International Journal of Mathematics and Soft Computing
Vol.5, No.1. (2015), 57 - 64.

ISSN Print : 2249 – 3328
ISSN Online: 2319 – 5215

Cosplitting and co-regular graphs

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Abstract

The graph $S(G)$ obtained from a graph $G(V,E)$, by adding a new vertex w for every vertex $v \in V$ and joining w to all neighbours of v in G, is called the splitting graph of G. The cosplitting graph $CS(G)$ is obtained from G, by adding a new vertex w for each vertex $v \in V$ and joining w to those vertices of G which are not adjacent to v in G. In this paper, we introduce the concept of cosplitting graph and characterise the graphs for which splitting and cosplitting graphs are isomorphic.

Keywords: Cosplitting graph, splitting graph, degree splitting graph, co – regular graph.

AMS Subject Classification (2010): 05C(Primary).

1 Introduction

Throughout this paper, we consider only finite, simple and undirected graphs. For notations and terminology, we follow [2]. A graph G is said to be *r - regular* if every vertex of G has degree r. For $r \neq k$, a graph G is said to be (r,k) – *biregular* if $d(v)$ is either r or k for any vertex v in G. A *1 – factor* of G is a 1 – regular spanning subgraph of G and it is denoted by F. For any vertex $v \in V$ in a graph $G(V,E)$, the *open neighbourhood* $N(v)$ of v is the set of all vertices adjacent to v. That is, $N(v) = \{u \in V / uv \in E\}$. The *closed neighbourhood* $N[v]$ of v is defined by $N[v] = N(v) \cup \{v\}$.

A vertex of degree one is called a *pendant vertex*. A vertex v is said to be a *k – regular adjacency vertex* (or simply a *k – RA vertex*) if $d(u) = k$ for all $u \in N(v)$. A vertex is called *an RA vertex* if it is a k – RA vertex for some $k \geq 1$. A graph G in which every vertex is an RA vertex, is said to be an *RA graph.* A *full vertex* of a graph G is a vertex which is adjacent to all other vertices of G.

Let G_1 and G_2 be any two graphs. The graph $G_1 \circ G_2$ obtained from one copy of G_1 and $|V(G_1)|$ copies of G_2 by joining each vertex in the ith copy of G_2 to the ith vertex of G_1 is called the *corona* of G_1 and G_2.

The cartesian product of G_1 and G_2 is denoted by $G_1 \times G_2$, whereas, the join of G_1 and G_2 is denoted by $G_1 \vee G_2$. $\gamma(G)$ denotes the domination number of a graph G and $\chi(G)$ denotes its chromatic number.

The concept of splitting graph was introduced by Sampath Kumar and Walikar [4]. The graph $S(G)$, obtained from G, by adding a new vertex w for every vertex $v \in V$ and joining w to all vertices of G adjacent to v, is called the s*plitting graph* of G. For example, a graph G and its splitting graph $S(G)$ are shown in Figure 1.

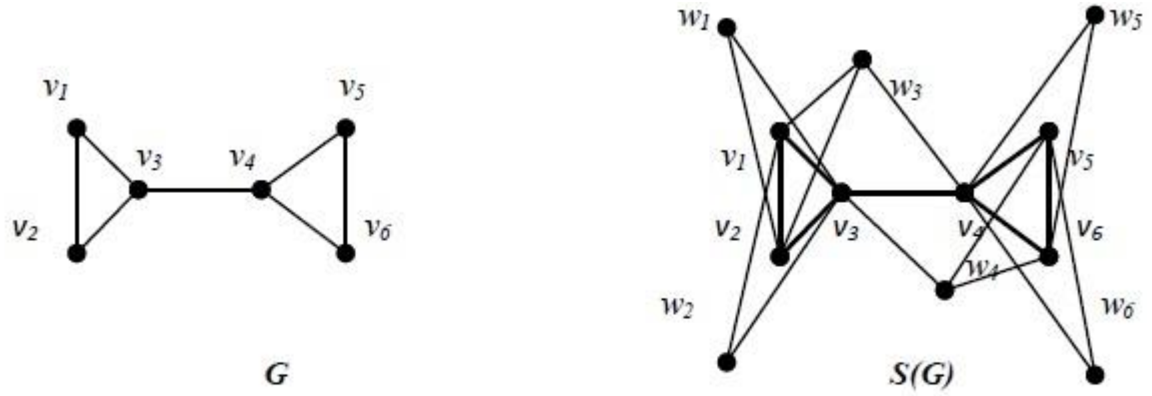

Figure 1: A graph G and its splitting graph $S(G)$.

In [4], the following result has been proved.

Result 1.1. [4] A graph G is a splitting graph if and only if $V(G)$ can be partitioned into two sets V_1 and V_2 such that there exists a bijective mapping f from V_1 to V_2 and $N(f(v)) = N(v) \cap V_1$, for any $v \in V_1$.

On a similar line, Ponraj and Somasundaram [3] have introduced the concept of degree splitting graph $DS(G)$ of a graph G. For a graph $G = (V, E)$ with vertex set partition $V_i = \{v \in V / d(v) = i\}$, the *degree splitting graph* $DS(G)$ is obtained from G, by adding a new vertex w_i for each partition V_i that contains at least two vertices and joining w_i to each vertex of V_i. For example, a graph G and its degree splitting graph $DS(G)$ are shown in Figure 2.

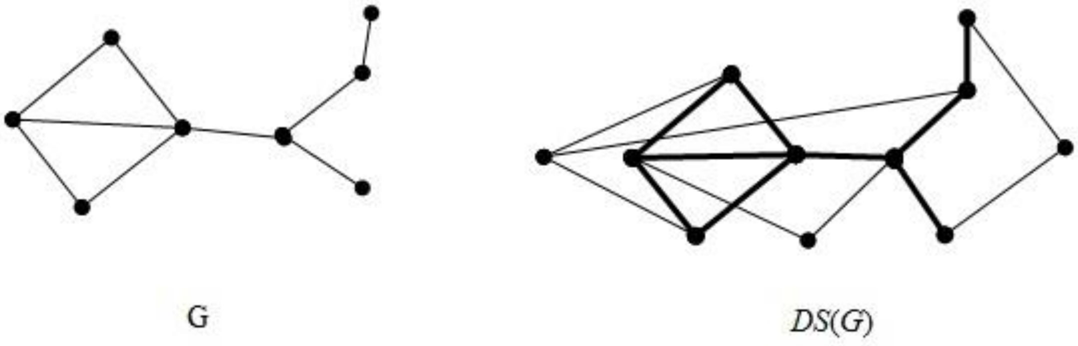

Figure 2: A graph G and its degree splitting graph $DS(G)$.

It is obvious that every graph is an induced subgraph of $DS(G)$. The following results on $DS(G)$ have been proved in [1]:

Result 1.2. [1] The degree splitting graph $DS(G)$ is regular if and only if $G \cong K_r$, $r \geq 1$or $(K_{2k} - F) \vee K_1$, where F is a 1-factor of K_{2k} and $k \geq 1$.

If $K_{n,2n+1}$ is the complete bipartite graph with bipartition (X,Y) where $X = \{v_1, v_2, ..., v_n\}$ and $Y = \{w_1, w_2, ..., w_{2n+1}\}$, then $K^*_{n,2n+1}$ denotes the graph obtained from $K_{n,2n+1}$ by deleting the edges v_iw_{2i-1} and v_iw_{2i} for all i, $1 \leq i \leq n$.

Result 1.3. [1] Let G be a connected graph. Then $DS(G)$ is a biregular RA graph if and only if $G \cong K_{1,n}$ or $K^*_{n,2n+1}$, where $n \geq 2$.

Result 1.4. [1] For any $n \geq 2$, there are n non isomorphic graphs whose degree splitting graphs are all isomorphic.

We define the cosplitting graph $CS(G)$ of a graph G as follows:

Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The *cosplitting graph* $CS(G)$ is the graph obtained from G, by adding a new vertex w_i for each vertex v_i and joining w_i to all vertices which are not adjacent to v_i in G. For example, a graph G and its cosplitting graph $CS(G)$ are shown in Figure 3.

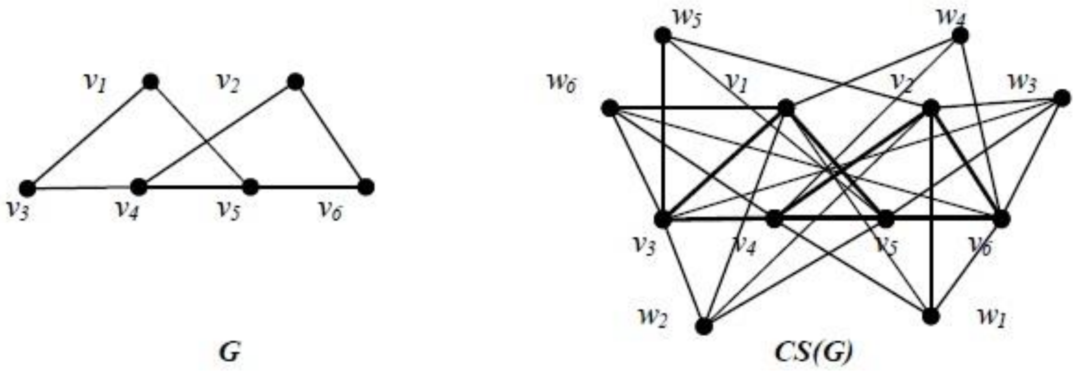

Figure 3: A graph G and its cosplitting graph $CS(G)$.

In this paper, we characterise the graphs for which the cosplitting graph is regular, biregular or bipartite. Also we give a necessary and sufficient condition for a graph to be a cosplitting graph. And finally we characterise the graphs for which the splitting graph and the cosplitting graph are isomorphic.

2 Properties of Cosplitting Graph

Let $K(m,n)$ denote the bipartite graph with vertex set bipartition (X,Y) where $X = \{u_1, u_2, ..., u_{m+n}\}$ and $Y = \{v_1, v_2, ..., v_{m+n}\}$ and edge set $E(K(m,n)) = \{u_iv_j \,/\, 1 \leq i \leq m \text{ and } 1 \leq j \leq m+n\} \cup \{u_iv_j \,/\, 1 \leq i \leq m+n \text{ and } 1 \leq j \leq n\}$. For example, the graph $K(2,3)$ is shown in Figure 4.

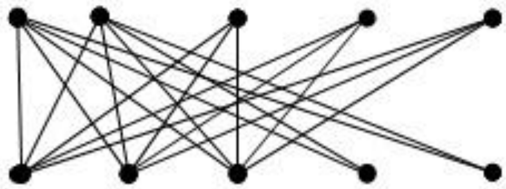

Figure 4: The graph $K(2,3)$.

For any graph G of order n, clearly $CS(G)$ contains $2n$ vertices. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and $w_1, w_2, \ldots, w_n$ be the corresponding newly added vertices in $CS(G)$. Let $d'(v)$ and $d^*(v)$ denote the degrees of a vertex v in $CS(G)$ and $S(G)$ respectively.

For the cosplitting graph $CS(G)$, the following results can be easily verified:

Result 2.1. $d'(v_i) = n$ and $d'(w_i) + d(v_i) = n$, for all i, $1 \leq i \leq n$.

Result 2.2. If G has n vertices and m edges, then $CS(G)$ has $2n$ vertices and $n^2 - m$ edges.

Result 2.3. For a connected graph G, $1 \leq d'(w_i) \leq n - 1$. $d'(w_i) = 1$ implies that v_i is a full vertex in G and $d'(w_i) = n - 1$ implies that v_i is a pendant vertex in G.

It is important to note that Result 2.3 is also true for any disconnected graph G unless G contains an isolated vertex. In other words, $d'(w_i) = n$ if and only if v_i is an isolated vertex. Hence $\Delta(CS(G)) = n$. Also $CS(G)$ contains $n + m$ vertices of degree n, if and only if G contains m isolated vertices. Let them be denoted by $u_1, u_2, \ldots, u_m$. Note that in such case, $CS(G)$ contains $K_{m,m}$ as an induced subgraph. The removal of the $2m$ vertices that induces $K_{m,m}$ from $CS(G)$ results in a graph which is isomorphic to $CS(G \setminus \{u_1, u_2, \ldots, u_m\})$.

Result 2.4. $CS(K_n) \cong K_n \circ K_1$, $CS(K_n^c) \cong K_{n,n}$ and $CS(K_{m,n}) \cong K(m,n)$.

It is easy to observe that $G \circ K_1$ is a spanning subgraph of $CS(G)$ and $G \circ K_1 = CS(G)$ if and only if $G \cong K_n$.

Result 2.5. Every graph G is an induced subgraph of its cosplitting graph $CS(G)$.

Result 2.6. In $CS(G)$, the subgraph induced by the set of all vertices of degree n is isomorphic to G.

Result 2.7. For any graph G, the cosplitting graph $CS(G)$ is always connected. But in case of splitting graph, $S(G)$ is connected if and only if G is connected.

Result 2.8. The cosplitting graph $CS(G)$ is r - regular if and only if $G \cong K_r^c$.

Result 2.9. The cosplitting graph $CS(G)$ is $(r, n - r)$ – biregular if and only if G is an r – regular graph for any positive integer r.

Result 2.10. In the cosplitting graph of a connected graph, every newly added vertex that corresponds to a non - full vertex lies on at least one new cycle.

Result 2.11. For any graph G, $\chi(CS(G)) = \chi(G)$ or $\chi(G) + 1$.

The following theorem gives a characterisation of cosplitting graphs.

Theorem 2.12. A graph G is a cosplitting graph if and only if $V(G)$ can be partitioned into two sets V_1 and V_2 such that there exists a bijection f from V_1 to V_2 which satisfies the following conditions: (i) $N(v) \cup N(f(v)) = V \setminus f(N(v))$ and

(ii) $N(v) \cap N(f(v)) = \phi$, for any $v \in V_1$.

Proof: Let G be a cosplitting graph of a graph H. To construct G from H, we add a new vertex w for each vertex v of H and join w with every vertex of H which is not adjacent to v. Let $V_1 = V(H)$ and $V_2 = V(G) \setminus V(H)$. For $v_i \in V_1$, let $w_i \in V_2$, be the corresponding newly added vertex where $1 \leq i \leq |V_1|$.

Now define a function $f : V_1 \to V_2$ by $f(v_i) = w_i$, $1 \leq i \leq |V_1|$. Then clearly f is a bijection from V_1 onto V_2. Also by definition $N(f(v_i)) = V_1 \setminus N(v_i)$. Hence (ii) is proved. In H, each v_i is adjacent not only to its neighbours in G, but also to all newly added vertices corresponding to its non-neighbours. Therefore we get $N(v_i) \cup N(f(v_i)) = V \setminus f(N(v_i))$.

Conversely, let the given conditions be true for a graph G. Let H be the subgraph of G induced by V_1. We claim that $CS(H) \cong G$. Since f is bijective, it is clear that for every vertex v_i in H, there is a unique vertex $f(v_i)$ in $G \setminus H$. Also by the assumptions (i) and (ii), v_i and $f(v_i)$ are adjacent for every i, $1 \leq i \leq n$ and every vertex in V_1 is a neighbour of either v_i or $f(v_i)$ but not both.

Let us prove that $< G \setminus H >$ contains no edge. Suppose not, let $f(v_i)$ and $f(v_j)$ be adjacent for some $i \neq j$. Then by assumption (ii), $f(v_j) \notin N(v_i)$. In other words, $v_i \notin N(f(v_j))$ which implies that $v_i \in N(v_j)$ which is a contradiction to (i) since $N(v_i) \cup N(f(v_i))$ does not contain any vertex of $f(N(v_i))$. Therefore $< G \setminus H >$ is a null graph. Hence if we consider $f(v_i)$ to be the corresponding newly added vertex for v_i, then G is the cosplitting graph of H. ■

The following theorem characterises all bipartite cosplitting graphs.

Theorem 2.13. For any graph G, $CS(G)$ is bipartite if and only if $G \cong K_{m,n}$ or K_n^c.

Proof: Let G be any graph for which CS(G) is bipartite. Since G is an induced subgraph of $CS(G)$, G is also bipartite. Let (X,Y) be the bipartition of G.

Case (i): Suppose G is connected. Let $x \in X$ and $y \in Y$. We claim that x and y are adjacent in G. Suppose not, then there exists an (x, y) – path P of odd length in G. Also the newly added vertex w corresponding to x, is adjacent to both x and y in $CS(G)$. Therefore the path P together with the edges xw and wy forms a cycle of odd length in $CS(G)$, which is a contradiction. Therefore every $x \in X$ is adjacent to any $y \in Y$ in G and we have $G \cong K_{m,n}$.

Case (ii): Suppose G is disconnected. If $G \ncong K_n^c$, then there is a component, say G_1 of G containing at least one edge xy. Let v be a vertex of G not in G_1 and let w be the newly added vertex corresponding to v in $CS(G)$. Clearly w is adjacent to both x and y in $CS(G)$. Thus $wxyw$ forms a triangle in $CS(G)$. This is a contradiction to the assumption that $CS(G)$ is bipartite. Hence $G \cong K_n^c$.

Conversely if $G \cong K_{m,n}$ or K_n^c, then $CS(G) \cong K(m,n)$ or $K_{n,n}$ respectively and hence the result follows. ■

Corollary 2.14. $CS(G)$ is a tree if and only if $G \cong K_{1,1}$ or K_1.

Proof: Suppose $CS(G)$ is a tree. Then $CS(G)$ is bipartite and G is acyclic. Therefore, by the above theorem, $G \cong K_{1,1}$ or K_1. And the converse is obvious. ■

From the above corollary, P_2 and P_4 are the only cosplitting trees.

Next we prove that $K_3 \circ K_1$ and C_4 are the only unicyclic cosplitting graphs.

Theorem 2.15. The cosplitting graph $CS(G)$ of a graph G is unicyclic if and only if $G \cong K_3$ *or* K_2^c.

Proof: Let G be any graph such that $CS(G)$ is unicyclic with the cycle C. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and $w_1, w_2, \ldots, w_n$ be the corresponding newly added vertices in $CS(G)$. Since $\{w_1, w_2, \ldots, w_n\}$ is independent, either $V(C) \subseteq V(G)$ or $w_i \in V(C)$ for some i.

Case (i): Suppose $V(C) \subseteq V(G)$.

It is clear that the cosplitting graph of a disconnected graph other than K_n^c contains more than one triangle. Hence G must be connected. Also by Result 2.10, every vertex of G is a full vertex and therefore the newly added vertices do not form any new cycle. Hence, $G \cong K_3$.

Case (ii): Suppose $w_i \in V(C)$ for some i.

Then G is acyclic and so every component of G is a tree. Since $CS(G)$ is unicyclic, by Result 2.10 every component of G contains only one non full vertex. This is possible only when G is empty. If G contains more than two isolated vertices, then $CS(G)$ is not unicyclic. Thus $G \cong K_2^c$.

Conversely, the cosplitting graphs of K_3 and K_2^c are $K_3 \circ K_1$ and C_4 respectively which are unicyclic. ■

Theorem 2.16. No two non – isomorphic graphs can have the same cosplitting graph.

Proof: Suppose there are two non-isomorphic graphs G_1 and G_2 such that $CS(G_1) \cong CS(G_2)$.

Case (i): Suppose G_1 has no isolated vertex. Then by Result 2.3, no newly added vertex in $CS(G_1)$ is of degree n. Therefore the subgraph induced by the set of all vertices of degree n in $CS(G_1)$ is isomorphic to G_1. Since $CS(G_1) \cong CS(G_2)$, we have $CS(G_2)$ also contains exactly n vertices of degree n, and the subgraph induced by them is isomorphic to G_2. This implies that $G_1 \cong G_2$, a contradiction.

Case (ii): Let $G_1 = H_1 \cup K_m^c$, where H_1 contains no isolated vertex. Then $CS(G_1)$ contains $n + m$ vertices of degree n and it contains $K_{m,m}$ as an induced subgraph. Since $CS(G_1) \cong CS(G_2)$, it is clear that $CS(G_2)$ also contains $n + m$ vertices of degree n. Therefore, $G_2 = H_2 \cup K_m^c$, for some graph H_2 which contains no isolated vertex. From Result 2.3, by removing $2m$ vertices that induces $K_{m,m}$ in $CS(G_1)$ and $CS(G_2)$, we get $CS(H_1)$ and $CS(H_2)$ respectively. This implies that $CS(H_1) \cong CS(H_2)$. Now using Case (i), we conclude that $H_1 \cong H_2$ and so $G_1 \cong G_2$, which is again a contradiction. Hence the result follows. ■

3 Co-regular Graphs

In this section, we define a new type of graphs called co – regular graphs and prove that co – regular graphs are the only graphs for which splitting and cosplitting graphs are isomorphic.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then the *co – regular graph* of G denoted by $CR(G)$ is the graph with vertex set $V(CR(G)) = \{u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\}$ and edge set $E(CR(G)) = \{u_iu_j, w_iw_j / v_iv_j \in E(G), i \neq j \text{ and } 1 \leq i, j \leq n\} \cup \{u_iw_j / v_iv_j \notin E(G) \text{ and } 1 \leq i, j \leq n\}$.

For example, a graph G and its co-regular graph $CR(G)$ are shown in Figure 5.

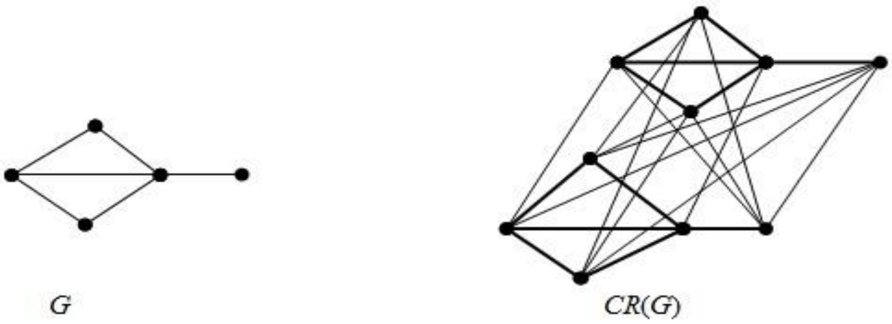

Figure 5: A graph G and its co-regular graph $CR(G)$.

The following results can be easily verified for a co-regular graph:

Result 3.1. A co – regular graph is an n – regular graph on $2n$ vertices.

Result 3.2. $G \times P_2$ is a spanning subgraph of $CR(G)$. In particular, $CR(K_n) = K_n \times P_2$.

Result 3.3. $CR(K_n^c) = K_n^c \vee K_n^c = K_{n,n}$.

Result 3.4. For any graph G, $CR(G)$ is connected.

For, if G is connected since $G \times P_2$ is a spanning subgraph of $CR(G)$, then $CR(G)$ is also connected. If G is disconnected, then every vertex in each component of one copy of G is adjacent to all vertices in the other components of another copy of G and hence $CR(G)$ is connected.

Result 3.5. For any graph G, $\gamma(CR(G)) = 2$.

For, $CR(G)$ does not contain a full vertex and hence $\gamma(CR(G)) \neq 1$, and $\{u_i, w_i\}$ is a minimum dominating set of $CR(G)$ for any i, $1 \leq i \leq n$.

Theorem 3.6. A graph G is co – regular if and only if its vertex set can be partitioned into two element subsets $\{u_i, w_i\}$, $1 \leq i \leq n$, such that for any i, $N(u_i)$ and $N(w_i)$ form a partition of $V(G)$, that is, such that $N(u_i) \cup N(w_i) = V(G)$ and $N(u_i) \cap N(w_i) = \phi$, for every $i = 1, 2, \ldots, n$.

Proof: Let G be the co – regular graph of some graph H. Let $V(G) = \{u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\}$ such that $<\{u_1, u_2, \ldots, u_n\}> \cong <\{w_1, w_2, \ldots, w_n\}> \cong H$. Without loss of generality, let u_i be the isomorphic image of w_i. Consider the pair $\{u_i, w_i\}$. By the definition of co – regular graph, any vertex u_j, $1 \leq j \leq n$, $i \neq j$, is adjacent to either u_i or w_i but not both. Similar condition holds with any w_j, $1 \leq j \leq n$, $i \neq j$. Since u_i and w_i are adjacent, $u_i \in N(w_i)$ and $w_i \in N(u_i)$. Therefore, the neighbour sets of u_i and w_i form a partition of $V(G)$.

Conversely, suppose the vertex set of any graph G can be partitioned into two element subsets such that any vertex in G is a neighbour of any one vertex but not to both in each subset. Therefore G contains even number of vertices. Let $V(G) = \{u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\}$ such that $\{u_1,w_1\}$, $\{u_2,w_2\}$, ..., $\{u_n,w_n\}$ be the partition of V(G).

First we claim that $<\{u_1, u_2, \ldots, u_n\}> \cong <\{w_1, w_2, \ldots, w_n\}>$. Suppose u_r is adjacent to u_s. Then $u_s \notin N(w_r)$ and hence $w_r \in N(w_s)$. In a similar way, we prove that if u_r and u_s are non adjacent, then w_r and w_s are non adjacent. Since r and s are arbitrary, $<\{u_1, u_2, \ldots, u_n\}> \cong <\{w_1, w_2, \ldots, w_n\}> \cong H$, say.

For $1 \le i \le n$, since $N(u_i) \cup N(w_i) = V(G)$, we have $u_i \in N(w_i)$. Hence, u_i is adjacent to w_i. Also since $N(u_i) \cap N(w_i) = \phi$, both u_i and w_i have no common neighbours. Combining the two conditions we get $[N(u_i)]^c = N(w_i)$. Thus we conclude that $G = CR(H)$. ■

Theorem 3.7. Let G be any graph of order n. Then $S(G) \cong CS(G)$ if and only if $G \cong CR(H)$ for some graph H.

Proof: Let G be any graph of order n such that its splitting graph $S(G)$ is isomorphic to its cosplitting graph $CS(G)$. Hence by Result 2.7, G is connected. For any vertex u in G, $d^*(u) = 2d(u)$ and $d'(u) = n$. Since $S(G) \cong CS(G)$, we have $d(u) = n/2$ for all $u \in V(G)$. That is, G is an $n/2$ – regular graph on n vertices.

Let $V(G) = \{u_1,u_2,...,u_n\}$ and let $v_1, v_2, ..., v_n$ be the newly added vertices in $S(G)$. From the definition of splitting graph, for every vertex v_i, there exists a unique vertex $u_k \notin N(v_i)$ in G such that $N(u_k) \cap V(G) = N(v_i)$ by Result 1.1. Since $S(G) \cong CS(G)$, there will be a one to one correspondence between the newly added vertices in $S(G)$ and $CS(G)$. Therefore from the definition of cosplitting graph, corresponding to every v_i, there exists a unique vertex $u_m \in N(v_i)$ in G such that $N(v_i) = V(G) \setminus N(u_m)$ by Theorem 2.12.

Combining the above two conditions we get $N(u_m) \cup N(u_k) = V(G)$, $N(u_m) \cap N(u_k) = \phi$. Then clearly u_k and u_m are adjacent. Thus u_k and u_m are two adjacent vertices in G, whose neighbour sets form a partition of $V(G)$. In a similar manner, we can pair off vertices of G such that each pair has distinct neighbour set whose union is $V(G)$ itself. Thus by the above theorem, G is isomorphic to $CR(H)$ for some H.

Conversely, assume that G is a co – regular graph of a graph H. Let $V(G) = \{u_1, u_2, ..., u_n, w_1, w_2, ..., w_n\}$ such that $<\{u_1, u_2, ..., u_n\}> \cong <\{w_1, w_2, ..., w_n\}> \cong H$. Without loss of generality, let u_i be the isomorphic image of w_i. Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ and $c_1, c_2, ..., c_n, d_1, d_2, ..., d_n$ be the newly added vertices in $S(G)$ and $CS(G)$ respectively corresponding to the vertices $u_1, u_2, ..., u_n, w_1, w_2, ..., w_n$. Then a function $f : S(G) \to CS(G)$ defined by $f(u_i) = u_i$, $f(w_i) = w_i$, $f(a_i) = d_i$, $f(b_i) = c_i$, where $1 \le i \le n$, can be easily verified to be an isomorphism. Hence the theorem is proved. ■

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