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Computation of Efficient Nash Equilibria for experimental economic games

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Abstract

Nash equilibrium is the central solution concept with diverse applications for most games in game theory. For games with multiple equilibria, different equilibria can have different rewards for the players thus causing a challenge on their choice of strategies. The computation of most efficient Nash Equilibrium in games can be applied to most situations in competitive Economic environment that are faced with multiple choices on which strategy is optimal.

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1 Introduction

Game theory is the formal study of conflict and cooperation. The concept of game theory provides a language to formulate, structure, analyze and understand strategic scenarios. The games studied in game theory are well defined mathematical objects with a set of players, a set of moves (strategies) available to those players and a specification of payoffs or costs for each combination of strategies [1]. Game theory has been broadly classified into four main subcategories: Classical game theory, combinatorial game theory, dynamic game theory and other topics such as evolutionary game theory, experimental game theory and economic game theory [4].

A strong solution concept, which is applicable to all games in game theory, is the Nash equilibrium which captures the notion of a stable solution. As much as some experimental economic games have a unique Nash equilibrium, others have none whereas the rest have multiple equilibria. Multiple Nash Equilibria is one of the fundamental problems in game theory. For the games with multiple Nash equilibria, it becomes difficult to predict what strategies will be chosen by the players and there is need for players to make the best choices so as to optimize from the outcomes of the game.

On the other hand, most experiments that have been conducted involve two players. Two-person games do not take us very far because many of the games that are most important in real world involve considerably more than two players, for example, economic competition, highway congestion, over-exploitation of the environment and monetary exchange. For this study, a game modelled as an experimental economic game with more than two players and multiple equilibria was considered and the most efficient Nash equilibrium for the game computed.

This paper described and carried out an experiment on a game that was modelled as a three-player experimental economic game. The results were recorded and by the best response sets method we identified all the Pure Nash equilibria and computed the most efficient Nash equilibrium for the modelled game. Using the Brauwer's fixed point theorem we verified the existence of mixed Nash equilibrium in the game. An individual whose aim was to minimize risks played the risk dominant strategies whereas those aiming to maximize their profits, the payoff dominant strategies were played to achieve their most efficient Nash equilibrium.

2 Nash Equilibrium

To represent a game, we will use the notation

$$\Gamma = \langle N, (S_i), (u_i) \rangle \tag{2.1}$$

where N is the number of players, S_i the available strategies, u_i the payoff to the player i and $i = 1, 2, 3$. Every finite game has an equilibrium point [9]. Nash (1951) proved that every game with a finite number of players, each having a finite set of strategies, has a Nash Equilibrium of mixed strategies [5].

More formally, a strategy vector $\mathbf{s} \in \mathbf{S}$ is said to be a Nash equilibrium if for all players i and each alternate strategy $\mathbf{s}'_i \in \mathbf{S}$, we have that

$$u_i(\mathbf{s}_i, \mathbf{s}_j) \geq u_i(\mathbf{s}'_i, \mathbf{s}_j). \tag{2.2}$$

A dominant strategy solution is a Nash equilibrium. Moreover if the solution is strictly dominating, it is also a unique Nash equilibrium. A game can have either a pure strategy and/or a mixed strategy Nash equilibria [3].

Given a game (2.1) with pure strategies, the strategy profile

$$\mathbf{s}^* = (s_1^*, \cdots, s_n^*)^T$$

is said to be a pure strategy Nash equilibrium of (2.1) if

$$u_i(\mathbf{s}_i^*, \mathbf{s}_j^*) \geq u_i(\mathbf{s}_i, \mathbf{s}_j^*) \tag{2.3}$$

$\forall \mathbf{s}_i \in \mathbf{S}_i \ \forall \quad i = 1, 2 \cdots n$. That is each player's Nash equilibrium strategy is a best response to the Nash equilibrium strategies of the other players. Therefore for the game (2.1), the strategy profile $(s_1^*, \cdots, s_n^*)^T$ is a Nash equilibrium if and only if $\mathbf{s}_i^* \in B_i(\mathbf{s}_j^*) \quad \forall i = 1, \cdots, n$.

Definition 2.1. A strategy profile $(\mathbf{s}_i^*, \mathbf{s}_j^*)$ is a strict Nash Equilibrium if for every player i, $u_i(\mathbf{s}_i^*, \mathbf{s}_j^*) > u_i(\mathbf{s}_i, \mathbf{s}_j^*)$, for every $\mathbf{s}_i^* \neq \mathbf{s}_j^*$.

Consider a pure strategy game: (2.1). A pure strategy or a deterministic strategy for player i specifies the deterministic choice $s_i(I)$ at each information set I. Let $\mathbf{S}_i$ be finite for each $i = 1, 2 \cdots n$. If player i randomly chooses one element of the set S_i, we have a mixed strategy or a randomized strategy.

More formally, given a player i with $\mathbf{S}_i$ as the set of pure strategies, a mixed strategy ρ_i for player i is a probability density function over $\mathbf{S}_i$. That is, $\rho_i : \mathbf{S}_i \mapsto [0, 1]$ assigns to each pure strategy $\mathbf{s}_i \in \mathbf{S}_i$, a probability $\rho_i(\mathbf{s}_i)$ such that

$$\sum_{\mathbf{s}_i \in \mathbf{S}_i} \rho_i(\mathbf{s}_i) = 1. \tag{2.4}$$

A mixed strategy profile is a Nash equilibrium if the mixed strategy for each player is a best response to the mixed strategies of the rest; that is, it attains the maximum possible utility among all possible mixed strategies of this player. The support of a mixed strategy is the set of all pure strategies that have non-zero probability in it. A mixed strategy is a best response if and only if all pure strategies in its support are best responses. If each player in n-player game has a finite number of pure strategies, then there exists at least one equilibrium in mixed strategy[5]. If there are no pure strategy equilibria, there must be a unique mixed strategy equilibrium. However, it is possible for pure strategy and mixed strategy Nash equilibria to co-exist in games.

3 Description of the Game

3.1 Stag Hunt Game

The original stag hunt game was described by the philosopher Jean-Jacques Rousseau in the year 1755 . This game is a well known coordination game in which two players go out to hunt together. If they cooperate they have a chance of capturing a stag, constituting a high reward. On their own, the hunters can only hope to capture a hare yielding a lower payoff. Should one player try to cooperate, while the other chooses to hunt alone (defects), the cooperator will fail and get nothing, whereas the defector can still get a hare. In order to make the stag hunt game to be more applicable in real world, it was generalized into an N-player Stag hunt game [7].

For this study, the following social cooperation situation was modelled as three- player stag hunt game: In a certain High School, students were given two assignments to attempt. The first assignment (stag) was quite challenging and for an individual to succeed he must have the cooperation of one of his partners. The second assignment (hare) is simpler and can be done by any one student without any problem.

The students were required to make a choice between attempting the first assignment or attempting the second assignment. Attempting the first assignment together and obtaining a correct solution was more rewarding than individually finding a solution to the second assignment. Any student who cooperated with any other to correctly complete the first assignment was given a payoff of 10 and whoever cooperated with any other student to complete the second assignment correctly was given a payoff of 7 points. Attempting the first assignment individually was doomed to failure and had a payoff of zero.

The following assumptions were made:

(i) All the players (students) were rational as they made their choice.

(ii) All the players had the same ability in making choices.

(iii) All the players had the same strategy profile.

We denote the above game as

$$\Gamma_1 = \langle N, (S_i), (u_i) \rangle \tag{3.1}$$

where N is the number of players, S_i the available strategies, u_i the payoffs to the players and $i = 1, 2, 3$. The game (3.1) is a pure strategy game with the strategy profile $\beta = (\beta_1, \beta_2)$ where: β_1 represents the first pure strategy (choosing the first assignment - stag), β_2 represents the second pure strategy (choosing the second assignment - hare. Note that $\beta \in S_i$.

3.2 Stages of the Game (3.1)

The stag hunt game and the game modelled as the stag hunt game, (3.1), was explained to students so that they had the full knowledge of the game and they made their choices independently. It was a dynamical stage game in that the students were allowed to play it for a finite number of repetitions as they varied their strategies as well. The students had complete information about the game since all the parameters and the rules of the game were well known by all of them.

The game modelled in (3.1) had three players (three students) in each group. The two strategies available to the players were:

(β_1) and (β_2). The following steps were followed:

(i) Students were asked to choose their strategies, β_1 or β_2. Note that players chose their strategies independently.

(ii) The outcomes were recorded before they were allowed to repeat the same game.

(iii) The payoffs for all the possible outcome cells from the game were calculated.

3.3 Outcomes of the Game (3.1)

In the three person stag hunt game (3.1) modelled above, each player had two choices, attempting the first assignment or attempting the second assignment. This resulted to eight possible outcomes (cells) for the three players, (Player 1, Player 2, Player 3) respectively as: $(\beta_1, \beta_1, \beta_1)$; $(\beta_1, \beta_1, \beta_2)$; $(\beta_1, \beta_2, \beta_1)$; $(\beta_1, \beta_2, \beta_2)$; $(\beta_2, \beta_1, \beta_1)$; $(\beta_2, \beta_1, \beta_2)$; $(\beta_2, \beta_2, \beta_1)$ and $(\beta_2, \beta_2, \beta_2)$.
Therefore to find the number of possible outcomes we use the expression S^N where S represents the number of strategies available to the players and N represents the number of players. Thus $2^3 = 8$.
Payoffs were calculated by examining each pair-wise payoff set among players, and the payoffs for three players were calculated by considering the type of interaction they had. For example, the payoffs for three players for $(\beta_1, \beta_2, \beta_1)$ was as follows: Player 1 received 0 points for the interaction with player 2 and 10 points for cooperating with player 3. Player 2 received 7 points for not cooperating with player 1 and 7 points for not cooperating with player 3. Player 3 received 10 points for cooperating with player 1 and 0 points for not cooperating with player 2. Therefore $(\beta_1, \beta_2, \beta_1) = (10, 14, 10)$. This implies that $u_1(\beta_1, \beta_2, \beta_1) = 10$, $u_2(\beta_1, \beta_2, \beta_1) = 14$, $u_3(\beta_1, \beta_2, \beta_1) = 10$ where u_1, u_2 and u_3 are the payoffs of player 1, player 2 and player 3 respectively. Applying the same rules, we have the summary for the payoffs to the three players as per the eight possible outcomes as shown below:
$(\beta_1, \beta_1, \beta_1) = (20, 20, 20)$; $(\beta_1, \beta_1, \beta_2) = (10, 10, 14)$; $(\beta_1, \beta_2, \beta_1) = (10, 14, 10)$; $(\beta_1, \beta_2, \beta_2) = (0, 14, 14)$; $(\beta_2, \beta_1, \beta_1) = (14, 10, 10)$; $(\beta_2, \beta_1, \beta_2) = (14, 0, 14)$; $(\beta_2, \beta_2, \beta_1) = (14, 14, 0)$; $(\beta_2, \beta_2, \beta_2) = (14, 14, 14)$. (see Figure 1.)

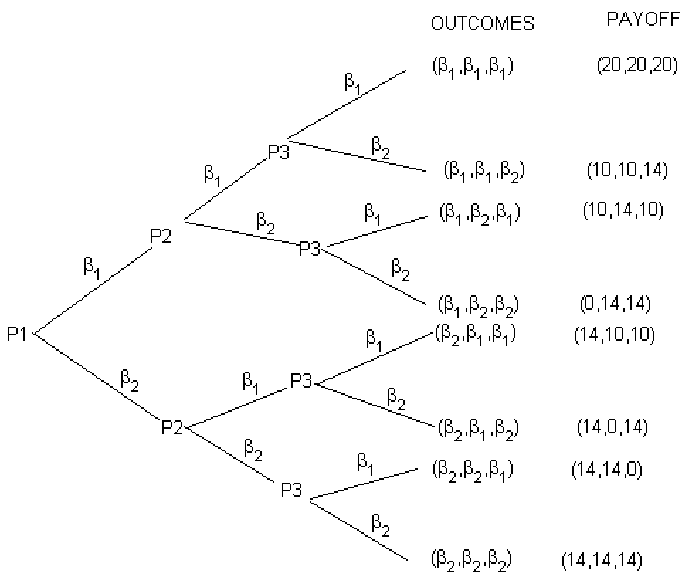

Figure 1: Tree Diagram on outcomes and payoffs.

The payoffs for all the three players, $u_i(\beta)$ will be as shown below:
$u_1(\beta_1,\beta_1,\beta_1) = 20$, $u_2(\beta_1,\beta_1,\beta_1) = 20$ and $u_3(\beta_1,\beta_1,\beta_1) = 20$
$u_1(\beta_1,\beta_1,\beta_2) = 10$, $u_2(\beta_1,\beta_1,\beta_2) = 10$ and $u_3(\beta_1,\beta_1,\beta_2) = 14$
$u_1(\beta_1,\beta_2,\beta_1) = 10$, $u_2(\beta_1,\beta_2,\beta_1) = 14$ and $u_3(\beta_1,\beta_2,\beta_1) = 10$
$u_1(\beta_1,\beta_2,\beta_2) = 0$, $u_2(\beta_1,\beta_2,\beta_2) = 14$ and $u_3(\beta_1,\beta_2,\beta_2) = 14$
$u_1(\beta_2,\beta_1,\beta_1) = 14$, $u_2(\beta_2,\beta_1,\beta_1) = 10$ and $u_3(\beta_2,\beta_1,\beta_1) = 10$
$u_1(\beta_2,\beta_1,\beta_2) = 14$, $u_2(\beta_2,\beta_1,\beta_2) = 0$ and $u_3(\beta_2,\beta_1,\beta_2) = 14$
$u_1(\beta_2,\beta_2,\beta_1) = 14$, $u_2(\beta_2,\beta_2,\beta_1) = 14$ and $u_3(\beta_2,\beta_2,\beta_1) = 0$
$u_1(\beta_2,\beta_2,\beta_2) = 14$, $u_2(\beta_2,\beta_2,\beta_2) = 14$ and $u_3(\beta_2,\beta_2,\beta_2) = 14$

4 Computation and Identification of Efficient Nash Equilibria

4.1 Identification of Pure NE in the Game (3.1)

The game (3.1) which is an example of non-cooperative coordination game has eight different action profiles:
$(\beta_1,\beta_1,\beta_1) = (20,20,20)$, $(\beta_1,\beta_1,\beta_2) = (10,10,14)$, $(\beta_1,\beta_2,\beta_1) = (10,14,10)$, $(\beta_1,\beta_2,\beta_2) = (0,14,14)$, $(\beta_2,\beta_1,\beta_1) = (14,10,10)$, $(\beta_2,\beta_1,\beta_2) = (14,0,14)$, $(\beta_2,\beta_2,\beta_1) = (14,14,0)$ and $(\beta_2,\beta_2,\beta_2) = (14,14,14)$ for the three players, (Player 1, Player 2, Player 3), respectively. Since the game has only a few actions, we found Nash equilibria for the game by examining each action profile in turn to determine if it satisfied the conditions for equilibrium.
The best response sets for the game (3.1) were:

(i) $B_i(\beta_1; i = 1,2,3) = \beta_1$, that is, the best response for player i when he or she plays β_1 is β_1 and

(ii) $B_i(\beta_2; i = 1,2,3) = \beta_2$ which means that the best response for player i when β_2 is played is β_2.

Therefore, $B_1(\beta_1) = \beta_1$; $B_1(\beta_2) = \beta_2$, $B_2(\beta_1) = \beta_1$; $B_2(\beta_2) = \beta_2$ and $B_3(\beta_1) = \beta_1$; $B_3(\beta_2) = \beta_2$. B_1, B_2 and B_3 are the best response for player 1, 2 and 3 respectively.
Since $\beta_1 \subset B_1(\beta_1)$, $\beta_1 \subset B_2(\beta_1)$ and $\beta_1 \subset B_3(\beta_1)$, then $(\beta_1,\beta_1,\beta_1) = (20,20,20)$ is a pure Nash Equilibrium.
Similarly, since $\beta_2 \subset B_1(\beta_2)$,$\beta_2 \subset B_2(\beta_2)$ and $\beta_2 \subset B_3(\beta_2)$, then $(\beta_2,\beta_2,\beta_2) = (14,14,14)$ is a pure Nash equilibrim.

The other profiles: $(\beta_1,\beta_1,\beta_2) = (10,10,14)$, $(\beta_1,\beta_2,\beta_1) = (10,14,10)$, $(\beta_1,\beta_2,\beta_2) = (0,14,14)$, $(\beta_2,\beta_1,\beta_1) = (14,10,10)$, $(\beta_2,\beta_1,\beta_2) = (14,0,14)$ and $(\beta_2,\beta_2,\beta_1) = (14,14,0)$ are not pure Nash equilibria since, $\beta_1 \subsetneq B_i(\beta_2)$ and $\beta_2 \subsetneq B_i(\beta_1)$.
In summary, the results of the game (3.1) are as shown below:

(i) $(\beta_1, \beta_1, \beta_1)$ is a pure Nash equilibrium because it is better off remaining attentive to the pursuit of a stag, β_1, than running after a hare, β_2, if all other players remain attentive).

(ii) $(\beta_2, \beta_2, \beta_2)$ is a pure Nash equilibrium because it is better off catching a hare, (β_2), than pursuing a stag, (β_1), if no one else pursues a stag.

(iii) No other profile is a pure Nash equilibrium because in any other profile at least one player chooses a stag, (β_1), and at least one player chooses a hare, (β_2), so that any player choosing (β_1) is better off switching to (β_2)

Since $(\beta_1, \beta_1, \beta_1)$ is a strict NE, then β_1 is evolutionary stable. $(\beta_2, \beta_2, \beta_2)$ is also a strict NE, therefore β_2 is also evolutionary stable.

4.2 Mixed Nash Equilibrium in the Game (3.1)

Since this game had multiple equilibrium points, the optimal choice is a mixed strategy. Thus randomization of the pure strategies was done as shown below:
Suppose (ρ_1, ρ_2, ρ_3) is a mixed strategy profile. This means that ρ_1 is a probability density function on $S_1 = \{\beta_1, \beta_2\}$, ρ_2 is a probability density function on $S_2 = \{\beta_1, \beta_2\}$ and ρ_3 is a probability density function on $S_3 = \{\beta_1, \beta_2\}$.
Let us represent $\rho_1 = (\rho_1(\beta_1)\rho_1(s_2))$, $\rho_2 = (\rho_2(\beta_1)\rho_2(s_2))$ and $\rho_3 = (\rho_3(\beta_1)\rho_3(s_2))$. We have,
$S = S_1 \times S_2 \times S_3$
$= \{(\beta_1,\beta_1,\beta_1)(\beta_1,\beta_1,\beta_2)(\beta_1,\beta_2,\beta_1)(\beta_1,\beta_2,\beta_2)(\beta_2,\beta_1,\beta_1)(\beta_2,\beta_1,\beta_2)(\beta_2,\beta_2,\beta_1)(\beta_2,\beta_2,\beta_2).\}$
We computed the payoff functions u_1, u_2 and u_3.
Note that, $u_i(\rho_1, \rho_2, \rho_3) = \sum_{S_1,S_2,S_3 \in S} \rho(S_1, S_2, S_3)u_i(S_1, S_2, S_3)$ for $i = 1, 2, 3$. That is:

$$\begin{aligned} u_1(\rho_1,\rho_2,\rho_3) =& \rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_1)u_1(\beta_1,\beta_1,\beta_1) + \rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_2)u_1(\beta_1,\beta_1,\beta_2)+ \\ & \rho_1(\beta_1)\rho_2(\beta_2)\rho_3(\beta_1)u_1(\beta_1,\beta_2,\beta_1) + \rho_1(\beta_1)\rho_2(\beta_2)\rho_3(\beta_2)u_1(\beta_1,\beta_2,\beta_2)+ \\ & \rho_1(\beta_2)\rho_2(\beta_1)\rho_3(\beta_1)u_1(\beta_2,\beta_1,\beta_1) + \rho_1(\beta_2)\rho_2(\beta_1)\rho_3(\beta_2)u_1(\beta_2,\beta_1,\beta_2)+ \\ & \rho_1(\beta_2)\rho_2(\beta_2)\rho_3(\beta_1)u_1(\beta_2,\beta_2,\beta_1) + \rho_1(\beta_2)\rho_2(\beta_2)\rho_3(\beta_2)u_1(\beta_2,\beta_2,\beta_2). \end{aligned} \tag{4.1}$$

$$\begin{aligned} u_1(\rho_1,\rho_2,\rho_3) = & 20\rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_1) + 10\rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_2)+ \\ & 10\rho_1(\beta_1)\rho_2(\beta_2)\rho_3(\beta_1) + 14\rho_1(\beta_2)\rho_2(\beta_1)\rho_3(\beta_1)+ \\ & 14\rho_1(\beta_2)\rho_2(\beta_1)\rho_3(\beta_2) + 14\rho_1(\beta_2)\rho_2(\beta_2)\rho_3(\beta_1)+ \\ & 14\rho_1(\beta_2)\rho_2(\beta_2)\rho_3(\beta_2). \end{aligned} \tag{4.2}$$

$$\begin{aligned} u_1(\rho_1,\rho_2,\rho_3) = & 20\rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_1) + 10\rho_1(\beta_1)\rho_2(\beta_1)(1-\rho_3)(\beta_1)+ \\ & 10\rho_1(\beta_1)(1-\rho_2)(\beta_1)\rho_3(\beta_1) + 14(1-\rho_1)(\beta_1)\rho_2(\beta_1)\rho_3(\beta_1)+ \\ & 14(1-\rho_1)(\beta_1)\rho_2(\beta_1)(1-\rho_3)(\beta_1) + 14(1-\rho_1)(\beta_1)(1-\rho_2)(\beta_1)\rho_3(\beta_1)+ \\ & 14(1-\rho_1)(\beta_1)(1-\rho_2)(\beta_1)(1-\rho_3)(\beta_1). \end{aligned} \tag{4.3}$$

$$u_1(\rho_1,\rho_2,\rho_3) = 14-14\rho_1(\beta_1)+10\rho_1(\beta_1)\rho_2(\beta_1)+10\rho_1(\beta_1)\rho_3(\beta_1). \tag{4.4}$$

Similarly,

$$u_2(\rho_1,\rho_2,\rho_3) = 14-14\rho_2(\beta_1)+10\rho_1(\beta_1)\rho_2(\beta_1)+10\rho_2(\beta_1)\rho_3(\beta_1). \tag{4.5}$$

$$u_3(\rho_1,\rho_2,\rho_3) = 14-14\rho_3(\beta_1)+10\rho_1(\beta_1)\rho_3(\beta_1)+10\rho_2(\beta_1)\rho_3(\beta_1). \tag{4.6}$$

Basing on the assumption that all the students were rational, they had the same strategy profile to choose from and their ability in making choices were the same, we let

$$\rho_1 = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \rho_2 = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \rho_3 = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right).$$

Then, $u_1(\rho_1,\rho_2,\rho_3) = \frac{104}{9}, u_2(\rho_1,\rho_2,\rho_3) = \frac{104}{9}, u_3(\rho_1,\rho_2,\rho_3) = \frac{104}{9}$.

Suppose (ρ_1,ρ_2,ρ_3) is a mixed strategy profile. It can be seen that

$$\begin{aligned}
u_1(\rho_1,\rho_2,\rho_3) &= 14-14\rho_1(\beta_1)+10\rho_1(\beta_1)\rho_2(\beta_1)+10\rho_1(\beta_1)\rho_3(\beta_1),\\
u_2(\rho_1,\rho_2,\rho_3) &= 14-14\rho_2(\beta_1)+10\rho_1(\beta_1)\rho_2(\beta_1)+10\rho_2(\beta_1)\rho_3(\beta_1) \quad \text{and}\\
u_3(\rho_1,\rho_2,\rho_3) &= 14-14\rho_3(\beta_1)+10\rho_1(\beta_1)\rho_3(\beta_1)+10\rho_2(\beta_1)\rho_3(\beta_1).
\end{aligned}$$

Let $(\rho_1^*,\rho_2^*,\rho_3^*)$ be a mixed strategy equilibrium. Then

$$\begin{aligned}
&u_1(\rho_1^*,\rho_2^*,\rho_3^*) \ge u_1(\rho_1,\rho_2^*,\rho_3^*) \quad \forall \quad \rho_1 \in \Delta(S_1);\\
&u_2(\rho_1^*,\rho_2^*,\rho_3^*) \ge u_1(\rho_1^*,\rho_2,\rho_3^*) \quad \forall \quad \rho_2 \in \Delta(S_2);\\
&u_3(\rho_1^*,\rho_2^*,\rho_3^*) \ge u_1(\rho_1^*,\rho_2^*,\rho_3) \quad \forall \quad \rho_3 \in \Delta(S_3).
\end{aligned} \tag{4.7}$$

The inequalities (4.7) are equivalent to:

$$\begin{aligned}
&14-14\rho_1^*(\beta_1)+10\rho_1^*(\beta_1)\rho_2^*(\beta_1)+10\rho_1^*(\beta_1)\rho_3^*(\beta_1)\\
&\ge 14-14\rho_1(\beta_1)+10\rho_1(\beta_1)\rho_2^*(\beta_1)+10\rho_1(\beta_1)\rho_3^*(\beta_1)\\
&\forall \rho_1 \in \Delta(S_1);\\
\\
&14-14\rho_2^*(\beta_1)+10\rho_1^*(\beta_1)\rho_2^*(\beta_1)+10\rho_2^*(\beta_1)\rho_3^*(\beta_1)\\
&\ge 14-14\rho_2(\beta_1)+10\rho_1^*(\beta_1)\rho_2(\beta_1)+10\rho_2(\beta_1)\rho_3^*(\beta_1)\\
&\forall \rho_2 \in \Delta(S_2);\\
\\
&14-14\rho_3^*(\beta_1)+10\rho_1^*(\beta_1)\rho_3^*(\beta_1)+10\rho_2^*(\beta_1)\rho_3^*(\beta_1)\\
&\ge 14-14\rho_3(\beta_1)+10\rho_1^*(\beta_1)\rho_3(\beta_1)+10\rho_2^*(\beta_1)\rho_3(\beta_1)\\
&\forall \rho_3 \in \Delta(S_3).
\end{aligned} \tag{4.8}$$

These inequalities (4.8) are equivalent to:

$$
\begin{aligned}
&10\rho_1^*(\beta_1)\rho_2^*(\beta_1) + 10\rho_1^*(\beta_1)\rho_3^*(\beta_1) - 14\rho_1^*(\beta_1) \geq 10\rho_1(\beta_1)\rho_2^*(\beta_1) + 10\rho_1(\beta_1)\rho_3^*(\beta_1) - 14\rho_1(\beta_1)\\
&\forall \rho_1 \in \Delta(S_1);\\
&10\rho_1^*(\beta_1)\rho_2^*(\beta_1) + 10\rho_2^*(\beta_1)\rho_3^*(\beta_1) - 14\rho_2^*(\beta_1) \geq 10\rho_1^*(\beta_1)\rho_2(\beta_1) + 10\rho_2(\beta_1)\rho_3^*(\beta_1) - 14\rho_2(\beta_1)\\
&\forall \rho_2 \in \Delta(S_2);\\
&10\rho_1^*(\beta_1)\rho_3^*(\beta_1) + 10\rho_2^*(\beta_1)\rho_3^*(\beta_1) - 14\rho_3^*(\beta_1) \geq 10\rho_1^*(\beta_1)\rho_3(\beta_1) + 10\rho_2^*(\beta_1)\rho_3(\beta_1) - 14\rho_3(\beta_1)\\
&\forall \rho_3 \in \Delta(S_3).
\end{aligned}
\tag{4.9}
$$

In turn the inequalities (4.9) are equivalent to:

$$
\begin{aligned}
&\rho_1^*(\beta_1)\{10\rho_2^*(\beta_1) + 10\rho_3^*(\beta_1) - 14\} \geq \rho_1(\beta_1)\{10\rho_2^*(\beta_1) + 10\rho_3^*(\beta_1) - 14\}\\
&\forall \rho_1 \in \Delta(S_1);\\
&\rho_2^*(\beta_1)\{10\rho_1^*(\beta_1) + 10\rho_3^*(\beta_1) - 14\} \geq \rho_2(\beta_1)\{10\rho_1^*(\beta_1) + 10\rho_3^*(\beta_1) - 14\}\\
&\forall \rho_2 \in \Delta(S_2);\\
&\rho_3^*(\beta_1)\{10\rho_1^*(\beta_1) + 10\rho_2^*(\beta_1) - 14\} \geq \rho_3(\beta_1)\{10\rho_1^*(\beta_1) + 10\rho_2^*(\beta_1) - 14\}\\
&\forall \rho_3 \in \Delta(S_3).
\end{aligned}
\tag{4.10}
$$

Some of the possible cases are:

(i) $\frac{5}{7}\{\rho_2^*(\beta_1) + \rho_3^*(\beta_1)\} > 1$ which leads to the pure strategy profile $\beta_1, \beta_1, \beta_1$ that is a NE.

(ii) $\frac{5}{7}\{\rho_2^*(\beta_1) + \rho_3^*(\beta_1)\} < 1$ which leads to the pure strategy profile $\beta_2, \beta_2, \beta_2$ that is a NE.

(iii) $\frac{5}{7}\{\rho_2^*(\beta_1) + \rho_3^*(\beta_1)\} = 1$ which leads to a mixed strategy profile that we indeed showed that it was also a NE.

4.2.1 Verification of Existence of Equilibria in the Game (3.1)

Considering the game (3.1) analyzed above, the two multiple equilibria (pure Nash equilibria) were $(\beta_1, \beta_1, \beta_1) = (20, 20, 20)$ and $(\beta_2, \beta_2, \beta_2) = (14, 14, 14)$. We proved the existence of mixed Nash equilibrium using the Brauwer's fixed point theorem as shown below:
We had the game (3.1),

$$\Gamma_1 = \langle N, (S_i), (u_i) \rangle,$$

where N is the number of players and $S_i = S_1 \times S_2 \times S_3$ is the action set for the players. All the action sets S_i are finite.

We let $\Delta = \Delta_1 \times \cdots \times \Delta_N$ denote the set of mixed strategies for the players in the game (3.1). The finiteness of S_i ensures the compactness of Δ.

We then defined the gain function for player i, G_i. For a mixed strategy $\rho \in \Delta$, we let the gain for player i on action $\beta \in S_i$ be

$$G_i(\rho, \beta) = max\{0, u_i(\beta, \rho_j) - u_i(\rho_i, \rho_j)\},$$

where ρ_i is the mixed strategy for player i and ρ_j is the mixed strategy for all other players in the game (3.1). The gain function represents the benefit a player gets by unilaterally changing his strategy.

We now define $g = (g_1, \cdots, g_N)$ where $g_i(\rho, \beta) = \rho_i(\beta) + G_i(\rho, \beta)$ for $\rho \in \Delta$, $\beta \in S_i$.
We see that

$$\sum_{\beta \in S_i} g_i(\rho, \beta) = \sum_{\beta \in S_i} \rho_i(\beta) + G_i(\rho, \beta) = 1 + \sum_{\beta \in S_i} G_i(\rho, \beta) > 0.$$

We now use g to define $f : \Delta \mapsto \Delta$ as follows:
Let

$$f_i(\rho, \beta) = \frac{g_i(\rho, \beta)}{\sum_{\beta \in S_i} g_i(\rho, \beta)}$$

for $\beta \in S_i$.

It is easy to see that f_i is a valid mixed strategy in Δ_i. It is also easy to check that each f_i is a continuous function of ρ, and hence f is a continuous function. Now Δ is the cross product of a finite number of compact convex sets, and so we get that Δ is also compact and convex. Therefore we may apply the Brouwer fixed point theorem to f. So f has a fixed point in Δ, call it ρ^*.

We claim that ρ^* is a Nash equilibrium in the game (3.1). For this purpose it suffices to show that

$$\forall \quad 1 \leq i \leq N, \forall \quad \beta \in S_i, G_i(\rho^*, \beta) = 0.$$

This simply states that each player gains nothing by unilaterally changing his strategy, which is exactly the necessary condition for Nash equilibrium.

Now assume that the gains are not zero. $\exists \quad i, 1 \leq i \leq N$ and $\beta \in S_i$ such that

$$G_i(\rho^*, \beta) > 0.$$

Note then that

$$\sum_{\beta \in S_i} g_i(\rho^*, \beta) = 1 + \sum_{\beta \in S_i} G_i(\rho^*, \beta) > 1.$$

So let

$$C = \sum_{\beta \in S_i} g_i(\rho^*, \beta).$$

We denote $G(i, \cdot)$ as the gain vector indexed by actions in S_i. Since

$$f(\rho^*) = \rho^*,$$

we clearly have that

$$f_i(\rho^*) = \rho_i^*.$$

Therefore we see that

$$\rho_i^* = \frac{g_i(\rho^*)}{\sum_{\beta \in S_i} g_i(\rho^*, \beta)}.$$

$$\Rightarrow \rho_i^* = \frac{\rho_i^* + G_i(\rho^*, \cdot)}{C}$$

$$C\rho_i^* = \rho_i^* + G_i(\rho^*, \cdot)$$

$$(C-1)\rho_i^* = G_i(\rho^*, \cdot)$$

$$\rho_i^* = (\frac{1}{C-1})G_i(\rho^*, \cdot).$$

Since $C > 1$, we have that ρ_i^* is some positive scaling of the vector $G_i(\rho^*, \cdot)$.

Now we claim that

$$\rho_i^*(\beta)(U_i(\beta_i, \rho_j^*) - U_i(\rho_i^*, \rho_j^*)) = \rho_i^*(\beta)G_i(\rho^*, \beta)\forall\beta \in S_i.$$

To see this, we first note that if

$$G_i(\rho^*, \beta) > 0,$$

then this is true by the definition of the gain function.

We assume that

$$G_i(\rho^*, \beta) = 0.$$

By our previous statements we have that

$$\rho_i^*(\beta) = \frac{1}{C-1}G_i(\rho^*, \beta) = 0,$$

and so the left term is zero, giving the entire expression as 0 as needed.

So finally we have that

$$\begin{aligned}
0 =&(U_i(\beta_i, \rho_j^*) - U_i(\rho_i^*, \rho_j^*) \\
=& \sum_{\beta \in S_i} (\rho_i^*(\beta) U_i(\beta_i, \rho_j^*) - U_i(\rho_i^*, \rho_j^*)) \\
=& \sum_{\beta \in S_i} (\rho_i^*(\beta)(U_i(\beta_i, \rho_j^*) - U_i(\rho_i^*, \rho_j^*)) \\
=& \sum_{\beta \in S_i} \rho_i^*(\beta) G_i(\rho^*, \beta) \text{ by the previous statements.} \\
=& \sum_{\beta \in S_i} (C-1)\rho_i^*(\beta)^2 > 0
\end{aligned}$$

where the last inequality follows since ρ_i^* is a non-zero vector. But this is a clear contradiction, so all the gain must indeed be zero.

Therefore ρ^* is a mixed Nash equilibrium for the game (3.1) as needed.

More often, most situations involve population of players and to study multi-player games effectively we need to deviate from classical game theory to Evolutionary Game Theory. Edgar (2012)[6] presented an approach that deviates from classical game theory in regard to rationality of players, belief about the behaviour of other players and the alignment of such beliefs across players. This is important because in a multi-player game, some players may make their choices irrationally. Evolutionary Game Theory will effectively enable us determine equilibria of games played by a population of players, where the fitness (payoff) of the players is derived from the success each player has in playing the game.

Together with Evolutionary Game Theory, new concepts were developed such as the Evolutionary Stable Strategy which is applied to study the stability of populations [8]. ESS is an equilibrium refinement of NE. It is a NE that is evolutionary stable in the sense that if adopted by a population of players in a given environment, it cannot be invaded by any alternative strategy that is initially rare. It is known that any ESS is an asymptotically stable strategy [2]. In particular, in games with multiple ESS, we resolve the problem of equilibrium selection by choosing the one that is stochastically stable.

Suppose in the game (3.1), a third pure strategy (attempting the third assignment, β_3) is introduced such that attempting the first assignment (β_1) together is still more rewarding than individually attempting either the second assignment (β_2) or the third assignment (β_3.) We denote the new game as

$$\Gamma_2 = \langle N, (S_i), (u_i) \rangle \tag{4.11}$$

where N is the number of players, S_i the available strategies, u_i the payoffs to the players and $i = 1, 2, 3$.

The game (4.11) is a pure strategy game with the strategy profile $\beta = (\beta_1, \beta_2, \beta_3)$, where
β_1 represents the first pure strategy (choosing the first assignment),
β_2 represents the second pure strategy (choosing the second assignment),
β_3 represents the third pure strategy (choosing the third assignment) and N is the number of players.
The rewards for β_1 and β_2 are maintained as in the game (3.1). However the third assignment (β_3) has the lowest reward of 5. We considered two cases: where the third assignment (β_3) could be completed successfully on its own and where the reward for β_3 depends on cooperation among the students. The result was twenty seven possible outcome cells and their respective payoffs were calculated by examining each pair-wise payoff set among players, and the payoffs for three players were calculated by considering the type of interaction they had as was done in the game (3.1). The first eight possible outcomes of this game and their respective payoffs for the three players were the same as the outcomes in the game (3.1). However the other 19 possible outcome cells and their respective payoffs for the three players were calculated and the results were as displayed in Table (1).

Table 1: Possible outcomes and their respective payoffs for the game (4.11)

OUTCOMES	FIRST PAYOFF, $U_i(\beta)$	SECOND PAYOFF,$U_i(\beta)$
$\beta_3, \beta_3, \beta_3$	$(10, 10, 10)$	$(10, 10, 10)$
$\beta_3, \beta_3, \beta_1$	$(10, 10, 0)$	$(5, 5, 0)$
$\beta_3, \beta_1, \beta_1$	$(10, 10, 10)$	$(0, 10, 10)$
$\beta_3, \beta_1, \beta_3$	$(10, 0, 10)$	$(5, 0, 5)$
$\beta_3, \beta_3, \beta_2$	$(10, 10, 14)$	$(5, 5, 14)$
$\beta_3, \beta_2, \beta_2$	$(10, 14, 14)$	$(0, 14, 14)$
$\beta_3, \beta_2, \beta_3$	$(10, 14, 10)$	$(5, 14, 5)$
$\beta_1, \beta_2, \beta_3$	$(0, 14, 10)$	$(0, 14, 0)$
$\beta_3, \beta_2, \beta_1$	$(10, 14, 0)$	$(0, 14, 0)$
$\beta_2, \beta_3, \beta_1$	$(14, 10, 0)$	$(14, 0, 0)$
$\beta_2, \beta_3, \beta_3$	$(14, 10, 10)$	$(14, 5, 5)$
$\beta_1, \beta_3, \beta_3$	$(0, 10, 10)$	$(0, 5, 5)$
$\beta_1, \beta_3, \beta_1$	$(10, 10, 10)$	$(10, 0, 10)$
$\beta_1, \beta_1, \beta_3$	$(10, 10, 10)$	$(10, 10, 0)$
$\beta_1, \beta_3, \beta_2$	$(0, 10, 14)$	$(0, 0, 14)$
$\beta_2, \beta_1, \beta_3$	$(14, 0, 10)$	$(14, 0, 0)$
$\beta_3, \beta_1, \beta_2$	$(10, 0, 14)$	$(0, 0, 14)$
$\beta_2, \beta_2, \beta_3$	$(14, 14, 10)$	$(14, 14, 0)$
$\beta_2, \beta_3, \beta_2$	$(14, 10, 14)$	$(14, 0, 14)$

In the game (4.11), we had multiple equilibria. Assuming that all players acted rationally, the two pure Nash Equilibria are $(\beta_1, \beta_1, \beta_1)$ and $(\beta_2, \beta_2, \beta_2)$ profiles since no player had an incentive to deviate from either the first or second equilibria. Since the two pure NE are strict,

then the two pure strategies of the game (4.11) are evolutionary stable strategies. The mixed strategy that resulted from the two pure Nash Equilibria is also evolutionary stable.
In some cases, some students decided to behave irrationally by attempting the third assignment, β_3 which was less rewarding than the first two assignments. Since β_1 and β_2 were evolutionary stable strategies, any student with mutant behaviour who decided to adopt the third strategy, β_3, could not successfully invade this population of players.
More precisely, β_1 is an ESS if either:

(i) the payoff for playing β_1 against other players playing β_1 is greater than that of playing any other strategy β_3 against players playing β_1, for example,

$$U_i(\beta_1, \beta_1, \beta_1) > U_i(\beta_3, \beta_1, \beta_1),$$

(ii) the payoff of playing β_1 against itself is equal to that of playing β_3 against β_1 but the payoff of playing β_3 against β_3 is less than that of playing β_1 against β_3, for example

$$U_i(\beta_1, \beta_1, \beta_1) = U_i(\beta_3, \beta_1, \beta_1)$$

and

$$U_i(\beta_1, \beta_1, \beta_3) > U_i(\beta_3, \beta_3, \beta_3).$$

Alternatively, β_2 is an ESS if either:

(i) the payoff for playing β_2 against other players playing β_2 is greater than that of playing any other strategy β_3 against players playing β_2, for example,

$$U_i(\beta_2, \beta_2, \beta_2) > U_i(\beta_3, \beta_2, \beta_2),$$

(ii) the payoff of playing β_2 against itself is equal to that of playing β_3 against β_2 but the payoff of playing β_3 against β_3 is less than that of playing β_2 against β_3, for example

$$U_i(\beta_2, \beta_2, \beta_2) = U_i(\beta_3, \beta_2, \beta_2)$$

and

$$U_i(\beta_2, \beta_2, \beta_3) > U_i(\beta_3, \beta_3, \beta_3).$$

Note that for both evolutionary stable strategies, either (i) or (ii) will do and that the former is a stronger condition than the latter. It is most likely that players will always adopt the evolutionary stable strategies since no mutant strategy can successfully invade this game.

4.3 Identification of Efficient Nash Equilibria in the Game (3.1)

The game (3.1) modelled in this study is an example of coordination game with multiple Nash equilibria. Some equilibria may give higher payoffs, some may be naturally more salient, others may be safer and/or fairer. When there are several NE, how will a rational agent decide on which of the several equilibria is the right one to settle upon? Attempts to resolve this problem have produced a number of refinements to the concept of NE. This necessitated the need to identify which equilibria is efficient in the case of multiple equilibria.

Risk dominance and payoff dominance are two related refinements of NE solution concept in game theory. A NE is considered payoff dominant if it is Pareto superior to all other NE in the game. When faced with a choice among equilibria, all players would agree on the payoff dominant equilibrium since it offers each player at least as much payoff as the other NE. This implies that

$$u_i(\beta_1, \beta_1, \beta_1) > u_i(\beta_2, \beta_2, \beta_2).$$

In the game modelled in (3.1), $(\beta_1, \beta_1, \beta_1)$ is a payoff dominant equilibrium because each player prefers this profile to that in which she chooses β_2 alone. A player is better off remaining attentive in attempting the first assignment, β_1, than attempting the second assignment, β_2, if all other players remain attentive since this will give them a higher reward.

Conversely, a NE equilibrium is considered risk dominant if it has the largest basin of attraction. In the game (3.1), $(\beta_2, \beta_2, \beta_2)$ is a risk dominant equilibrium because each player prefers this profile to that in which she attempts the first assignment, (β_1), alone. A player is better off attempting the second assignment, (β_2), than the first assignment, (β_1), if no one else attempts the first assignment because this option is less risky. Pareto dominant and the risk dominant strategies are both ESS.

5 Conclusion

A major contribution that this study has made is that since most situations in economics such as cooperative projects and security dilemma are usually faced with multiple choices which challenge players in this field, and if Economics strives to be a predictive Science, then multiplicity of equilibria is a problem that needs to be dealt with. More often, in selecting from multiple equilibria, economists make use of efficiency considerations and that not only equilibria that are payoff dominant should be chosen, but also risk dominance should be considered as well.

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