International Journal of Mathematics and Soft Computing
Vol.5, No.1 (2015), 79 - 92.

ISSN Print : 2249 - 3328
ISSN Online: 2319 - 5215

An alternative approach for solving fuzzy matrix games

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Abstract

In this paper, two-person matrix games is considered whose elements of pay-off matrix are triangular fuzzy numbers (TFNs). To solve such game a new method based on α-cut of TFN is developed for each of the players. In this method, two auxiliary bi-objective linear programming (BOLP) models are derived. Then using average weighted approach these two BOLP models are decomposed into two auxiliary crisp linear programming (LP) problems. Finally, the value of the matrix game for each player is obtained by solving two corresponding auxiliary LP problems using the existing simplex method. Validity and applicability of this method are illustrated with practical example compared to existing methods.

Keywords: Fuzzy game theory, Mathematical programming, Triangular fuzzy numbers, Interval number, Multi-objective optimization, Linear programming.
AMS Subject Classification(2010): 90C70.

1 Introduction

Game theory is a mathematical tool for analysis of conflicting interests situations, which includes players or decision makers (DM) who select various strategies from the set of available strategies. Although the traditional game theory assumes well suited information and precise data it is difficult to assess information exactly in real game situation due to lack of information on the exact values on some parameters and fuzzy understanding of situations by players. In such situations, the fuzzy sets introduced by, Zadeh [24], is very useful tool in game theory.

Various attempts have been made in the literature to study fuzzy game theory. Campos [4] introduced fuzzy linear programming model to solve fuzzy matrix game. Sakawa and Nishizaki [21] introduced max-min solution procedure for multi-objective fuzzy games. Nayak and Pal [16, 17] studied the interval

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and fuzzy matrix games. Li [9] implemented multi-objective programming approach to solve fuzzy games. Maeda [15] studied about the characterization of the equilibrium strategy of the fuzzy games. Bector et al. [2] studied matrix games with fuzzy goals and utilized fuzzy linear programming duality ([1]), to solve such games by solving the equivalent primal-dual pair of fuzzy linear programming problems. This approach has further been extended by Bector et al. [3] to matrix games with fuzzy pay offs where duality in linear programming with fuzzy parameters becomes the basic tool. Vijay et al. [22, 23] extended their duality results using aspiration level approach and also studied a generalized fuzzy relation approach to solve matrix games with fuzzy goals and fuzzy pay-offs. Some recent references on fuzzy matrix games are ([14], [5], [8], [11, 13]).

Obviously, in the aforementioned methods ([4], [3]) though the pay-offs matrix is fuzzy as it's elements are TFNs, the value of the game for each players are assumed to be crisp numbers. But from logical point of view it seems natural that if the pay-offs matrix is fuzzy the values of the game for players I and II should also be fuzzy. However, in the above methods defuzzification approaches are used whose solutions closely depend on ranking functions, therefore only defuzzified values of the game for both the players were obtained. In the methodology [22, 23] aspiration level approach and fuzzy preference relation approach is used which is more or less dependent in subjective factors such as attitude, judgment and preference of the players. In our proposed method, we do not have require any defuzzification function which otherwise may be difficult to decide in practice. Also the value of the game for each players obtained in this method as TFNs which is desirable. Li [10] solved the fuzzy matrix games with pay-offs of TFNs by solving auxiliary multi-objective programming models using lexicographic method in which the value of the game for each players' obtained as TFNs. Here, it may be noted that the solution of the fuzzy game in our approach is "close to" that of [10], though different methodologies are used to solve the game.

This paper is arranged as follows. Section 2 briefly reviews some concepts such as TFNs, α-cut sets, basic arithmetic and order relation of interval numbers. Section 3 formulates matrix games with payoffs of TFNs which are arbitrary and develops a new method based on α-cut sets of payoffs. In Section 4, the proposed method is illustrated with a numerical example and the results obtained is compared with existing one. Conclusion is made in Section 5.

2 Definition and Preliminaries

In this section, we recall some definitions and preliminaries. TFNs, their arithmetic operations and α-cut sets are defined according to [6].

Definition 2.1. A TFN $\tilde{a} = (\grave{a}, a, \acute{a})$ is a special type of fuzzy set on the set $\Re$ of real numbers, whose membership function is defined as follows

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x-\grave{a}}{a-\grave{a}}, & \grave{a} \le x \le a \\ \dfrac{\acute{a}-x}{\acute{a}-a}, & a < x \le \acute{a} \end{cases}$$

where a is the mean value of $\tilde{a}$, $\grave{a}$ and $\acute{a}$ represents the left and right spread respectively. Note that if $\grave{a} = \acute{a} = a$, then $\tilde{a}$ is reduced to a crisp number. The set of all TFNs are denoted by $\tilde{\mathcal{F}}(\Re)$.

Definition 2.2. (α-cut sets): A α-cut set of a TFN, $\tilde{a} = (\grave{a}, a, \acute{a})$ is a crisp subset of $\Re$, which is defined as $\tilde{a}_\alpha = \{x : \mu_{\tilde{a}}(x) \geq \alpha\}$ where $0 \leq \alpha \leq 1$. Using the membership function $\mu_{\tilde{a}}(x)$ it can be easily proved that $\hat{a}_\alpha$ is a closed interval and $\tilde{a}_\alpha = [\grave{a} + \alpha(a - \grave{a}), \acute{a} - \alpha(\acute{a} - a)] = [a_L^\alpha, a_R^\alpha]$. where $a_L^\alpha = \grave{a} + \alpha(a - \grave{a})$ and $a_R^\alpha = \acute{a} - \alpha(\acute{a} - a)$. The set of all α-cut values of TFNs is denoted by $\tilde{\mathcal{F}}_\alpha(\Re)$.

2.1 Basic Interval Arithmetic

Definition 2.3. ([20]): An interval number is defined as $\hat{a} = [a_L, a_R] = \{x \in \Re : a_L \leq x \leq a_R\}$, $\Re$ is the set of all real numbers. The numbers a_L, a_R are called respectively the lower and upper limits of the interval $\hat{a}$. An interval number $\hat{a}$ can also be represented in mean-width form as $\hat{a} = \langle m(a), w(a) \rangle$,

$$\text{where} \quad m(a) = \frac{1}{2}(a_L + a_R) \text{ and } w(a) = \frac{1}{2}(a_R - a_L)$$

are the mid point and half-width of the interval $\hat{a}$. The set of all interval numbers in $\Re$ is denoted by $I(\Re)$.

The basic interval arithmetic are given as follows. Let $\hat{a} = [a_L, a_R]$ and $\hat{b} = [b_L, b_R]$ be two interval numbers. Then

$$\hat{a} + \hat{b} = [a_L + b_L, a_R + b_R], \hat{a} - \hat{b} = [a_L - b_R, a_R - b_L] \text{ and } \lambda\hat{a} = \begin{cases} [\lambda a_L, \lambda a_R] & \text{if } \lambda \geq 0 \\ (\lambda a_R, \lambda a_L] & \text{if } \lambda < 0 \end{cases}$$

where λ is a real scalar.

The ranking order relation between intervals is a difficult problem. An extensive research and wide coverage on interval arithmetic and its application can be found in [20]. According to Moore's concept of set inclusion, a brief comparison on different interval orders is given in [17].

Definition 2.4. ([17]) The satisfactory crisp equivalent forms of interval inequality constraints $\hat{a}z \leq_I \hat{b}$ and $\hat{a}z \geq_I \hat{b}$ are defined as

$$\hat{a}z \leq_I \hat{b} \Rightarrow \begin{cases} a_R z \leq b_R \\ \dfrac{m(\hat{a}z) - m(b)}{w(\hat{a}z) + w(b)} \leq \beta \end{cases} \text{ and } \hat{a}z \geq_I \hat{b} \Rightarrow \begin{cases} a_L z \geq b_L \\ \dfrac{m(b) - m(\hat{a}z)}{w(\hat{a}z) + w(b)} \leq \beta. \end{cases} \tag{1}$$

where $\leq_I$ and $\geq_I$ denote the interval number inequalities and $\beta \in [0, 1]$ represents the minimal acceptance degree of the inequality constraints which may be allowed to violate.

Definition 2.5. ([7]) Let $\hat{a} = [a_L, a_R]$ be an interval. The maximization and minimization problem with the interval valued objective function are described as follows

$$\max\{\hat{a} | \hat{a} \in \Omega_1\} \text{ and } \min\{\hat{a} | \hat{a} \in \Omega_2\}$$

which are equivalent to the following bi-objective mathematical programming problems:

$$\max\{a_L, m(\hat{a}) | \hat{a} \in \Omega_1\} \text{ and } \min\{a_R, m(\hat{a}) | \hat{a} \in \Omega_2\}$$

where Ω_1 and Ω_2 are the set of constraints in which the variable $\hat{a}$ should satisfy according to requirements in real situations.

3 Mathematical Model of a Matrix Game

Let $i \in \{1, 2, \ldots, m\}$ be a pure strategy available for player I and $j \in \{1, 2, \ldots, n\}$ be a pure strategy available for player II. When player I chooses a pure strategy i and the player II chooses a pure strategy j, then a_{ij} is the payoff for player I and $-a_{ij}$ be a payoff for player II. The two-person zero-sum matrix game G can be represented as a pay-off matrix $A = (a_{ij})_{m\times n}$.

3.1 Mixed Strategy

Consider the game G with no saddle point, $i.e.$ $\max_i\{\min_j a_{ij}\} \neq \min_j\{\max_i a_{ij}\}$. To solve such game, Neumann [18] introduced the concept of mixed strategy in classical form. We denote the sets of all mixed strategies, called strategy spaces, available for the players I, II by

$$\begin{aligned} S_I &= \{\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \Re_+^m : x_i \geq 0; i = 1, 2, \ldots, m \text{ and } \sum_{i=1}^m x_i = 1\} \\ S_{II} &= \{\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \Re_+^n : y_j \geq 0; j = 1, 2, \ldots, n \text{ and } \sum_{j=1}^n y_j = 1\}, \end{aligned}$$

where $\Re_+^m$ denotes the $m-$dimensional non-negative Euclidean space. Thus by a crisp two-person zero-sum matrix game G we mean the triplet $G = (S_I, S_{II}, A)$. Since each player is uncertain about what strategy he/she will choose, he/she will choose a probability distribution over the set of alternatives available to him/her or a mixed strategy in terms of game theory.

3.2 Matrix Game with Fuzzy Pay-offs

Let S_I, S_{II} be the strategy spaces for player I and player II respectively and $\tilde{A} = (\tilde{a}_{ij})$ be the pay-off matrix where each $\tilde{a}_{ij} = (\grave{a_{ij}}, a_{ij}, \acute{a_{ij}})$ is a TFN defined as in Section 2. Then a two-person zero-sum matrix game with fuzzy pay-offs is the triplet $(S_I, S_{II}, \tilde{A})$.

In the following, we shall often call a two-person zero-sum matrix game with fuzzy pay-offs simply as fuzzy matrix game $\widetilde{FG} = (S_I, S_{II}, \tilde{A})$. The solution concept of $\widetilde{FG}$ is defined in the following.

Definition 3.1. ([3]) Let $\tilde{v}$ and $\tilde{w}$ be two TFNs. Then $(\tilde{v}, \tilde{w})$ is called an reasonable solution of the game $\widetilde{FG}$, if there exists $\mathbf{x}^* \in S_I$ and $\mathbf{y}^* \in S_{II}$, satisfying $\mathbf{x}^{*T}\tilde{A}\mathbf{y} \succeq \tilde{v} \ \forall\, \mathbf{y} \in S_{II}$ and $\mathbf{x}^T\tilde{A}\mathbf{y}^* \preceq \tilde{w} \ \forall\, \mathbf{x} \in S_I$. If $(\tilde{v}, \tilde{w})$ is a reasonable solution of $\widetilde{FG}$ then $\tilde{v}$ (respectively, $\tilde{w}$) is called reasonable value of the player I (respectively, player II).

It is worth noticing that Definition 3.1 only gives the notion of the reasonable solution rather than the notion of optimal solution.

Definition 3.2. ([3]) Let $\widetilde{V}$ and $\widetilde{W}$ denote the set of all reasonable values $\tilde{v}, \tilde{w}$ for players I and II respectively. Assume that there exist $\tilde{v}^* \in \widetilde{V}$ and $\tilde{w}^* \in \widetilde{W}$. If there do not exist any $\tilde{v} \in \widetilde{V}(\tilde{v} \neq \tilde{v}^*)$ and $\tilde{w} \in \widetilde{W}(\tilde{w} \neq \tilde{w}^*)$ such that they satisfy the following conditions $\tilde{v}^* \preceq \tilde{v}$ and $\tilde{w}^* \succeq \tilde{w}$, then $(\mathbf{x}^*, \mathbf{y}^*, \tilde{v}^*, \tilde{w}^*)$ is called an optimal solution of the $\widetilde{FG}$. $\mathbf{x}^*$ (respectively, $\mathbf{y}^*$) is called an optimal strategy for player I (respectively, player II) and $\tilde{v}^*$ (respectively, $\tilde{w}^*$) is termed as the value of the game $\widetilde{FG}$ for player I (respectively, player II).

By using the above definitions for the game $\widetilde{FG}$, we now construct the following pair of problems for player I and player II respectively.

According to above definitions, the optimal strategies $\mathbf{x}^* \in S_I$ for player I and $\mathbf{y}^* \in S_{II}$ for player II can be generated by solving the following fuzzy mathematical problems:

$$\max\{\tilde{v}\} \quad \text{subject to} \begin{cases} \mathbf{x}^T \tilde{A} \mathbf{y} \succeq \tilde{v} \ \ \forall \mathbf{y} \in S_{II} \\ \mathbf{x} \in S_I \\ \tilde{v} \in \tilde{F}(\Re) \end{cases} \quad \text{and} \quad \min\{\tilde{w}\} \quad \text{subject to} \begin{cases} \mathbf{x}^T \tilde{A} \mathbf{y} \preceq \tilde{w} \ \ \forall \mathbf{x} \in S_I \\ \mathbf{y} \in S_{II} \\ \tilde{w} \in \tilde{F}(\Re) \end{cases}$$

Since S_I and S_{II} are convex polytopes it is sufficient to consider only the extreme points (i.e. pure strategies) of S_I and S_{II}. This observation leads to the following two fuzzy mathematical programming (FMP) models as

$$\max\{\tilde{v}\} \quad \text{subject to} \begin{cases} \sum_{i=1}^{m} \tilde{a}_{ij} x_i \succeq \tilde{v} \ (j = 1, 2, \ldots, n) \\ \sum_{i=1}^{m} x_i = 1 \\ x_i \geq 0 \ (i = 1, 2, \ldots m) \end{cases} \quad \text{and} \quad \min\{\tilde{w}\} \quad \text{subject to} \begin{cases} \sum_{j=1}^{n} \tilde{a}_{ij} y_j \preceq \tilde{w} \ (i = 1, 2, \ldots, m) \\ \sum_{j=1}^{n} y_j = 1 \\ y_j \geq 0 \ (j = 1, 2, \ldots n) \end{cases}$$

respectively.

Using Definition 2.2, the above two FMPs, can be transformed into the following interval mathematical programming models as

$$\max\{\tilde{v}_\alpha\} \quad \text{subject to} \begin{cases} \sum_{i=1}^{m} (\tilde{a}_{ij} x_i)_\alpha \geq_I \tilde{v}_\alpha, \\ \qquad (j = 1, 2, \ldots, n) \\ \sum_{i=1}^{m} x_i = 1 \\ x_i \geq 0 \ (i = 1, 2, \ldots m) \end{cases} \quad \text{and} \quad \min\{\tilde{w}_\alpha\} \quad \text{subject to} \begin{cases} \sum_{j=1}^{n} (\tilde{a}_{ij} y_j)_\alpha \leq_I \tilde{w}_\alpha, \\ \qquad (i = 1, 2, \ldots, m) \\ \sum_{j=1}^{n} y_j = 1 \\ y_j \geq 0 \ (j = 1, 2, \ldots n) \end{cases}$$

respectively.

Let us consider the α-cut sets of $\tilde{v}, \tilde{w}$ and $\tilde{a}_{ij}$ as

$$\tilde{v}_\alpha = [v_L^\alpha, v_R^\alpha],\ \tilde{w}_\alpha = [w_L^\alpha, w_R^\alpha] \text{ and } (\tilde{a}_{ij})_\alpha = [a_{ijL}^\alpha, a_{ijR}^\alpha],$$

where $[v_L^\alpha, v_R^\alpha]$ is an interval and v_L^α, v_R^α denote lower and upper limits respectively.

Then the above two models can be written as

$$\max\{[v_L^\alpha, v_R^\alpha]\} \qquad\qquad \min\{[w_L^\alpha, w_R^\alpha]$$

$$\text{subject to} \begin{cases} \sum\limits_{i=1}^{m}[a_{ijL}^\alpha, a_{ijR}^\alpha]x_i \geq_I [v_L^\alpha, v_R^\alpha], \\ \qquad (j = 1, 2, \ldots, n) \\ \sum\limits_{i=1}^{m} x_i = 1 \\ x_i \geq 0\ (i = 1, 2, \ldots m) \end{cases} \text{and subject to} \begin{cases} \sum\limits_{j=1}^{n}[a_{ijL}^\alpha, a_{ijR}^\alpha]y_j \leq_I [w_L^\alpha, w_R^\alpha], \\ \qquad (i = 1, 2, \ldots, m) \\ \sum\limits_{j=1}^{n} y_j = 1 \\ y_j \geq 0\ (j = 1, 2, \ldots n) \end{cases}$$

respectively.

Using Definitions 2.4 and 2.5, the above two problems can be converted into the two bi-objective mathematical programming models as follows

$$\begin{aligned} & \max\left\{v_L^\alpha, \frac{v_L^\alpha + v_R^\alpha}{2}\right\} \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ijL}^\alpha x_i \geq v_L^\alpha\ (j = 1, 2, \ldots, n) \\ & \frac{v_R^\alpha + v_L^\alpha - \sum\limits_{i=1}^{m}(a_{ijL}^\alpha + a_{ijR}^\alpha)x_i}{v_R^\alpha - v_L^\alpha + \sum\limits_{i=1}^{m}(a_{ijR}^\alpha - a_{ijL}^\alpha)x_i} \leq \beta\ (j = 1, 2, \ldots, n) \\ & v_L^\alpha \leq v_R^\alpha \\ & \sum_{i=1}^{m} x_i = 1 \\ & x_i \geq 0\ (i = 1, 2, \ldots m) \end{aligned} \tag{2}$$

and

$$\begin{aligned} & \min\left\{w_R^\alpha, \frac{w_L^\alpha + w_R^\alpha}{2}\right\} \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ijR}^\alpha y_j \leq w_R^\alpha\ (i = 1, 2, \ldots, m) \end{aligned}$$

$$\frac{\sum_{j=1}^{n}(a_{ijL}^{\alpha}+a_{ijR}^{\alpha})y_j-(w_L^{\alpha}+w_R^{\alpha})}{w_R^{\alpha}-w_L^{\alpha}+\sum_{j=1}^{n}(a_{ijR}^{\alpha}-a_{ijL}^{\alpha})y_j}\leq\beta\ (i=1,2,\ldots,m) \tag{3}$$

$$w_L^{\alpha}\leq w_R^{\alpha}$$

$$\sum_{j=1}^{n}y_j=1$$

$$y_j\geq 0\ (j=1,2,\ldots n).$$

The equations (2) and (3) are bi-objective linear programming problems (BOLP) on the decision variables $v_L^{\alpha}, v_R^{\alpha}, x_i\ (i=1,2,\ldots,m)$ and $w_L^{\alpha}, w_R^{\alpha}, y_j\ (i=1,2,\ldots,n)$. There exists several solution methods for them. However, in this study we use average weighted approach ([11]) to solve equations (2) and (3) in the sense of pareto optimality.

Therefore, the above two BOLPs become

$$
\begin{aligned}
&\max\left\{\frac{3v_L^{\alpha}+v_R^{\alpha}}{4}\right\}\\
\text{subject to}\quad &\sum_{i=1}^{m}a_{ijL}^{\alpha}x_i\geq v_L^{\alpha}\ (j=1,2,\ldots,n)\\
&\sum_{i=1}^{m}\left\{(1+\beta)a_{ijR}^{\alpha}+(1-\beta)a_{ijL}^{\alpha}\right\}x_i\geq(1+\beta)v_L^{\alpha}+(1-\beta)v_R^{\alpha}\ (j=1,2,\ldots,n)\\
&v_L^{\alpha}\leq v_R^{\alpha}\\
&\sum_{i=1}^{m}x_i=1\\
&x_i\geq 0\ (i=1,2,\ldots m)
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
&\min\left\{\frac{3w_R^{\alpha}+w_L^{\alpha}}{4}\right\}\\
\text{subject to}\quad &\sum_{j=1}^{n}a_{ijR}^{\alpha}y_j\leq w_R^{\alpha}\ (i=1,2,\ldots,m)\\
&\sum_{j=1}^{n}\left\{(1+\beta)a_{ijL}^{\alpha}+(1-\beta)a_{ijR}^{\alpha}\right\}y_j\leq(1-\beta)w_L^{\alpha}+(1+\beta)w_R^{\alpha}\ (i=1,2,\ldots,m)\\
&w_L^{\alpha}\leq w_R^{\alpha}\\
&\sum_{j=1}^{n}y_j=1\\
&y_j\geq 0\ (j=1,2,\ldots n)
\end{aligned}
\tag{5}
$$

Now, if all TFNs $\tilde{a}_{ij} = (\grave{a}, a, \acute{a})$ are approximated to its α-cut interval $\tilde{a}^{\alpha}_{ij} = [a^{\alpha}_{ijL}, a^{\alpha}_{ijR}]$, then from Definition 2.2 we have $a^{\alpha}_{ijL} = \grave{a}_{ij} + \alpha(a_{ij} - \grave{a}_{ij})$ and $a^{\alpha}_{ijR} = \acute{a}_{ij} - \alpha(\acute{a}_{ij} - a_{ij})$. Therefore, the above two liner programming problems reduced to

$$
\begin{aligned}
& \max\left\{\frac{3v^{\alpha}_{L} + v^{\alpha}_{R}}{4}\right\} \\
\text{subject to}\quad & \sum_{i=1}^{m}\{\grave{a}_{ij} + \alpha(a_{ij} - \grave{a}_{ij})\}x_i \geq v^{\alpha}_{L}\ (j = 1, 2, \ldots, n) \\
& \sum_{i=1}^{m}[(1+\beta)\{\acute{a}_{ij} - \alpha(\acute{a}_{ij} - aij)\} + (1-\beta)\{\grave{a}_{ij} + \alpha(a_{ij} - \grave{a}_{ij})\}]x_i \\
& \qquad\qquad \geq (1+\beta)v^{\alpha}_{L} + (1-\beta)v^{\alpha}_{R}\ (j = 1, 2, \ldots, n) \\
& v^{\alpha}_{L} \leq v^{\alpha}_{R} \\
& \sum_{i=1}^{m} x_i = 1 \\
& x_i \geq 0\ (i = 1, 2, \ldots m)
\end{aligned}
\tag{6}
$$

and

$$
\begin{aligned}
& \min\left\{\frac{3w^{\alpha}_{R} + w^{\alpha}_{L}}{4}\right\} \\
\text{subject to}\quad & \sum_{j=1}^{n}\{\acute{a}_{ij} - \alpha(\acute{a'}_{ij} - a_{ij})\}y_j \leq w^{\alpha}_{R}\ (i = 1, 2, \ldots, m) \\
& \sum_{j=1}^{n}[(1+\beta)\{\grave{a}_{ij} + \alpha(a_{ij} - \grave{a}_{ij})\} + (1-\beta)\{\acute{a}_{ij} - \alpha(\acute{a}_{ij} - a_{ij})\}]y_j \\
& \qquad\qquad \leq (1-\beta)w^{\alpha}_{L} + (1+\beta)w^{\alpha}_{R}\ (i = 1, 2, \ldots, m) \\
& w^{\alpha}_{L} \leq w^{\alpha}_{R} \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \geq 0\ (j = 1, 2, \ldots n)
\end{aligned}
\tag{7}
$$

where the parameters α and β are given by the players/decision makers.

We take $\beta = 0$, which indicates that the inequality constraints are not allowed to violate. For given parameter α, using existing simplex method for linear programming problems, optimal solutions of equations (6) and (7) can be obtained. We denote them as $(x^*, v^{\alpha *}_{L}, v^{\alpha *}_{R})$ and $(y^*, w^{\alpha *}_{L}, w^{\alpha *}_{R})$ respectively. Thus the optimal strategy x^* and the corresponding upper and lower bounds of α-cut of the value of the game, $\tilde{v}^*$ for player I is obtained for given α. Similarly, for player II, optimal strategy y^* and the corresponding upper and lower bounds of α-cut of the value of the game, $\tilde{w}^*$ for player II is obtained for given α.

Remark: If all the fuzzy numbers are to be taken as crisp numbers, that is, $\tilde{a}_{ij} = a_{ij}$, $(\grave{a} = 0, \acute{a} = 0)$ then the fuzzy game $\widetilde{FG}$ reduces to the crisp two-person zero-sum game G. Then the pair of problems (6) and (7) reduce to the crisp linear programming problems. Thus, this approach is a generalization of crisp situation.

4 Numerical Example

Suppose that there are two companies I and II to enhance the market share of a new product by competing in advertising. The two companies are considering two different strategies to increase market share: strategy I (adv. by TV), II (adv. by Newspaper). Here it is assumed that the targeted market is fixed, i.e the market share of the one company increases while the market share of the other company decreases and also each company puts all its advertisements in one. The above problem may be regarded as matrix game. Namely, the company I and II are considered as players I and II respectively. The marketing research department of company I establishes the following pay-off matrix

$$\tilde{A} = \begin{array}{c c} & \begin{array}{cc} \text{adv. by TV} & \text{adv. by Newspaper} \end{array} \\ \begin{array}{c} \text{adv. by TV} \\ \text{adv. by Newspaper} \end{array} & \begin{pmatrix} (175, 180, 190) & (150, 156, 158) \\ (80, 90, 100) & (175, 180, 190) \end{pmatrix} \end{array}.$$

where the element $(175, 180, 190)$ in the matrix $\tilde{A}$ indicates that the sales amount of the company I increases by "about 180" units when the company I and II use the strategy I (adv. by TV) simultaneously. The other elements in the matrix $\tilde{A}$ can be explained similarly.

According to the equations (6) and (7) we get the linear programming models as

$$\begin{aligned}
& \max \left\{ \frac{3v_L^\alpha + v_R^\alpha}{4} \right\} \\
\text{subject to} \quad & (175 + 5\alpha)x_1 + (80 + 10\alpha)x_2 \geq v_L^\alpha \\
& (150 + 6\alpha)x_1 + \{175 + 5\alpha)x_2 \geq v_L^\alpha \\
& \{(1+\beta)(190 - 10\alpha) + (1-\beta)(175 + 5\alpha)\} x_1 \\
& \qquad + \{(1+\beta)(100 - 10\alpha) + (1-\beta)(80 + 10\alpha)\} x_2 \geq (1+\beta)v_L^\alpha + (1-\beta)v_R^\alpha \\
& \{(1+\beta)(158 - 2\alpha) + (1-\beta)(150 + 6\alpha)\} x_1 \\
& \qquad + \{(1+\beta)(190 - 10\alpha) + (1-\beta)(175 + 5\alpha)\} x_2 \geq (1+\beta)v_L^\alpha + (1-\beta)v_R^\alpha \\
& v_L^\alpha \leq v_R^\alpha \\
& x_1 + x_2 = 1 \\
& x_1, x_2 \geq 0
\end{aligned} \tag{8}$$

and

$$\min \left\{ \frac{3w_R^\alpha + w_L^\alpha}{4} \right\}$$

$$
\begin{aligned}
\text{subject to} \quad & (190-10\alpha)y_1 + (158-2\alpha)y_2 \le w_R^\alpha \\
& (100-10\alpha)y_1 + (190-10\alpha)y_2 \le w_R^\alpha \\
& \{(1+\beta)(175+5\alpha) + (1-\beta)(190-10\alpha)\}\, y_1 \\
& \quad + \{(1+\beta)(150+6\alpha) + (1-\beta)(158-2\alpha)\}\, y_2 \le (1-\beta)w_L^\alpha + (1+\beta)w_R^\alpha \\
& \{(1+\beta)(80+10\alpha) + (1-\beta)(100-10\alpha)\}\, y_1 \\
& \quad + \{(1+\beta)(175+5\alpha) + (1-\beta)(190-10\alpha)\}\, y_2 \le (1-\beta)w_L^\alpha + (1+\beta)w_R^\alpha \\
& w_L^\alpha \le w_R^\alpha \\
& y_1 + y_2 = 1 \\
& y_1, y_2 \ge 0
\end{aligned} \tag{9}
$$

$\beta = 0$ indicates that the inequality constraints are not allowed to violate. For given α we can solve the above two equations (8) and (9). The upper and lower bounds of α-cut sets of the value of the game $\tilde{v}^*$ (respectively, $\tilde{w}^*$) for player I (respectively, player II) and the corresponding optimal mixed strategies for any $\alpha \in [0,1]$ are shown in the following table.

Table 1: Solution of the game $\widetilde{FG}$ for different values of α.

α	x_1^*	x_2^*	y_1^*	y_2^*	$v_L^{\alpha*}$	$v_R^{\alpha*}$	$w_L^{\alpha*}$	$w_R^{\alpha*}$
0	0.7916667	0.2083333	0.2622951	0.7377049	155.21	164.67	156.56	166.39
0.1	0.7914573	0.2085427	0.2574257	0.7425743	155.79	164.31	157.01	165.83
0.2	0.7912458	0.2087542	0.2524917	0.7475083	156.38	163.95	157.46	165.27
0.3	0.7910321	0.2089679	0.2474916	0.7525084	156.96	163.58	157.91	164.72
0.4	0.7908163	0.2091837	0.2424242	0.7575758	157.54	163.22	158.36	164.18
0.5	0.7905983	0.2094017	0.2372881	0.7627119	158.13	162.86	158.81	163.64
0.6	0.7903780	0.2096220	0.2320819	0.7679181	158.71	162.50	159.26	163.11
0.7	0.7901554	0.2098446	0.2268041	0.7731959	159.23	162.14	159.71	162.59
0.8	0.7899306	0.2100694	0.2214533	0.7785467	159.88	161.78	160.16	162.07
0.9	0.7897033	0.2102967	0.2160279	0.7839721	160.47	161.41	160.61	161.56
1	0.7894737	0.2105263	0.2105263	0.7894737	161.05	161.05	161.05	161.05

In the above table we provide the optimal solution for the above two problems (6) and (7) for different values of α. In particular, for $\alpha = 0.8$, the optimal strategies for player I and player II are $\mathbf{x}^* = (0.7899306, 0.2100694)$ and $\mathbf{y}^* = (0.2214533, 0.7785467)$ and the cut set of the value of the game for player I and player II are the intervals $[159.88, 161.78]$ and $[160.16, 162.07]$, respectively.

It can be easily seen from Table 1 that larger the α values, lower the degree of uncertainty of the value of the game for both players. Moreover, when $\alpha = 0$ the cut set of the value of the game of the player I and II are the intervals [155.21,164.67] and [156.56,166.39] respectively, which are the widest. Thus, in this example it is impossible that the value of the game for player I falls out side of the interval

[155.21,164.67]. Again for $\alpha = 1$ the value of the game for player I is 161.05, which is the most likely value. Similarly, the value of the game for player II never falls outside of the interval [156.56,166.39] and the most likely value is 161.05. Therefore, the approximate values of the game for players I and II are obtained as follows:

$$\begin{aligned} & \tilde{v}^* = (155.21, 161.05, 164.67) \\ \text{and} \quad & \tilde{w}^* = (156.56, 161.05, 166.39) \end{aligned}$$

respectively, which are TFNs. It means that the sales amount increases of the company I is "approximately 161.05". In other words, company I's minimum reward is 155.21 while his maximum reward is 164.67. Similar interpretation can also be given to player II.

4.1 Results obtained by Li's Approach

Li[10] assumed that the value of the game for player I and for player II, respectively as TFNs, denoted by $\tilde{v}^l = (\grave{v}^l, v^l, \acute{v}^l)$ and $\tilde{w}^l = (\grave{w}^l, w^l, \acute{w}^l)$. Then, player I's LP problem in the level 1 is constructed as follows

$$\begin{aligned} & \max\{v^l\} \\ \text{subject to} \quad & 175x_1 + 80x_2 \geq \grave{v}^l \\ & 150x_1 + 175x_2 \geq \grave{v}^l \\ & 180x_1 + 90x_2 \geq v^l \\ & 156x_1 + 180x_2 \geq v^l \\ & 190x_1 + 100x_2 \geq \acute{v}^l \\ & 158x_1 + 190x_2 \geq \acute{v}^l \\ & x_1 + x_2 = 1 \\ & \grave{v}^l \leq v^l \leq \acute{v}^l \\ & x_1, x_2 \geq 0. \end{aligned} \tag{10}$$

Solving (10) by using the simplex method of linear programming we obtain the optimal solution $(x^{*l}, \grave{v}^{0l}, v^{*l}, \acute{v}^{0l})$, where $x^{*l} = (0.7895, 0.2108)^T$, $\grave{v}^{0l} = 61.398$, $v^{*l} = 161.05, \acute{v}^{0l} = 163.063$. Now, player I's LP problem in the level 2 are constructed as follows:

$$\begin{aligned} & \max\{\grave{v}^l\} \\ \text{subject to} \quad & \grave{v}^l \leq 175 \times 0.7895 + 80 \times 0.2105 \\ & \grave{v}^l \leq 150 \times 0.7895 + 175 \times 0.2105 \\ & \grave{v}^l \leq 161.05 \end{aligned} \tag{11}$$

and

$$\begin{aligned} & \max\{\acute{v}^l\} \\ \text{subject to} \quad & \acute{v}^l \leq 190 \times 0.7895 + 100 \times 0.2105 \\ & \acute{v}^l \leq 158 \times 0.7895 + 190 \times 0.2105 \\ & \acute{v}^l \geq 161.05 \end{aligned} \tag{12}$$

It can be seen that equations (11) and (12) have the optimal solutions $\grave{v}^{*l} = 155.0025$ and $\acute{v}^{*l} = 164.736$. Therefore, optimal mixed strategy and corresponding value of the game for player I are $x^{*l} = (0.7895, 0.2105)^T$ and $\tilde{v}^{*l} = (155.0025, 161.05, 164.736)$, respectively.

Similarly, it is easily computed that optimal mixed strategy and corresponding value of the game for player II are $y^{*l} = (0.2105, 0.7895)$ and $\tilde{w}^{*l} = (155.2655, 161.05, 171.055)$, respectively.

4.2 Discussion

Campos [4] solved the matrix games with payoff of TFNs and obtained crisp values of value of the game for player I and player II which is not desirable. Bector et al.'s [3] method was developed on certain duality of LP with fuzzy parameters. Campos's [4] method and Bector et al.'s [3] method are defuzzification approaches, which are close dependent on ranking functions, parameters. Moreover, two methods cannot explicitly describe the membership functions of the value of the game for two players. Also the approach followed by Vijoy et al. [22] to study fuzzy matrix games requires the two players to prescribe their aspiration levels.

However, in our proposed method and Li's [10] method, it is ensured that any matrix game with pay-offs of TFNs has TFNs type value and the membership functions of the value of the game can be explicitly defined. Although Li's model was developed on the basis of the ordering relations of TFNs. But, in our method no ranking criteria or any type of aspiration levels are required, which are difficult to do since different ranking criteria generates different solutions.

The proposed method in this paper can explicitly generates the fuzzy values of the game for player I and II as $\tilde{v}^* = (155.21, 161.05, 164.67)$ and $\tilde{w}^* = (156.56, 161.05, 166.39)$, respectively. Also Li's [10] method gives the fuzzy values of the game for player I and II as $\tilde{v}^{*l} = (155.0025, 161.05, 164.736)$ and $\tilde{w}^{*l} = (155.2625, 161.05, 171.055)$, respectively.

It can be noted that the solution of the fuzzy game in our approach is close to the solution that is reported in Li [10], although significantly different approaches are used to solve the game.

5 Conclusion

In this paper, two-person zero-sum matrix game is considered where each elements of the pay-off matrix is a TFN and a new approach is derived to solve such games based on α-cut sets of TFNs. For this, a pair of interval programming models is established for each player, which are transformed into bi-objective linear programming models based on the maximization and minimization problems with interval objective functions and using the interval inequality relations. Applying the weighted average method, two simpler auxiliary linear programming models are constructed to determine the upper and lower bounds of α-cut sets of the values of the game for player I and II respectively, for any given α. The main advantage of this method is that besides unifying the fuzzy matrix game theory it does not require any defuzzification function or any specification of aspiration levels which otherwise may be difficult to decide in practice.

However, the major limitation in this approach is that unlike crisp situations, we do not have any duality results for such linear programming approach. Therefore, these problems need a further investigation in the future.

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