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Some square graceful graphs

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Abstract

A (p,q) graph $G(V,E)$ is said to be a square graceful graph if there exists an injection $f:V(G)\to\{0,1,2,3,\dots, q^2\}$ such that the induced mapping $f_p:E(G)\to\{1,4,9,\dots, q^2\}$ by $f_p(uv)=|f(u)-f(v)|$ is a bijection. The function f is called a square graceful labeling of G. In this paper, we prove the graph obtained by the subdivision of the edges of stars of bistar $B_{m,n}$, the graph obtained by the subdivision of the edges of bistar $B_{m,n}$, the graph obtained by the subdivision of the edges of the path P_n in a comb $P_n\Theta K_1$, $<C_3 * K_{1,n}>$, $<S_n : m>$ and $<C_3, K_{1,n}>$ are square graceful graph.

Keywords: Square graceful graph, square graceful labeling.
AMS Subject Classification (2010): 05C69.

1 Introduction

All graphs in this paper are finite, simple and undirected graphs. Let (p,q) be a graph with $p=|V(G)|$ vertices and $q=|E(G)|$ edges. A detailed survey of graph labeling can be found in [1]. Terms not defined here are used in the sense of Harary in [2]. There are different types of graceful labelings in the graph labeling. The concept of square graceful labeling was first introduced in [5] and some results on square graceful labeling of graphs are discussed in [5]. In this paper, we investigate some more graphs for square graceful labeling. We use the following definitions in the subsequent sections.

Definition 1.1. [5] A (p,q) graph $G(V,E)$ is said to be a square graceful graph if there exists an injection $f:V(G)\to\{0,1,2,\dots, q^2\}$ such that the induced mapping $f_p:E(G)\to\{1,4,9,\dots, q^2\}$ defined by $f_p(uv)=|f(u)-f(v)|$ is a bijection. The function f is called a square graceful labeling of G.

Definition 1.2. [6] The corona $G_1\Theta G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1(which has p points) and p copies of G_2 and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2.

Definition 1.3. [1] A complete biparitite graph $K_{1,n}$ is called a star and it has $n+1$ vertices and n edges.

Definition 1.4. [1] The bistar graph $B_{m,n}$ is the graph obtained from a copy of star $K_{1,m}$ and a copy of star $K_{1,n}$ by joining the vertices of maximum degree by an edge.

Definition 1.5. [3] A subdivision of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions.

Definition 1.6.[1] The graph $< S_n : m >$ is the graph obtained by taking m disjoint copies of star S_n and joining a new vertex to the centres of the m copies of star S_n.

Definition 1.7. [4] The graph $< C_m * K_{1,n} >$ is the graph obtained from C_m and $K_{1,n}$ by identifying any one of the vertices of C_m with a pendent vertex of $K_{1,n}$ (that is a non-central vertex of $K_{1,n}$).

Definition 1.8. [4] The graph $< C_m , K_{1,n} >$ is the graph obtained from C_m and $K_{1,n}$ by identifying any one of the vertices of C_m with the central vertex of $K_{1,n}$.

2 Main Results

Theorem 2.1. The graph obtained by the subdivision of the edges of stars of the bistar $B_{m,n}$ is a square graceful graph.

Proof: Let $B_{m,n}$ be a bistar with $m+n+2$ vertices and $m+n+1$ edges . the vertex and edge sets are given by $V(B_{m,n}) = \{ u_i , v_j : 1 \le i \le m+1; 1 \le j \le n+1 \}$ and

$E(B_{m,n}) = \{u_i u_{m+1},\ v_j v_{n+1},\ u_{m+1} v_{n+1} : 1 \le i \le m; 1 \le j \le n\}$.

Let G be the graph obtained by the subdivision of the edges of stars of $B_{m,n}$. Let w_i divide $u_i u_{m+1}$ for $1 \le i \le m$ and z_j divide $v_j v_{n+1}$ for $1 \le j \le n$. Then the vertex and the edge set of G are given by

$$V(G) = \{ u_i , v_j : 1 \le i \le m+1, 1 \le j \le n+1 \} \cup \{ w_i , z_j : 1 \le i \le m, 1 \le j \le n \} \text{ and}$$

$$E(G) = \{ w_i u_{m+1},\ u_i w_i : 1 \le i \le m\} \cup \{ z_j v_{n+1},\ v_j z_j : 1 \le i \le m\} \cup \{u_{m+1} v_{n+1}\}$$

Case (i): $m < n$.

Define an injection $f : V(G) \to \{0,1,2,3,.......,(2m+2n+1)^2\}$ by

$f(u_{m+1}) = 1$; $f(v_{n+1}) = 0$;

For $1 \le i \le m$, $f(w_i) = (2m+n+2-i)^2 + 1$; $f(u_i) = (3m+2n+4-2i)(m)+1$.

For $1 \le j \le n$, $f(z_j) = (2m+2n+2-j)^2$; $f(v_j) = (2m+3n+4-2j)(2m+n)$.

Then, f induces a bijection $f_p : E(G) \to \{1,4,9,.......,(2m+2n+1)^2\}$.

In this case the edge labels of G are as follows:

$f_p(u_{m+1} v_{n+1}) = 1$;

$f_p(w_i u_{m+1}) = (2m+n+2-i)^2$ and $f_p(u_i w_i) = (m+n+2-i)^2$ for $1 \le i \le m$

$f_p(z_j v_{n+1}) = (2m+2n+2-j)^2$ and $f_p(v_j z_j) = (n+2-j)^2$ for $1 \le j \le n$.

Case(ii): $m = n$.

Define an injection $f : V(G) \to \{0,1,2,3,.......,(4n+1)^2\}$ by

$f(u_{n+1}) = 0$; $f(v_{n+1}) = 1$;

For $1 \le i \le n$, $f(w_i) = (4n+2-i)^2$; $f(u_i) = (5n+4-2i)(3n)$.

For $1 \le j \le n$, $f(z_j) = (3n+2-j)^2 + 1$; $f(v_j) = (5n+4-2j)(n)+1$.

Then, f induces a bijection $f_p : E(G) \to \{1,4,9,.......,(4n+1)^2$.

In this case the edge labels of G are as follows:

$f_p(u_{n+1}v_{n+1}) = 1$;

$f_p(w_iu_{n+1}) = (4n+2-i)^2$ and $f_p(u_iw_i) = (n+2-i)^2$ for $1 \le i \le n$.

$f_p(z_jv_{n+1}) = (3n+2-j)^2$ and $f_p(v_jz_j) = (2n+2-j)^2$ for $1 \le j \le n$.

Case(iii): $m > n$.

Define an injection $f : V(G) \to \{0,1,2,3,.......,(2m+2n+1)^2\}$ by

$f(u_{m+1}) = 0$; $f(v_{n+1}) = 1$;

For $1 \le i \le m$, $f(w_i) = (2m+2n+2-i)^2$; $f(u_i) = (3m+2n+4-2i)(m+2n)$.

For $1 \le j \le n$, $f(z_j) = (m+2n+2-j)^2 + 1$; $f(v_j) = (2m+3n+4-2j)(n)+1$.

Then, f induces a bijection $f_p : E(G) \to \{1,4,9,.......,(2m+2n+1)^2\}$.

In this case the edge labels are as follows:

$f_p(u_{m+1}v_{n+1}) = 1$;

$f_p(w_iu_{m+1}) = (2m+2n+2-i)^2$ and $f_p(u_iw_i) = (m+2-i)^2$ for $1 \le i \le m$.

$f_p(z_jv_{n+1}) = (m+2n+2-j)^2$ and $f_p(v_jz_j) = (m+n+2-j)^2$ for $1 \le j \le n$.

Hence, the graph obtained by the subdivision of the edges of stars of the bistar $B_{m,n}$ is a square graceful graph. ■

Example 2.2. A square graceful labeling of the graph obtained by the subdivision of the edges of stars of bistar $B_{7,5}$ is shown in Figure 1.

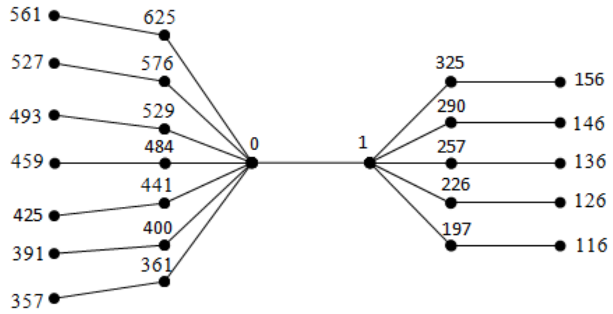

Figure 1: Square graceful labeling of the graph obtained by the subdivision of the edges of the stars of $B_{7,5}$.

Theorem 2.3. The graph obtained by the subdivision of the edges of the bistar $B_{m,n}$ is a square graceful graph.

Proof: Let $B_{m,n}$ be a bistar with $m+n+2$ vertices and $m+n+1$ edges. The vertex and edge sets are given by, $V(B_{m,n})=\{u_i, v_j : 1\le i\le m+1, 1\le j\le n+1\}$ and

$E(B_{m,n})=\{u_iu_{m+1}, v_jv_{n+1}, u_{m+1}v_{n+1} : 1\le i\le m, 1\le j\le n\}$.

Let G be the graph obtained by the subdivision of the edges of the bistar $B_{m,n}$. Let w_i divide u_iu_{m+1} for $1\le i\le m$ and z_j divide v_jv_{n+1} for $1\le j\le n$. Let v divide $u_{m+1}v_{n+1}$. Then the vertex and the edge set of G are given by

$$V(G)=\{u_i, v_j : 1\le i\le m+1, 1\le j\le n+1\}\cup\{w_i, z_j : 1\le i\le m, 1\le j\le n\}\cup\{v\} \text{ and}$$

$$E(G)=\{w_iu_{m+1}, u_iw_i : 1\le i\le m\}\cup\{z_jv_{n+1}, v_jz_j : 1\le i\le m\}\cup\{vu_{m+1}, vv_{n+1}\}$$

Case (i): $m<n$.

Define an injection $f:V(G)\to\{0,1,2,3,....,(2m+2n+2)^2\}$ by

$f(u_{m+1})=5$; $f(v_{n+1})=0$; $f(v)=1$;

For $1\le i\le m$, $f(w_i)=(2m+n+3-i)^2+5$; $f(u_i)=(3m+2n+6-2i)(m)+5$.

For $1\le j\le n$, $f(z_i)=(2m+2n+3-i)^2$; $f(v_i)=(2m+3n+6-2i)(2m+n)$.

Then, f induces a bijection $f_p:E(G)\to\{1,4,9,....,(2m+2n+2)^2\}$.

In this case the induced edge labels of G are as follows:

$f_p(vu_{m+1})=4$; $f_p(vv_{n+1})=1$;

$f_p(w_iu_{m+1})=(2m+n+3-i)^2$ and $f_p(u_iw_i)=(m+n+3-i)^2$ for $1\le i\le m$.

$f_p(z_jv_{n+1})=(2m+2n+3-j)^2$ and $f_p(v_jz_j)=(n+3-j)^2$ for $1\le j\le n$.

Case (ii): $m=n$.

Define an injection $f:V(G)\to\{0,1,2,3,....,(4n+2)^2\}$ by

$f(u_{n+1})=5$; $f(v_{n+1})=0$; $f(v)=1$.

For $1\le i\le n$, $f(w_i)=(3n+3-i)^2+5$; $f(u_i)=(5n+6-2i)(n)+5$.

For $1\le j\le n$, $f(z_j)=(4n+3-j)^2$; $f(v_j)=(5n+6-2j)(3n)$.

Then, f induces a bijection $f_p:E(G)\to\{1,4,9,....,(4n+2)^2\}$.

In this case the edge labels of G are as follows:

$f_p(vu_{n+1})=4$; $f_p(vv_{n+1})=1$; $f_p(w_iu_{n+1})=(3n+3-i)^2$ and $f_p(u_iw_i)=(2n+3-i)^2$ for $1\le i\le n$.

$f_p(z_jv_{n+1})=(4n+3-j)^2$ and $f_p(v_jz_j)=(n+3-j)^2$ for $1\le j\le n$.

Case (iii): $m>n$.

Define an injection $f:V(G)\to\{0,1,2,3,....,(2m+2n+2)^2\}$ by

$f(u_{m+1})=0$; $f(v_{n+1})=5$; $f(v)=1$.

For $1 \le i \le m$, $f(w_i) = (2m+2n+3-i)^2$; $f(u_i) = (3m+2n+6-2i)(m+2n)$;

For $1 \le j \le n$, $f(z_j) = (m+2n+3-j)^2 + 5$; $f(v_j) = (2m+3n+6-2j)(n)+5$.

Then , f induces a bijection $f_p : E(G) \to \{1,4,9,....,(2m+2n+2)^2\}$.

In this case the edge labels of G are as follows : $f_p(v\,u_{m+1}) = 1$; $f_p(vv_{n+1}) = 4$;

$f_p(w_i u_{m+1}) = (2m+2n+3-i)^2$ and $f_p(u_i w_i) = (m+3-i)^2$ for $1 \le i \le m$.

$f_p(z_j v_{n+1}) = (m+2n+3-j)^2$ and $f_p(v_j z_j) = (m+n+3-j)^2$ for $1 \le j \le n$. ■

Example 2.4. A square graceful labeling of the graph obtained by the subdivision of the edges of bistar $B_{3,7}$ and $B_{9,4}$ are shown in Figure 2 and Figure 3 respectively.

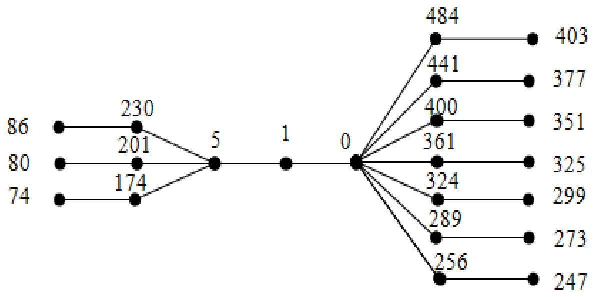

Figure 2: Square graceful labeling of the graph obtained by the subdivision of the edges of $B_{3,7}$.

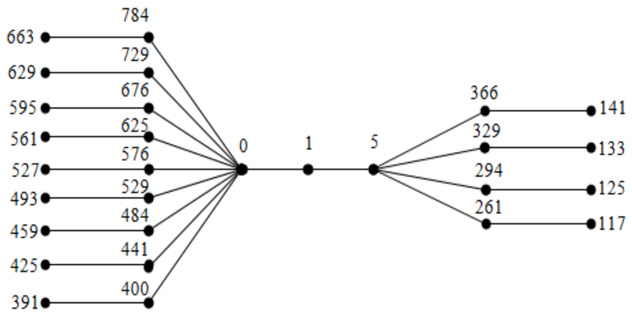

Figure 3: Square graceful labeling of the graph obtained by the subdivision of the edges of $B_{9,4}$.

Theorem 2.5. The graph obtained by the subdivision of the edges of the path P_n in comb $P_n \Theta K_1$ is a square graceful graph.

Proof: Let G be the graph obtained by the subdivision of the edges of the path P_n in comb $P_n \Theta K_1$.

Let $V(G) = \{u_i, v_j, w_k : 1 \le i \le n, 1 \le j \le n, 1 \le k \le n-1\}$ and

$E(G) = \{u_i w_k, w_k u_{i+1} : 1 \le i \le n-1, 1 \le k \le n-1\} \cup \{u_i v_j : 1 \le i \le n, 1 \le j \le n\}$

Define an injection $f : V(G) \to \{0,1,2,3,.......,(3n-2)^2\}$ by

$f(u_1) = (3n-2)^2$;

$f(u_{\frac{i+2}{2}}) = \dfrac{i(i-1)(2i-1)}{6}$ for $i = 2,4,6,....,2n-2$.

$f(w_{\frac{k+1}{2}}) = \dfrac{k(k-1)(2k-1)}{6}$ for $k = 1,3,5,....,2n-3.$

$f(v_{\frac{j+2}{2}}) = \dfrac{j(j-1)(2j-1)}{6} + \left(\dfrac{4n-2+j}{2}\right)^2$ for $j = 2,4,6,....,2n-4.$

$f(v_1) = (n-1)(5n-3)$; $f(v_n) = \dfrac{(n-1)(8n^2-10n+3)}{3}$.

Then, f induces a bijection $f_p : E(G) \to \{1,4,9,.......,(3n-2)^2\}$.

The edge labels are as follows: $f_p(u_1 w_1) = (3n-2)^2$; $f_p(u_n v_n) = (2n-2)^2$;

$f_p(w_k u_{i+1}) = (2i-1)^2$ for $1 \le i \le n-1,\ 1 \le k \le n-1$;

$f_p(u_i w_k) = 4i^2$ for $2 \le i \le n-1,\ 2 \le k \le n-1$;

$f_p(u_i v_j) = (2n-2+i)^2$ for $1 \le i \le n-1, 1 \le j \le n-1$. ■

Example 2.6. A square graceful labeling of the graph obtained by the subdivision of the edges of the path P_5 in comb $P_5 \Theta K_1$ is shown in Figure 4.

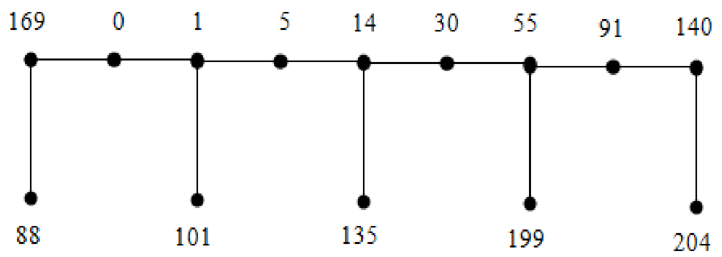

Figure 4

Theorem 2.7: The graph $< C_3 * K_{1,n} >$ is a square graceful graph for $n \ge 3$.

Proof: Let the vertex sets of C_3 and $K_{1,n}$ be given by $V(C_3) = \{u_i : 1 \le i \le 3\}$ and $V(K_{1,n}) = \{v_j : 1 \le j \le n+1\}$ where v_{n+1} is the centre of the star. Identify u_1 of C_3 with v_n of $K_{1,n}$ to get $< C_3 * K_{1,n} >$.

Then the vertex and edge sets of $< C_3 * K_{1,n} >$ are given by,

$V(C_3 * K_{1,n}) = \{u_i : 2 \le i \le 3 ; v_j : 1 \le j \le n+1\}$

$u_1 = v_n$. Let $E(C_3 * K_{1,n}) = \{u_1 u_2, u_2 u_3, u_3 u_1\} \cup \{v_j v_{n+1} : 1 \le j \le n\}$

Define an injection $f : V(C_3 * K_{1,n}) \to \{0,1,2,3,....,(n+3)^2\}$ by

$f(u_1 = v_n) = 0$; $f(u_2) = 16$; $f(u_3) = 25$; $f(v_{n+1}) = (n+3)^2$

$f(v_{n-2}) = (n+1)(n+5)$; $f(v_{n-1}) = n^2 + 6n + 8$;

$f(v_j) = (2n+6-j)j$ for $1 \le j \le n-3$.

Then, f induces a bijection $f_p : E(C_3 * K_{1,n}) \to \{1,4,9,....,(n+3)^2\}$.

The induced edge labels of $< C_3 * K_{1,n} >$ are as follows:

$f_p(u_1 u_2) = 16$; $f_p(u_2 u_3) = 9$; $f_p(u_1 u_3) = 25$; $f_p(v_{n-1} v_{n+1}) = 1$;

$f_p(u_1 v_{n+1}) = (n+3)^2$; $f_p(v_j v_{n+1}) = (n+3-j)^2$ for $1 \le j \le n-3$; $f_p(v_{n-2} v_{n+1}) = 4$.

Hence, the graph $< C_3 * K_{1,n} >$ is a square graceful graph for $n \ge 3$. ■

Example 2.8. A square graceful labeling of $< C_3 * K_{1,8} >$ is shown in Figure 5.

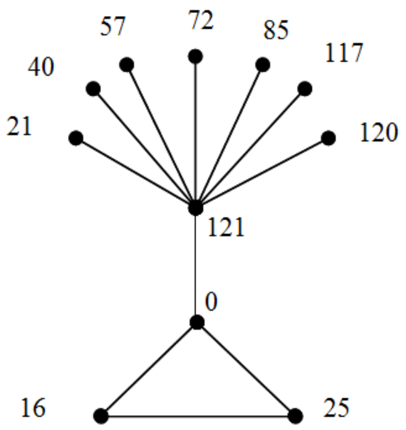

Figure 5: Square graceful labeling of $< C_3 * K_{1,8} >$.

Theorem 2.9. The graph $< S_n : m >$ is a square graceful graph.

Proof: Let $v_{0_j}, v_{1_j}, v_{2_j}, \cdots v_{n_j}$ be the vertices of the j^{th} copy of the star S_n in $< S_n : m >$ where v_{0_j} is the centre of the star where $1 \le j \le m$.

$V(< S_n : m >) = \{v, v_{i_j} : 0 \le i \le n, 1 \le j \le m\}$.

Let $E(< S_n : m >) = \begin{cases} v v_{0_j} & : 1 \le j \le m \\ v_{0_j} v_{i_j} & : 1 \le i \le n, \ 1 \le j \le m \end{cases}$

Define an injection $f : V(< S_n : m >) \to \{0,1,2,3,....,(mn+m)^2\}$ by

$f(v) = 1$; $f(v_{0_1}) = 0$; $f(v_{0_j}) = j^2 + 1$ *if* $2 \le j \le m$;

$f(v_{i_1}) = (mn+m+1-i)^2$ *if* $1 \le i \le n$;

$f(v_{i_j}) = [mn+m+n+1-nj-i]^2 + j^2 + 1$ *if* $1 \le i \le n$ *and* $2 \le j \le m$.

Then, f induces a bijection $f_p : E(< S_n : m >) \to \{1,4,9,....,(mn+m)^2\}$.

The induced edge labels of $< S_n : m >$ are as follows:

$f_p(v v_{0_j}) = j^2$ *if* $1 \le j \le m$;

$f_p(v_{0_j} v_{i_j}) = [mn+m+n+1-nj-i]^2$ *if* $1 \le i \le n$, $1 \le j \le m$.

Hence the graph $< S_n : m >$ is a square graceful graph. ■

Example 2.10: A square graceful labeling of $[S_5 : 4]$ is shown in Figure 6.

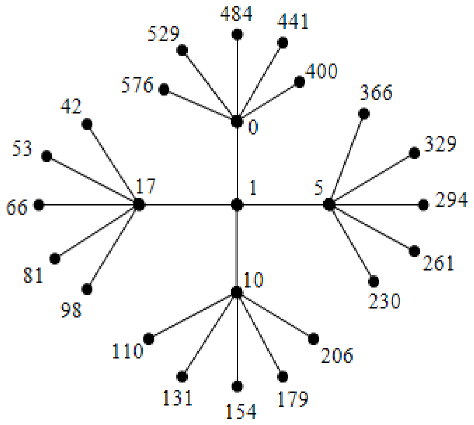

Figure 6: Square graceful labeling of $[S_5 : 4]$.

Theorem 2.11: The graph $< C_3, K_{1,n} >$ is a square graceful graph.

Proof: Let $V(< C_3, K_{1,n} >) = \{ u_i \ ; \ 1 \le i \le 3 \ ; \ v_j : 1 \le j \le n+1 \}$.

Take $u_1 = v_{n+1}$. Let $E(< C_3, K_{1,n} >) = \begin{cases} u_1u_2 \ ; \ u_1u_3 \ ; u_2u_3 \ ; \\ u_1v_j \ : 1 \le j \le n \end{cases}$

Define an injection $f : V(< C_3, K_{1,n} >) \to \{0,1,2,3,.......,(n+3)^2\}$ by

$f(u_1) = 0 \ ; \ f(u_2) = 16; \ f(u_3) = 25; \ f(v_{n-1}) = 4; \ f(v_n) = 1;$

$f(v_j) = (n+3-j)^2 \quad if \ \ 1 \le j \le n-2$.

Then, f induces a bijetion $f_p : E(< C_3, K_{1,n} >) \to \{1,4,9,.......,(n+3)^2\}$.

The edge labels of $< C_3, K_{1,n} >$ are as follows:

$f_p(u_1u_2) = 16; \ f_p(u_1u_3) = 25; \ f_p(u_2u_3) = 9; \ f_p(u_1v_{n-1}) = 4;$

$f_p(u_1v_n) = 1 \ ; \ f_p(u_1v_j) = (n+4-j)^2 \quad if \ \ 1 \le j \le n-2.$ ■

Example 2.12: A square graceful labeling of $< C_3, K_{1,8} >$ and $< C_3, K_{1,5} >$ are shown in Figure 7(a) and Figure 7(b) respectively.

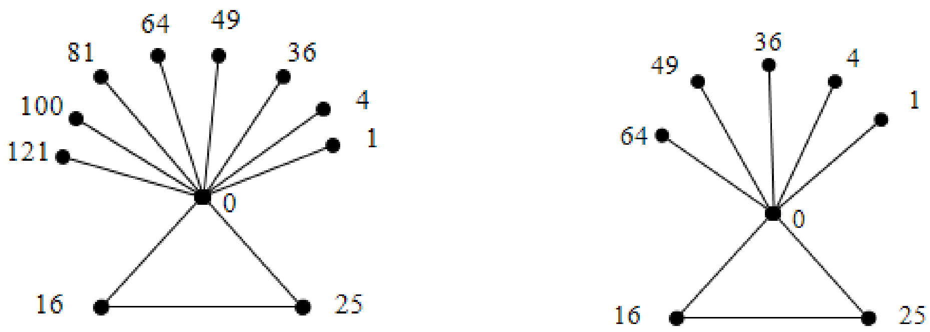

Figure 7: Square graceful labeling of $< C_3, K_{1,8} >$ and $< C_3, K_{1,5} >$.

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