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Vertex covering transversal domination in graphs

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Abstract

A set $D \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. A set $C \subseteq V$ of vertices in G is called a vertex covering set if every edge of G is incident to at least one vertex in C. Also C is said to be a minimum vertex covering set if there is no other vertex covering set C' such that $|C'| < |C|$. A dominating set which intersects every minimum vertex covering set in G is called a vertex covering transversal dominating set. The minimum cardinality of a vertex covering transversal dominating set is called vertex covering transversal domination number of G and is denoted by $\gamma_{vct}(G)$. In this paper, we begin with an investigation of this parameter.

Keywords: Dominating set, vertex covering set, vertex covering transversal dominating set.

AMS Subject Classification(2010): 05C69.

1 Introduction

Graph theory is one of the most interesting and application-oriented branch of mathematics. The most emerging topic in graph theory during the recent years is domination in graphs. Independent transversal domination in graphs was introduced by I. Sahul Hamid [5]. He defined it using maximum independent set in a Graph. In this paper, we introduce vertex covering transversal domination in graphs using minimum vertex covering set in a Graph. We provide the vertex covering transversal dominating sets in complete graphs and those sets of minimum cardinality which gives the vertex covering transversal domination number of them. We estabilish the vertex covering transversal domination number of some standard graphs. Further, we provide bounds for the vertex covering transversal domination number through the domination number in graphs and other parameters.

Consider a simple graph $G = (V, E)$. A set $D \subseteq V$ of vertices in the graph G is called a *dominating set* if every vertex in V - D is adjacent to a vertex in D. A set $C \subseteq V$ of vertices in G is called a *vertex covering set* (or simply covering set) if every edge of G is incident to at least one vertex in C. Also C is said to be a *minimum vertex covering set* if there is no other vertex covering set C' such that $|C'| <|C|$. The cardinality of a minimum vertex covering set is called the *vertex covering number* and is denoted by $\alpha_0(G)$.

It is clear that in any simple connected graph, every vertex covering set is a dominating set, but every dominating set need not be a vertex covering set. A dominating set which intersects every minimum vertex covering set in G is called a vertex covering transversal dominating set. The minimum cardinality of a vertex covering transversal dominating set is called vertex covering transversal domination number of G and is denoted by $\gamma_{vct}(G)$.

As an illustration, consider the graph G shown in Figure 1. $D_1 = \{a, b, f\}$, $D_2 = \{a, e, f\}$, $D_3 = \{b, e, \}$, $D_4 = \{b, f\}$, $D_5 = \{b, g\}$ and $D_6 = \{a, e\}$ are some dominating sets. $C_1 = \{b, e, g\}$, $C_2 = \{b, e, f\}$, $C_3 = \{a, b, e, f\}$, $C_4 = \{a, b, e, g\}$, $C_5 = \{a, c, d, f, g\}$, $C_6 = \{a, c, d, e, f\}$ and $C_7 = \{b, c, d, f, g\}$ are some vertex covering sets of G.

Of these, $C_1 = \{b, e, g\}$ and $C_2 = \{b, e, f\}$ are the minimum vertex covering sets in G. It is clear that D_1 intersects C_1 and C_2. Similarly, D_2, D_3, D_4, D_5 and D_6 also intersect C_1 and C_2. Hence D_1, D_2, D_3, D_4, D_5 and D_6 are vertex covering transversal dominating sets. Of these, D_3, D_4, D_5 and D_6 are of minimum cardinality two and so $\gamma_{vct}(G) = 2$. Also note that $\gamma(G) = 2$.

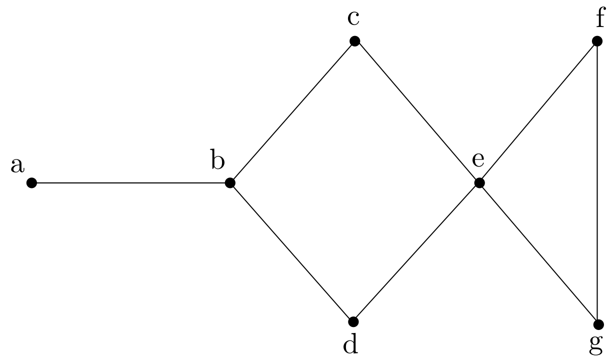

Figure 1

2 Preliminaries

In a simple graph G = (V, E), the open neighbourhood of a vertex $u \in V$ is $N(u) = \{v \in V : uv \in E\}$, that is, the set of all vertices adjacent to u. The closed neighbourhood of u is $N[u] = \{v \in V : uv \in E\} \cup \{u\}$. An independent set in G is the set of all vertices such that no two of them are adjacent in G. The maximum cardinality of an independent set is called the independence number and is denoted by $\beta_0(G)$. We use the following notations throughout the paper.

α_0-set to denote minimum vertex covering set, β_0-set to denote maximum independent set, γ-set to denote a dominating set of minimum cardinality and γ_{vct}-set to denote vertex covering transversal dominating set of minimum cardinality.

3 Vertex Covering Transversal Domination Number

In this section, we provide the vertex covering transversal domination number of some standard graphs such as complete graphs, complete bipartite graphs, paths, cycles and wheels.

Definition 3.1. A dominating set which intersects every minimum vertex covering set in G is called a vertex covering transversal dominating set.

Definition 3.2. The minimum cardinality of a vertex covering transversal dominating set is called vertex covering transversal domination number of G and is denoted by $\gamma_{vct}(G)$.

Example 3.3. If K_n is a complete graph with $n \geq 2$ vertices, then
(i) $\gamma_{vct}(K_n) = 2$ since any two-element subset of the vertex set of K_n is a γ_{vct}-set.
(ii)The total number of vertex covering transversal dominating sets in K_n is $2^n - n - 1$.
(iii) The number of γ_{vct}-sets in K_n is nC_2.

Example 3.4. If $K_{m,n}$ is a complete bipartite graph, then

$$\gamma_{vct}(K_{m,n}) = \begin{cases} 1 & \text{if } m = 1 \text{ and } n>1 \\ 2 & \text{otherwise} \end{cases}$$

Remark 3.5. It is noted that, for most of the graphs the domination number and the vertex covering transversal domination number are equal. Yet, this happens only if the γ-set itself intersects every minimum vertex covering set and hence becomes a γ_{vct}-set. Otherwise, the γ_{vct}-set contains more number of vertices than the γ-set and in this case the domination number is less than the vertex covering transversal domination number.

Example 3.6. If G is a star, then $\gamma_{vct}(G) = 1$ and if it is a bistar, then $\gamma_{vct}(G) = 2$.

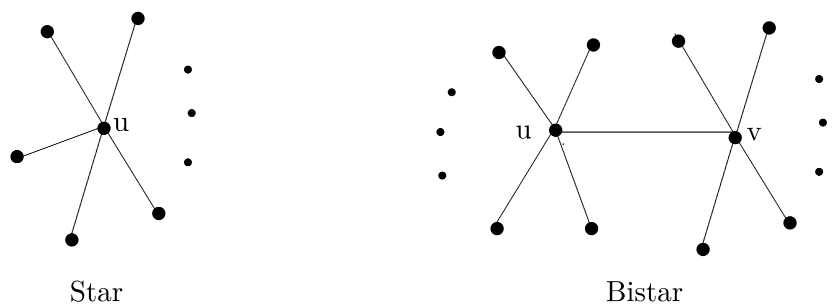

Figure 2

Example 3.7. If W_n is a wheel on n vertices with $n \geq 5$, then $\gamma_{vct}(W_n) = 1$ since $\{u\}$ is the only γ_{vct}-set as u is included in every minimum vertex covering set of W_n.

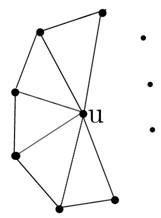

Figure 3

Theorem 3.8. If P_n is a path of order n, then $\gamma_{vct}(P_n) = \begin{cases} 2 & \text{if } n = 2, 4 \\ \lceil \frac{n}{3} \rceil & \text{otherwise} \end{cases}$

Proof: Let $P_n = \{v_1, v_2, ..., v_n\}$ be a path of order n. Clearly $\gamma_{vct}(P_2) = 2$.
Suppose $n = 4$. Then $S_1 = \{v_1, v_3\}$, $S_2 = \{v_2, v_3\}$ and $S_3 = \{v_2, v_4\}$ are the minimum vertex covering sets of P_4. Now $D = \{v_2, v_3\}$ is a dominating set of P_4 which intersects S_1, S_2 and S_3. Thus D is a vertex covering transversal dominating set of minimum cardinality. Therefore, $\gamma_{vct}(P_4) = 2$.
Now assume that $n \neq 2, 4$.
Case 1: $n \equiv 0(mod 3)$.
Then $D = \{v_{3i-1}$: $1 \leq$ i $\leq \frac{n}{3}$ $\}$ is the γ-set of P_n.
Subcase(i): n is odd.
Then S = $\{v_2, v_4, v_6,..., v_{n-1}\}$ is the only α_0-set. Clearly, D intersects S.
Subcase(ii): n is even.
Then $S_1 = \{v_1, v_3, v_5, ..., v_{n-1}\}$, $S_2 = \{v_2, v_3, v_5, ..., v_{n-1}\}$, $S_3 = \{v_2, v_4, v_6, ..., v_n\}$ and S_4 $= \{v_2, v_4, v_6, ..., v_{n-1}\}$ are the only α_0-sets. Clearly, D intersects S_1, S_2, S_3 and S_4. and D is the vertex covering transversal dominating set of minimum cardinality. Therefore, $\gamma_{vct}(P_n)$= $\gamma(P_n)$= $\lceil \frac{n}{3} \rceil$.
Case 2: $n \equiv 1(mod 3)$.
Then $D = \{v_{3i+1} : 0 \leq i \leq \frac{n-1}{3}\}$ is the γ-set of P_n. The minimum vertex covering sets of P_n are those discussed in the subcases(i) and (ii) of Case 1. Thus in this case also, D intersects the sets mentioned in the subcases of Case 1. Hence $\gamma_{vct}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.
Case 3: $n \equiv 2(mod 3)$.
Then $D = \{v_{3i+1} : 0 \leq i \leq \frac{n-2}{3}\}$ is the γ-set of P_n.
D is the vertex covering transversal dominating set of minimum cardinality. Thus in this case also, $\gamma_{vct}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$. ■

Theorem 3.9. If C_n is a cycle of order n, then

$$\gamma_{vct}(C_n) = \begin{cases} 2 & \text{if } n = 3, 4 \\ 3 & \text{if } n = 5 \\ \lceil \frac{n}{3} \rceil & \text{otherwise} \end{cases}$$

Theorem 3.10. Let a and b be two positive integers with $b \geq 2a + 3$. Then there exists a connected graph G on b vertices such that $\gamma_{vct}(G) = a$.

Proof: Let $b = 2a + r, r \geq 3$ and let H be any connected graph on a vertices. Choose a vertex of maximum degree in H and let it be v_1. So let $V(H) = \{v_1, v_2, ..., v_a\}$. Let G be the graph obtained from H by including $r + 1$ pendant edges at v_1 and one pendant edge at each v_i for $i \geq 2$ so that $|V(G)| = 2a + r = b$. Let u_i, $i \geq 2$ be the pendant vertex in G adjacent to v_i. Then $S = \{v_1,\ u_2,\ u_3,\ ...,\ u_a\}$ is a γ-set in G. Also every minimum vertex covering set of G must include v_1 as its degree is increased by $r + 1$ in G. For, otherwise any vertex covering set excluding v_1 will contain $r + 1$ more vertices and so it will not be minimum. Therefore, S is a vertex covering transversal dominating set of minimum cardinality. Hence $\gamma_{vct}(G) = a$. ∎

Theorem 3.11. Let T be a tree on n ≥ 5 vertices which is neither a path nor a star. Then $\gamma_{vct}(T) \leq \lfloor \frac{n}{2} \rfloor$.

Proof: Since T is not a path, T must have at least 3 pendant vertices. Let u be a vertex of maximum degree in T and so deg $u \geq 3$.
Then u is included in every minimum vertex covering set S of T. For, otherwise any vertex covering set excluding u must include two or more vertices than those in S and hence is not minimum.
Since T is not a star, $0 < |V(T) - N[u]| \leq n - 4$. Then these vertices are dominated by atmost $\lceil \frac{n-4}{2} \rceil$ vertices. Let D be such a dominating set and also of minimum cardinality. Then $|D| \leq \lceil \frac{n-4}{2} \rceil$. Now $D' = D \cup \{u\}$ is a γ-set of T which intersects every minimum vertex covering set of T. So D' is a γ_{vct} -set of T. Thus $\gamma_{vct}(T) = |D'| \leq \lfloor \frac{n}{2} \rfloor$. ∎

4 Bounds of γ_{vct}

In this section, we estabilish bounds for the vertex covering transversal domination number through the domination number in graphs and also through several other parameters.

Theorem 4.1. For any graph G, $\gamma(G) \leq \gamma_{vct}(G) \leq \gamma(G) + \Delta(G)$.

Proof: Let D be a γ_{vct}-set of G. Then D itself is a dominating set. Therefore, $\gamma(G) \leq |D| = \gamma_{vct}(G)$. Let S be a γ-set in G and u be a vertex of maximum degree $\Delta(G)$ in G. Then every minimum vertex covering set of G contains a vertex of $N[u]$. So $S \cup N[u]$ is a vertex covering transversal dominating set. Since S intersects $N[u]$, $|S \cup N[u]| \leq \gamma(G) + \Delta(G)$. Thus $\gamma_{vct}(G) \leq \gamma(G) + \Delta(G)$. ∎

Remark 4.2. The upper bound for γ_{vct} proved in Theorem 4.1 is so large, but it is attained for the graph P_2.

Remark 4.3. It is obvious that for any graph on n vertices, $1 \leq \gamma_{vct}(G) \leq n$.

Theorem 4.4. If G is a graph with $diam(G) = 2$, then $\gamma_{vct}(G) \leq \delta(G) + 1$.

Proof: Let u be a vertex of G with deg $u = \delta(G)$. Then $N[u]$ is a dominating set of G since $diam(G) = 2$. It is obvious that every minimum vertex covering set of G contains a vertex of $N[u]$ which implies that $N[u]$ is a vertex covering transversal dominating set of G. Hence $\gamma_{vct}(G) \leq \delta(G) + 1$. ■

Theorem 4.5. If G is a graph without any isolates, then $\gamma_{vct}(G) \leq \beta_0(G) + 1$.

Proof: Let S be a maximum independent set of G. Then S is a dominating set and $V - S$ is a minimum vertex covering set. Set $D = S \cup \{u\}$ where $u \in V - S$. Then D is a vertex covering transversal dominating set of G. Hence $\gamma_{vct}(G) \leq \beta_0(G) + 1$. ■

Remark 4.6. Eventhough the bound for γ_{vct} proved in Theorem 4.3 is very large, it is attained for complete graphs.

Corollary 4.7. If G is a graph on n vertices without any isolated vertices, then $\gamma(G)+\gamma_{vct}(G) \leq n + 1$.

Proof: Since $\gamma(G) \leq \alpha_0(G)$ and $\alpha_0(G) + \beta_0(G) = n$, it follows from Theorem 4.2 that $\gamma(\mathrm{G}) + \gamma_{vct}(\mathrm{G}) \leq \mathrm{n} + 1$. ■

Theorem 4.8. If G is a connected non-complete graph with clique number ω, then $\gamma_{vct}(G) \leq n - \omega + 1$.

Proof: Let H be a maximum clique in G. Let $u \in V(H)$. Then $S = V(G) - V(H - u)$ is a dominating set of G. Since H is a maximum clique, we get every minimum vertex covering set of G intersects S. Hence S is a vertex covering transversal dominating set so that $\gamma_{vct}(G) \leq n - \omega + 1$. ■

Theorem 4.9. Let G be a connected, non-complete graph with $\delta(G) \geq 2$. Then $\gamma_{vct}(\mathrm{G}) \leq \beta_0(\mathrm{G})$.

Proof: Let S be an α_0-set of G. Then $V - S$ is a β_0-set and is also a dominating set of G. Since $\delta(G) \geq 2$, there exists a vertex u in $V - S$ such that $|N(u) \cap S| \geq 2$. Let v and w be two neighbours of u in S. Since $\delta(G) \geq 2$, it follows that every neighbour of u in S is adjacent to at least one vertex other than u in V - S and hence $D = (V - S) - \{u\}$ is a dominating set of $G - \{u\}$. Then $D \cup \{w\}$ is a vertex covering transversal dominating set of G. (For, $S - \{w\}$ is the only set in the complement of $D \cup \{w\}$ which is not a vertex covering set). Hence $\gamma_{vct}(G) \leq n - \alpha_0(G) = \beta_0(G)$. ■

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