheels $W_n \cup W_m$ is a neighborhood-prime graph for all n and m, which we prove in Theorem 3.8. Note that every wheel graph is neighborhood-prime follows from Theorem 2.1 in [5]

Theorem 3.8. The graph $G = W_n \cup W_m$ is neighborhood-prime.

Proof: Let $\{v_0, v_1, \ldots, v_n\}$ and $\{u_0, u_1, \ldots, u_m\}$ denote the vertex sets of W_n and W_m respectively where v_0 and u_0 are the apex (central) vertices of the two wheels. Suppose p is a Bertrand's prime such that $\frac{n+2}{2} < p \leq n+2$. Now consider an arbitrary bijection $f : V(W_n) \to \{2, 3, \ldots, n+2\}$ and an arbitrary bijection $g : V(W_m) \to \{1, n+3, n+4, \ldots, n+m+2\}$ such that $f(v_0) = p$ and $g(u_0) = 1$. Define $h : V(W_n \cup W_m) \to \{1, 2, 3, \ldots, n+m+2\}$ by

$$h(w) = \begin{cases} f(w), & \text{if } w \in W_n \\ g(w), & \text{if } w \in W_m. \end{cases}$$

Now for an arbitrary vertex w_0 of $W_n \cup W_m$, consider the set $S = \{f(w) : w \in N(w_0)\}$. We observe that if w_0 is an apex vertex, then S contains two consecutive integers and if w_0 is a rim vertex, then S contains either p or 1. From this, it follows that h is a neighborhood-prime labeling on $W_n \cup W_m$. ∎

Example 3.9. The following figure shows a neighborhood-prime labeling of $W_{12} \cup W_9$.

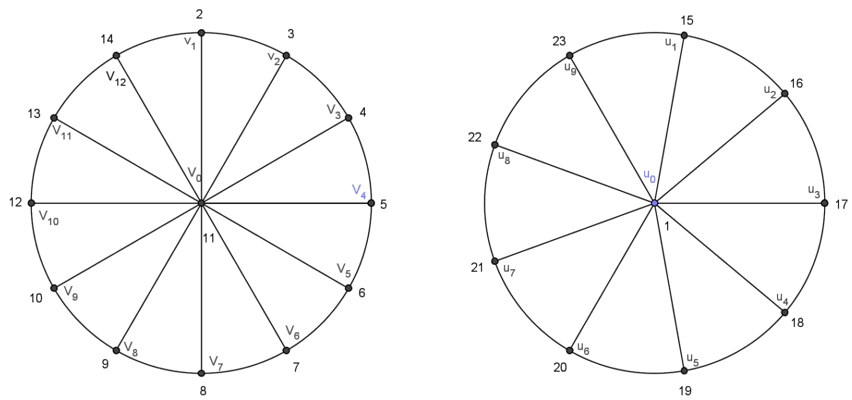

Figure 4: Neighborhood-prime labeling of $W_{12} \cup W_9$.

Theorem 3.10. Let $n_1, n_2, \ldots, n_k$ be any positive integers. Then the graph $G = P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k}$ is neighborhood-prime.

Proof: Let $v_1, v_2, \ldots, v_{n_1}$ be the consecutive vertices of P_{n_1}, $v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2}$ be the consecutive vertices of P_{n_2}, and for $3 \leq j \leq k$, let $v_{n_1+\cdots+n_{j-1}+1}, v_{n_1+\cdots+n_{j-1}+2}, \ldots, v_{n_1+\cdots+n_{j-1}+n_j}$ be the consecutive vertices of P_{n_j}.

Thus if we put $n_1 + n_2 + \cdots + n_k = m$, then the vertex set of G is $\{v_1, v_2, \ldots, v_m\}$. Define $f : V(G) \to \{1, 2, 3, \ldots, m\}$ as follows:

Case 1: m is even.

$$\begin{aligned} f(v_{2j-1}) &= \frac{m}{2} + j, \ \ 1 \le j \le \frac{m}{2} \\ f(v_{2j}) &= j, \qquad\quad 1 \le j \le \frac{m}{2}. \end{aligned}$$

Case 2: m is odd.

$$\begin{aligned} f(v_{2j-1}) &= \frac{m-1}{2} + j, 1 \le j \le \frac{m+1}{2} \\ f(v_{2j}) &= j, \qquad\qquad 1 \le j \le \frac{m-1}{2}. \end{aligned}$$

Clearly f is a neighborhood-prime labeling for G because for every $u \in V(G)$ with $deg(u) > 1$, the labels of the two vertices in $N(u)$ are consecutive integers. ■

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