Chi-square mixture of t-moment distribution

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Abstract

In this study, chi-square mixture of t-moment distribution has been defined and also determined its different moments and shape characteristics. Mean of this mixture distribution is zero. This distribution is found to be symmetrical about mean and bimodal.

Keywords: t-moment distribution, mixture distribution, Chi-square mixture.

AMS Subject Classification (2010): 46N30.

1   Introduction

According to Blischke [1] a mixture of distributions is a weighted average of probability distributions with positive weight that sum to one. The distributions that are mixed are called the components of the mixture. The weights themselves are probability distribution called mixing distribution and hence the mixture is also a probability distribution. Mixtures occur most commonly when the parameter (say θ) of a family of distributions, given by the density function $f(x;\theta)$ is itself subject to chance variation. The general formula for the finite mixture is $\sum_{k=1}^{p} f(x;\theta_k)g(\theta_k)$ and the infinite analogue is $\int f(x;\theta)g(\theta)d\theta$ where $g(\theta)$ is a density function.

Probability distribution of this type arises when observed phenomena can be the consequence of two or more related but usually each of which leads to a different probability distributions. Mixtures and related structures often arise in the construction of probabilistic models.

Pearson [9] was the first researcher in the field of mixtured distributions who consider the mixture of two normal distributions. After the work of Pearson there was a long gap in the field of mixture distributions. Robbins [10] studied some basic properties of mixtured distributions. After Robbins, two statisticians Mendenhall and Hader [8] studied on the estimation of parameters of mixed exponentially distributed failure time distributions from censored lifetime data. When the subsets of a big dataset follow different types of distributions separately then the whole dataset may follows a mixture distribution suitably.

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Alternatively we can say a random variable X is said to have a mixture distribution if the distribution of X depends on a quantity that also has a distribution. A typical example of mixture distribution is given below.

Let us consider the following hierarchy

$$X/K \sim \chi^2_{p+2k}$$

$$K \sim \text{Poison }(\lambda)$$

The marginal density of X is obtained as

$$P[X=x] = \sum_{k=0}^{\infty} P(X = x \mid K = k)$$

$$= \sum_{k=0}^{\infty} P(X = x, K = k) \cdot P[K=k]$$

Now since X/(K=k) is χ^2_{p+2k} and K is Poisson (λ), the marginal density of X can be obtained as

$$f(x \mid \lambda; p) = \sum_{k=0}^{\infty} \frac{\chi^{\frac{p}{2}+k-1} e^{-\frac{x}{2}}}{2^{\frac{p}{2}+k} \Gamma(\frac{p}{2}+k)} \cdot \frac{e^{-\lambda} \lambda^k}{k!} \quad ; \quad 0 < x < \infty \tag{1}$$

which is nothing but the non-central chi-square distribution with degrees of freedom p and non-centrality parameter λ. Thus non-central chi-square distribution is considered as a Poisson mixture of chi-square distributions.

A notable introductory idea about the theory and applications of mixture distributions can be found in Everitt and Hand [2], Titterington et *al*. [13], McLanchlan and Basford [6], Lindsay [5], McLachlan and Peel [7] and Fruhwirth-Schnatter [3]. Mixture distributions are widely used in the fields of environment, business, economics, agriculture, psychology, medicine etc. The main objective of the study is to derive the chi-square mixture of t-moment distributions and to have their different characteristics.

The outline of this paper is as follows. We present definition of some mixture distributions in section 2. In this section general form of t-moment distribution and chi-square mixtured distribution are also discussed. Chi-square mixture of t-moment distribution is defined section 3. In section 3.1 some characteristics of the defined distribution are determined. Graphical presentation of this mixture distribution is shown in section 3.2. In section 3.3 we discussed some important properties of this mixture distribution. Conclusion is presented in section 4.

2 Preliminaries

Definition 2.1. [11] A random variable X is said to have a Poisson mixture of distribution if its density function is given by

$$\sum_{\theta=0}^{\infty} \frac{e^{-\lambda} \lambda^{\theta}}{\theta!} f(x; \theta) \tag{2}$$

Definition 2.2. [12] A random variable X is said to have a negative binomial mixture of normal moment distribution if its density function is given by

$$\sum_{k=0}^{n} \binom{k+r-1}{r-1} p^r q^k \frac{e^{-\frac{x^2}{2}} x^{2r}}{2^{r+1/2} \Gamma(r+1/2)} ; \quad -\infty < x < \infty \tag{3}$$

Definition 2.3. [14] A random variable X is defined to be a chi-square mixture of binomial distribution with v degrees of freedom and parameters n and p, if its density function is defined by

$$f(x;v,n,p) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma \frac{v}{2}} \binom{n+\chi^2}{x} p^x (1-p)^{n+\chi^2-x} d\chi^2 ; \ x = 0, 1, 2, \ldots, \text{n}+\chi^2 \text{ and } 0 < p < 1. \tag{4}$$

Definition 2.4. [15] A random variable X is defined to be a chi-square mixture of Erlang distribution with v degrees of freedom and parameters α and β, if its density function is defined by

$$f(x;v,\alpha,\beta) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma \frac{v}{2}} \frac{(\alpha\beta)^{\alpha+\chi^2}}{\Gamma(\alpha+\chi^2)} e^{-\alpha\beta x} x^{\alpha+\chi^2-1} d\chi^2 ; \ 0 < x < \infty \ \text{and} \ \alpha\beta > 1. \tag{5}$$

Definition 2.5. [4] A random variable χ^2 is said to have a Rayleigh mixture of chi-square distribution with parameter σ^2 with degrees of freedom n if its probability density function is defined by

$$f(\chi^2,\sigma^2,n) = \int_0^\infty \frac{r e^{-r^2/2\sigma^2}}{\sigma^2} \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\left(\frac{n}{2}\right)+r-1}}{2^{\left(\frac{n}{2}\right)+r} \Gamma\left(\frac{n}{2}+r\right)} dr ; \quad 0 < \chi^2 < \infty. \tag{6}$$

Definition 2.6. The general form of t-moment distribution can be defined as

$$f(t;n,k) = \frac{1}{\beta\left(\frac{1}{2}+k,\frac{n}{2}\right)} \frac{t^{2k}}{n^{\frac{1}{2}+k}\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}+k}} ; \quad -\infty < t < \infty. \tag{7}$$

where n and k are integers. The name "t-moment distribution" is due to the fact that the form of the distribution is a density proportional to

$$\frac{t^{2k}}{(1+\frac{t^2}{n})^{\frac{n+1}{2}+k}} \tag{8}$$

The main characteristics of the distribution are mean=0, Variance = $\dfrac{n(2k+1)}{(n-2)}$ (for $n>2$), Skewness = 0

and Kurtosis = $\dfrac{(2k+3)(n-2)}{(2k+1)(n-4)}$ (for $n > 4$). When $k=0$, then the distribution in equation (7) becomes a t-distribution with n degrees of freedom. It is observed that t-moment distribution is symmetrical and bimodal. As the value of k increases, the modes of the distribution shift to both sides of the mean.

Definition 2.7. A random variable X having the density function

$$f(x;v,\theta) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}} \Gamma \frac{v}{2}} g(x;\theta) d\chi^2 \quad ; \quad -\infty < x < \infty \tag{9}$$

is said to have a chi-square mixtured distribution with n degrees of freedom where g($x;\theta$) is a density function.

The name chi-square mixture of distribution is due to the fact that integral values $f(x;v,\theta)$ in the derived distribution in Equation (9) is equal to one with weights equal to the ordinates of chi-square distribution having v degrees of freedom.

3   Main Results

Definition 3.1. A random variable t is defined to be a chi-square mixture of t-moment distribution with v and n degrees of freedom if its density function is defined by

$$f(t;v,n) = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)} \frac{t^2\chi^2}{n^{\frac{1}{2}+\chi^2}\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}+\chi^2}} d\chi^2; \quad -\infty < t < \infty \tag{10}$$

3.1 Some Characteristics

A theorem of chi-square mixture of t-moment distribution is derived below. Hence also we derive mean, variance, skewness and kurtosis of this distribution.

Theorem 3.1. If t follows a chi-square mixture of t-moment distribution with v and n degrees of freedom then

$$\mu_{2r+1}=\mu'_{2r+1}=0$$

$$\mu_{2r} = \mu'_{2r} = n^r \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{\beta\left(\chi^2+r+\frac{1}{2},\frac{n}{2}-r\right)}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)} d\chi^2 \tag{11}$$

Hence, Mean=0, Variance=$\dfrac{n(1+2v)}{n-2}$; $n>2$, $\beta_1=0$ and $\beta_2=\dfrac{(n-2)}{(n-4)}\left\{1+\dfrac{12v+2}{(2v+1)^2}\right\}$; $n>4$.

Proof: The (2r+1) th raw moment (odd order moment) about origin is given by

$$\mu'_{2r+1}=E[t^{2r+1}] \; = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)n^{\frac{1}{2}+\chi^2}} \int_{-\infty}^\infty \frac{t^{2\chi^2+2r+1}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}+\chi^2}} dt \, d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)n^{\frac{1}{2}+\chi^2}} \int_{-\infty}^\infty \varphi(t) \, dt \, d\chi^2$$

$$= 0 \text{ [Since } \varphi(t)=\frac{t^{2\chi^2+2r+1}}{(1+\frac{t^2}{n})^{\frac{n+1}{2}+\chi^2}} \text{ is an odd function of t].}$$

Therefore, $\mu_{2r+1}=\mu'_{2r+1}=0$. Hence all odd order moments are zero. Hence, the mean of this distribution is 0.

Now the $2r^{\text{th}}$ raw moment about   origin is given by

$$\mu'_{2r} = \mu_{2r} = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)n^{\frac{1}{2}+\chi^2}} \int_{-\infty}^\infty \frac{t^{2(\chi^2+r)}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}+\chi^2}} dt \, d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)n^{\frac{1}{2}+\chi^2}} \int_{-\infty}^\infty \varphi(t) \, dt \, d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)n^{\frac{1}{2}+\chi^2}} \cdot 2\int_0^\infty \varphi(t)\, dt\, d\chi^2$$

Since $\varphi(t) = \dfrac{t^{2(\chi^2+r)}}{\left(1+\dfrac{t^2}{n}\right)^{\frac{n+1}{2}+\chi^2}}$ is an even function over t,

$$\mu'_{2r} = \mu_{2r} = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)n^{\frac{1}{2}+\chi^2}} \int_0^\infty \frac{(t^2)^{(\chi^2+r-\frac{1}{2})}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}+\chi^2}} dt^2\, d\chi^2$$

$$= n^r \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{\beta\left(\chi^2+r+\frac{1}{2},\frac{n}{2}-r\right)}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)} d\chi^2 \qquad (12)$$

If $r = 1$, then Equation (12) becomes,

$$\mu_2 = \mu'_2 = n\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{\beta\left(\chi^2+\frac{3}{2},\frac{n}{2}-1\right)}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)} d\chi^2$$

$$= n\int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{\left(\chi^2+\frac{1}{2}\right)}{\left(\frac{n}{2}-1\right)} d\chi^2$$

$$= \frac{n}{(n-2)}\{2v+1\}; \quad n > 2$$

Thus the variance is, $\mu_2 = \dfrac{n}{(n-2)}\{2v+1\}$; $n > 2$.

If $r = 2$, then Equation (12) becomes,

$$\mu_4 = \mu'_4 = n^2 \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{\beta\left(\chi^2+\frac{5}{2},\frac{n}{2}-2\right)}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)} d\chi^2$$

$$= n^2 \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \frac{\left(\chi^2+\frac{3}{2}\right)\left(\chi^2+\frac{1}{2}\right)}{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)} d\chi^2$$

$$= \frac{n^2}{(n-2)(n-4)} \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} (2\chi^2+3)(2\chi^2+1)\, d\chi^2$$

$$= \frac{n^2}{(n-2)(n-4)}(4v^2+16v+3)$$

Now skewness (β_1) and kurtosis (β_2) of this distribution are,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0. \quad \text{(Since all odd order moments are zero) and}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\dfrac{n^2}{(n-2)(n-4)}(4v^2+16v+3)}{\left\{\dfrac{n^2}{(n-2)^2}(1+2v)^2\right\}}$$

$$= \frac{(n-2)}{(n-4)}\left\{1+\frac{12v+2}{(2v+1)^2}\right\}; \quad n>4.$$

Hence the theorem is proved.   ∎

The co-efficient of skewness is $\gamma_1 = \sqrt{\beta_1} = 0$.

The co-efficient of kurtosis is $\gamma_2 = \beta_2 - 3 = \dfrac{(n-2)}{(n-4)}\left\{1 + \dfrac{12v+2}{(2v+1)^2}\right\}$.

3.2 Area Curves

In order to examine the shape characteristics of this distribution, different area curves were constructed for different values of parameter and degrees of freedom. These curves are presented in Figure 1. It is observed that this mixture distribution is symmetrical about zero and the distribution is bimodal.
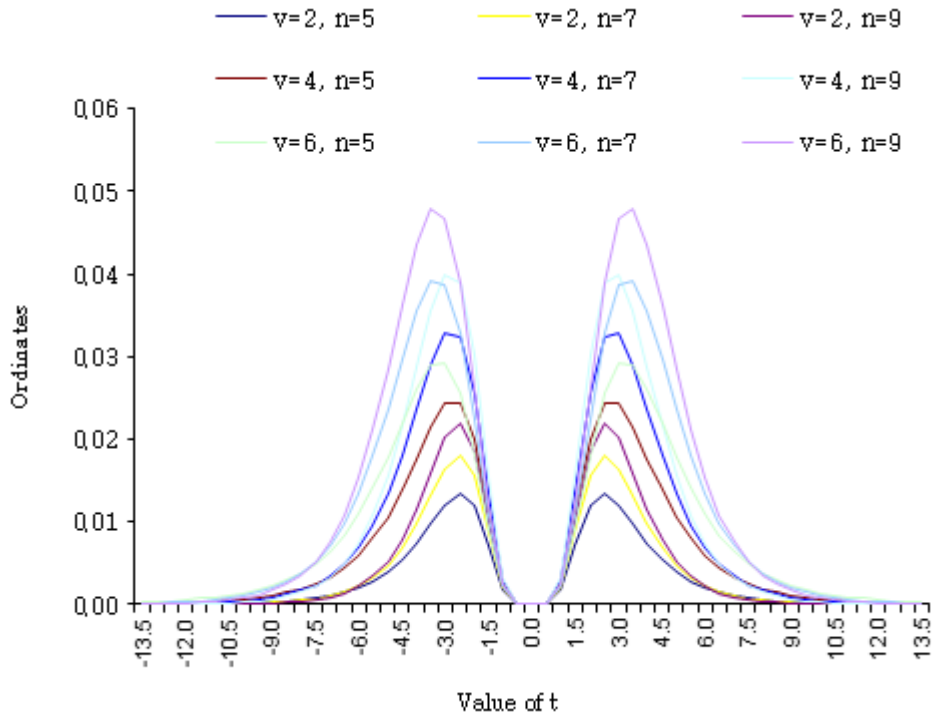

Figure 1: The Area curve of the chi-square mixture of t-moment distribution.

3.2 Important Properties

1. For this distribution total probability is unity.

That is, $\displaystyle\int_0^\infty f(t; v, n)\,dt = \int_{-\infty}^\infty \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)}\frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)}\frac{t^2\chi^2}{n^{\frac{1}{2}+\chi^2}\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}+\chi^2}}\,d\chi^2\,dt$

$\displaystyle = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)}\frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)}\int_{-\infty}^\infty \frac{t^2\chi^2}{n^{\frac{1}{2}+\chi^2}\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}+\chi^2}}\,dt\,d\chi^2$

$\displaystyle = \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)}\frac{1}{\beta\left(\frac{1}{2}+\chi^2,\frac{n}{2}\right)}2\int_0^\infty \frac{t^2\chi^2}{n^{\frac{1}{2}+\chi^2}\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}+\chi^2}}\,dt\,d\chi^2$

(since $\dfrac{t^{2}\chi^{2}}{n^{\frac{1}{2}+\chi^{2}}(1+\dfrac{t^{2}}{n})^{\frac{n+1}{2}+\chi^{2}}}$ is even function of t)

$$= \int_{0}^{\infty}\frac{e^{-\frac{\chi^{2}}{2}}(\chi^{2})^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{2}{\beta((\frac{1}{2}+\chi^{2},\frac{n}{2})n^{\frac{1}{2}+\chi^{2}}}\int_{0}^{\infty}\frac{t^{2}\chi^{2}}{\left(1+\frac{t^{2}}{n}\right)^{\frac{n+1}{2}+\chi^{2}}}\,dt\,d\chi^{2}$$

$$= \int_{0}^{\infty}\frac{e^{-\frac{\chi^{2}}{2}}(\chi^{2})^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{2}{\beta(\frac{1}{2}+\chi^{2},\frac{n}{2})n^{\frac{1}{2}+\chi^{2}}}\int_{0}^{\infty}\frac{(ny)\chi^{2}}{(1+y)^{\frac{n+1}{2}+\chi^{2}}}\frac{ndy}{2\sqrt{ny}}\,d\chi^{2}\quad [t^{2}=ny,\text{ say}]$$

$$= \int_{0}^{\infty}\frac{e^{-\frac{\chi^{2}}{2}}(\chi^{2})^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\frac{1}{\beta(\frac{1}{2}+\chi^{2},\frac{n}{2})n^{\frac{1}{2}+\chi^{2}}}\beta\left(\frac{1}{2}+\chi^{2},\frac{n}{2}\right)n^{\frac{1}{2}+\chi^{2}}\,d\chi^{2}$$

$$= \int_{0}^{\infty}\frac{e^{-\frac{\chi^{2}}{2}}(\chi^{2})^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\,d\chi^{2}$$

$$=1.$$

2. The mean and variance of the distribution be equal to zero and $\dfrac{n(1+2v)}{(n-2)}$ respectively.

3. If $v=0$, then the distribution reduces to the student t-distribution with n degrees of freedom.

4. The co-efficient of skewness is zero. That is, the distribution is always symmetrical about mean for any value of v and n.

5. The co-efficient of kurtosis is $\dfrac{(n-2)}{(n-4)}\left\{1+\dfrac{12v+2}{(2v+1)^{2}}\right\}-3.$ So kurtosis of this mixture distribution depends on the value of the parameters n and v. The shape of distribution becomes normal with the increase of v and platykurtic with the increase of n.

4 Conclusion

We have presented chi-square mixture of t-moment distribution using the chi-square distribution as a weight function. The different moments and shape characteristics of this distribution are also studied. Mean of this mixture distribution is zero. This distribution is found to be symmetrical about mean and bimodal. The further research can be done to check the suitability of the application of this distribution in the situations where the residuals of a regression model do not follow normal distribution.

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