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Spectral conditions for a graph to contain some subgraphs

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Abstract

In this paper, using the upper bound for the spectral radius for a graph obtained by Cao, we present sufficient conditions based on the spectral radius for a graph to contain some subgraphs.

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. The largest and smallest degrees of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The eigenvalues of a graph are defined as the eigenvalues of its adjacency matrix. The largest eigenvalue of a graph G, denoted $\rho(G)$, is called the spectral radius of G. If no confusion arises, we may drop G for those invariants. We use C_k to denote a cycle of length k. We also call C_3 as a triangle. The circumference of a graph is defined as the length of the longest cycle in the graph.

Cao [3] obtained the following upper bound for the spectral radius of a graph.

Theorem 1.1. [3] Let G be a graph of order n and size e with minimum degree $\delta \geq 1$ and maximum degree Δ. Then $\rho(G) \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta}$ with equality if and only if G is regular, a star plus copies of K_2, or a complete graph plus a regular graph with smaller degree of vertices.

2 Main Results

Using Theorem 1.1, Li [4] obtained sufficient conditions which are based on the spectral radius for some Hamiltonian properties of graphs. In this note, we use some of the ideas in [4] to obtain spectral conditions for a connected graph to contain some subgraphs.

Theorem 2.1. Let G be a connected graph of order n and size e. Suppose $k \geq 2$ is an integer. If $\rho > \sqrt{\left(1 - \frac{1}{k}\right) n^2 - \delta(n-1) + (\delta - 1)\Delta}$, then G contains K_{k+1}.

Proof: Let G be a connected graph satisfying the conditions in Theorem 2.1. Turán [6] proved that if a graph G does not contain K_{k+1} then $e \leq \left(1 - \frac{1}{k}\right) \frac{n^2}{2}$.

Suppose that G does not contain K_{k+1}. Then, by Theorem 1.1, we have that

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta} \leq \sqrt{\left(1 - \frac{1}{k}\right) n^2 - \delta(n-1) + (\delta - 1)\Delta}\,,$$

which is a contradiction. This completes the proof. ∎

Let $k = 2$ in Theorem 2.1. Then we have the following corollary.

Corollary 2.2. Let G be a connected graph of order n and size e. If $\rho > \sqrt{\frac{n^2}{2} - \delta(n-1) + (\delta - 1)\Delta}\,$, then G contains a triangle.

Let $H = K_{r,r}$, where $r \geq 2$. Then, for any $\epsilon > 0$,

$$r = \rho(H) > r - \epsilon = \sqrt{\frac{(n(H))^2}{2} - \delta(H)(n(H) - 1) + (\delta(H) - 1)\Delta(H)} - \epsilon,$$

and H does not contain a triangle. Thus Corollary 2.2 is best possible.

Theorem 2.3. Let G be a connected graph of order n and size e. Suppose G is not bipartite. If

$$\rho > \sqrt{\frac{(n-1)^2}{2} + 2 - \delta(n-1) + (\delta - 1)\Delta}\,,$$

then G contains a triangle.

Proof: Let G be a connected graph satisfying the conditions in Theorem 2.3. By Exercise $7.3.3(c)$ on Page 111 in [2], we have that if a non-bipartite graph G does not contain a triangle then $e \leq \frac{(n-1)^2}{4} + 1$. Suppose that the non-bipartite graph G does not contain a triangle. Then, by Theorem 1.1, we have

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta} \leq \sqrt{\frac{(n-1)^2}{2} + 2 - \delta(n-1) + (\delta - 1)\Delta}\,,$$

which is a contradiction. This completes the proof. ∎

Theorem 2.4. Let G be a connected graph of order n and size e. If

$$\rho > \sqrt{\frac{n}{2}(1 + \sqrt{4n-3}) - \delta(n-1) + (\delta - 1)\Delta}\,,$$

then G contains C_4.

Proof: Let G be a connected graph satisfying the given conditions. Reiman [5] proved that if a graph G does not contain C_4, then $e \leq \frac{n}{4}(1+\sqrt{4n-3})$. Suppose that G does not contain C_4. Then, by Theorem 1.1, we have that

$$\rho \leq \sqrt{2e-\delta(n-1)+(\delta-1)\Delta} \leq \sqrt{\frac{n}{2}(1+\sqrt{4n-3})-\delta(n-1)+(\delta-1)\Delta}\,,$$

which is a contradiction. This completes the proof. ■

Theorem 2.5. Let G be a connected graph of order n and size e. If

$$\rho > \sqrt{n\sqrt{(r-1)n}+\frac{n}{2}-\delta(n-1)+(\delta-1)\Delta}\,,$$

then G contains $K_{2,\,r}$ $(r \geq 2)$.

Proof: Let G be a connected graph satisfying the given conditions. By Exercise $7.3.4(b)$ on Page 111 in [2], we have that if a graph G does not contain $K_{2,\,r}$ $(r \geq 2)$ then $e \leq \frac{n\sqrt{(r-1)n}}{2}+\frac{n}{4}$.

Suppose that G does not contain $K_{2,\,r}$. Then, by Theorem 1.1, we have

$$\rho \leq \sqrt{2e-\delta(n-1)+(\delta-1)\Delta} \leq \sqrt{n\sqrt{(r-1)n}+\frac{n}{2}-\delta(n-1)+(\delta-1)\Delta}\,,$$

which is a contradiction. This completes the proof. ■

Theorem 2.6. Let G be a connected graph of order n and size e. If

$$\rho > \sqrt{(r-1)^{\frac{1}{r}}n^{2-\frac{1}{r}}+(r-1)n-\delta(n-1)+(\delta-1)\Delta}\,,$$

then G contains $K_{r,\,r}$.

Proof: Let G be a connected graph satisfying the given conditions. By Exercise $7.3.5$ on Page 112 in [2], we have that if a graph G does not contain $K_{r,\,r}$ then $e \leq \frac{(r-1)^{\frac{1}{r}}n^{2-\frac{1}{r}}}{2}+\frac{(r-1)n}{2}$.

Suppose that G does not contain $K_{r,\,r}$. Then by Theorem 1.1, we have

$$\rho \leq \sqrt{2e-\delta(n-1)+(\delta-1)\Delta} \leq \sqrt{(r-1)^{\frac{1}{r}}n^{2-\frac{1}{r}}+(r-1)n-\delta(n-1)+(\delta-1)\Delta}\,,$$

which is a contradiction. This completes the proof. ■

Theorem 2.7. Let G be a connected graph of order n and size e. Suppose c satisfies $3 \leq c \leq n$. If $\rho \geq \sqrt{(n-1)(c-1-\delta)+(\delta-1)\Delta+2}$, then the circumference of G is at least c.

Proof: Let G be a connected graph satisfying the given conditions. By Theorem 4.9 on Page 137 in [1], we have that if the circumference of a graph G is less than c then $e < \frac{(c-1)(n-1)}{2}+1$.

Suppose that the circumference of G is less than c. Then, by Theorem 1.1, we have

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta} < \sqrt{(n-1)(c-1-\delta) + (\delta-1)\Delta + 2}\,,$$

which is a contradiction. This completes the proof. ■

Theorem 2.8. Let G be a connected graph of order n and size e. Suppose c is the circumference of G. If

$$\rho > \sqrt{\frac{c(2n-c)}{2} - \delta(n-1) + (\delta-1)\Delta}\,,$$

then G contains C_r for each r with $3 \leq r \leq c$.

Proof: Let G be a connected graph satisfying the given conditions. By Theorem 5.2 on Page 149 in [1], we have that if G does not contain C_r for some r with $3 \leq r \leq c$ then $e \leq \frac{c(2n-c)}{4}$. Suppose that G does not contain C_r for some r with $3 \leq r \leq c$. Then, by Theorem 1.1, we have that

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta-1)\Delta} \leq \sqrt{\frac{c(2n-c)}{2} - \delta(n-1) + (\delta-1)\Delta}\,,$$

which is a contradiction. This completes the proof. ■

Obviously, Theorem 2.8 has the following corollary.

Corollary 2.9. Let G be a connected graph of order n and size e. Suppose G is Hamiltonian. If

$$\rho > \sqrt{\frac{n^2}{2} - \delta(n-1) + (\delta-1)\Delta}\,,$$

then G contains C_r for each r with $3 \leq r \leq n$.

Let $H = K_{r,r}$, where $r \geq 2$. Then H is Hamiltonian and, for any $\epsilon > 0$,

$$r = \rho(H) > r - \epsilon = \sqrt{\frac{(n(H))^2}{2} - \delta(H)(n(H)-1) + (\delta(H)-1)\Delta(H)} - \epsilon,$$

and H does not contain C_s when s is odd such that $3 \leq s \leq n(H)$. Thus Corollary 2.9 is possible.

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