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Triple simultaneous Fourier series equations involving associated Legendre function

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Abstract

In this paper, we have considered the two sets of triple series equations of the first kind and second kind, which are the generalization of dual series equations involving Associated Legendre functions and solved the two sets of series equations involving a spherical or spheroidal boundary when different conditions are prescribed on different parts of the boundary.

Keywords: Legendre functions, Series equations, Fourier series, associated Legendre polynomials.

AMS Subject Classification (2010): 42A16.

1 Introduction

Triple series equations are useful in finding the solution of mixed boundary value problems of electrostatics, elasticity and other fields of mathematical physics [8]. Collins [2] solved the triple series equations, for the first time and since then authors like Lowndes [5] Dwivedi and Gupta [3, 4], Parihar [7], Melrose and Tweed [6] obtained the solutions of the triple series equations involving different types of special polynomials.

In this paper, we are concerned with triple series equations of the form:

(i) Triple Series Equations of the First Kind

$$\sum_{n=0}^{\infty}(2n+2m+1)A_n T_{m+n}^{-m}(cos\theta) = f_1(\theta), \qquad 0 \le \theta < \alpha \tag{1.1}$$

$$\sum_{n=0}^{\infty} A_n T_{m+n}^{-m}(cos\theta) = f_2(\theta), \qquad \alpha \le \theta < \beta \tag{1.2}$$

$$\sum_{n=0}^{\infty}(2n+2m+1)A_n T_{m+n}^{-m}(cos\theta) = f_3(\theta), \qquad \beta \le \theta < \pi \tag{1.3}$$

(ii) Triple Series Equation of the Second Kind

$$\sum_{n=0}^{\infty} B_n T_{m+n}^{-m}(cos\theta) = g_1(\theta), \qquad 0 \le \theta < \alpha \tag{1.4}$$

$$\sum_{n=0}^{\infty} (2n+2m+1) B_n T_{m+n}^{-m}(Cos\theta) = g_2(\theta), \qquad \alpha \le \theta < \beta \tag{1.5}$$

$$\sum_{n=0}^{\infty} B_n T_{m+n}^{-m}(cos\theta) = g_3(\theta), \qquad \beta \le \theta < \pi \tag{1.6}$$

In above equations $f_i(\theta)$ and $g_i(\theta)$, i =1, 2, 3 are prescribed functions of the variable θ and the equations (1.1) to (2.3) are to be solved for the unknown coefficients A_n and B_n. $T_{m+n}^{-m}(\cos\theta)$ is the associated Legendre function of degree $n + m$ and order $-m$ of the first kind.

2 Preliminary Results

2.1 Inversion Theorem for Associated Legendre Polynomials

If the expansion

$$f(\theta) = \sum_{n=0}^{\infty} (2n+2m+1) C_n T_{m+n}^{-m}(cos\theta) \tag{2.1}$$

is valid for $0 \le \theta < \pi$ and that it can be integrated term by term, the coefficient C_n are given by

$$C_n = \frac{1}{2}(-1)^m \int_0^x f(x) T_{m+n}^{-m}(cosx) sinxdx \tag{2.2}$$

Also we have the result,

$$S_m(\theta,\ \mathrm{x}) = \sum_{n=0}^{\infty} T_{m+n}^{-m}(cos\theta) T_{m+n}^{m}(cosx) \tag{2.3}$$

$$= \frac{(-1)^m}{2\pi} \sum_{n=0}^{\infty} \int_0^{2x} P_n(cosr) cosm\Psi d\Psi \tag{2.4}$$

$$= \frac{2(-1)^m}{\pi (s_1 s_2)^m} \int_0^{\min(s_1,\, s_2)} \frac{s^{2m} ds}{\left[\left(s_1^2 - s^2\right)\left(s_2^2 - s^2\right)\right]^{1/2}} \tag{2.5}$$

where $S_1 = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}x$, $S_2 = 2\sin\frac{x}{2}\cos\frac{\theta}{2}\theta$ and both S_1 and S_2 are positive for all θ and x, since $0 < \theta, x < \pi$ and $S = 2\cos\frac{\theta}{2}\cos\frac{x}{2}\tan\frac{u}{2}$.

Finally, in order to get an integral representation of the associate Legendre polynomial $T_{m+n}^{-m}(\cos\theta)$, we make use of the following results:

If $f(\theta) = \int_{\alpha}^{\theta} \frac{g(u)}{(\cos u - \cos\theta)^{\frac{1}{2}}} du$, $\alpha < \theta < \beta$ (2.6) and $$f'(\theta)=\int_{\theta}^{\beta}\frac{g'(u)}{(\cos\theta-\cos u)^{\frac{1}{2}}}du,\ \alpha<\theta<\beta \tag{2.7}$$

then $$g(u)=\frac{1}{\pi}\frac{d}{du}\int_{\alpha}^{u}\frac{f(\theta)\sin\theta}{(\cos\theta-\cos u)^{\frac{1}{2}}}d\theta,\ \alpha<u<\beta \tag{2.8}$$

and $$g'(u)=\frac{1}{\pi}\frac{d}{du}\int_{u}^{\beta}\frac{f'(\theta)\sin\theta}{(\cos u-\cos\theta)^{\frac{1}{2}}}d\theta \quad ,\ \alpha<u<\beta \tag{2.9}$$

3 The Solutions of the Triple Series

3.1 Equations of the First kind

In order to solve the triple series equations of the first kind, we set

$$\sum_{n=0}^{\infty}(2n+2m+1)A_n T_{m+n}^{-m}(cos\theta)=h(\theta), \qquad \alpha\le\theta<\beta \tag{3.1.1}$$

Using Equations (2.1) and (2.2) in equation (1.1), (1.3) and (3.1.1), we get

$$A_n=\frac{1}{2}(-1)^m\left[\int_0^{\alpha}f_1(x)+\int_{\alpha}^{\beta}h(x)+\int_{\beta}^{\pi}f_3(x)\right]T_{m+n}^{-m}(cosx)sinxdx \tag{3.1.2}$$

Substituting the expression (3.1.2) for A_n in equation (1.5), we get

$$\sum_{n=0}^{\infty}T_{m=n}^{-m}(cos\theta)\frac{1}{2}(-1)^m\left[\int_0^{\alpha}f_1(x)+\int_{\alpha}^{\beta}h(x)+\int_{\beta}^{\pi}f_3(x)\right],$$

$$\times T_{m+n}^{m}(cosx)sinxdx=f_2(\theta), \qquad \alpha\le\theta<\beta \tag{3.1.3}$$

Interchanging the order of summation and integration, we get

$$\frac{1}{2}(-1)^m\left[\int_0^{\alpha}f_1(x)+\int_{\alpha}^{\beta}h(x)+\int_{\beta}^{\pi}f_3(x)\right]\sum_{n=0}^{\infty}T_{m+n}^{-m}(cos\theta)T_{m+n}^{m}(cosx)\times sinxdx=f_2(\theta),\ \alpha\le\theta<\beta \tag{3.1.4}$$

Using the expression given by (2.3) in (3.1.4), we obtain

$$\frac{1}{2}(-1)^m\int_{\alpha}^{\beta}h(x)S_m(\theta,\ \mathrm{x})sindx=F(\theta), \qquad \alpha<\theta<\beta \tag{3.1.5}$$

where $$F(\theta)=f_2(\theta)-\frac{1}{2}(-1)^m\left[\int_0^{\alpha}f_1(x)S_m(\theta,\ \mathrm{x})sindx+\int_{\beta}^{\pi}f_3(x)S_m(\theta,\ \mathrm{x})sindx\right] \tag{3.1.6}$$

Now using the summation result in terms of integral (2.5) in the equation (3.1.5), we get

$$\frac{2.\frac{1}{2}(-1)^{2m}}{\pi(s_1s_2)^m}\left[\int_{\alpha}^{\beta}h(x)\int_0^{\min(s_1,\,s_2)}\frac{s^{2m}ds}{\left[\left(s_1^2-s^2\right)\left(s_2^2-s^2\right)\right]^{1/2}}sindx\right]=F(\theta),\ \alpha<\theta<\beta \tag{3.1.7}$$

After breaking into parts, and noting that $S_1>S_2$ when $\theta>x$ and $S_1<S_2$ when $x>\theta$ we have

$$\frac{sin^{-m}\theta}{\pi}\left[\int_{\alpha}^{\theta} h(x)sin^{1-m}x \times \int_0^{s_2} \frac{s^{2m}ds}{\left[\left(s_1^2-s^2\right)\left(s_2^2-s^2\right)\right]^{1/2}}dx\right.$$
$$\left.+\int_{\theta}^{\beta} h(x)sin^{1-m}x \times \int_0^{s_1} \frac{s^{2m}ds}{\left[\left(s_1^2-s^2\right)\left(s_2^2-s^2\right)\right]^{1/2}}dx\right] = F(\theta),\ \alpha<\theta<\beta \quad (3.1.8)$$

Making change of variables by putting the values of s, s_1 and s_2, we get

$$\frac{sin^{-m}\theta}{\pi}\left[\int_{\alpha}^{\theta} h(x)sin^{1-m}xdx.\frac{1}{2}\int_0^{x} \frac{\left(2\cos\frac{1}{2}\theta.\cos\frac{1}{2}x.\tan\frac{1}{2}u\right)^{2m}}{\left[(\cos u-\cos\theta)(\cos u-\cos x)\right]^{1/2}}du\right.$$
$$\left.+\int_{\theta}^{\beta} h(x)sin^{1-m}xdx.\frac{1}{2}\int_0^{\theta} \frac{\left(2\cos\frac{1}{2}\theta.\cos\frac{1}{2}x.\tan\frac{1}{2}u\right)^{2m}}{\left[(\cos u-\cos\theta)(\cos u-\cos x)\right]^{1/2}}du\right] = F(\theta),\ \alpha<\theta<\beta \quad (3.1.9)$$

Now inverting the order of integration, we get

$$\frac{2^{-m}sin^{-m}\frac{\theta}{2}.cos^{-m}\frac{\theta}{2}.2^{2m}cos^{2m}\frac{\theta}{2}}{2\pi}\left[\int_0^{\alpha}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\times\int_{\alpha}^{\theta}\frac{2^{-m}h(x)cot^{+m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx\int_{\alpha}^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\right.$$
$$\times\int_{\alpha}^{\theta}\frac{2^{-m}h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx+\int_0^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du$$
$$\left.\times\int_{\theta}^{\beta}\frac{2^{-m}h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx\right]=F(\theta),\qquad \alpha<\theta<\beta \quad (3.1.10)$$

Breaking the last term in to part, on the left hand side, we get

$$\frac{2^{2m}\tan^{-m}\frac{1}{2}\theta cos^{2m}\frac{1}{2}\theta}{2\pi}\left\{\int_0^{\alpha}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\times\int_{\alpha}^{\theta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx+\int_{\alpha}^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\right.$$
$$\times\int_{u}^{\theta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx+\int_0^{\alpha}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\times\int_{\theta}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx+\int_{\alpha}^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du$$
$$\left.\times\int_{\theta}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx\right\}=F(\theta),\quad \alpha<\theta<\beta \quad (3.1.11)$$

$$\frac{1}{2\pi}\int_{\alpha}^{\theta}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\int_{u}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx \qquad =\tan^{m}\frac{\theta}{2}.F(\theta)-\frac{1}{2\pi}\int_{0}^{\alpha}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du$$

$$\int_{\alpha}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx,\quad \alpha<\theta<\beta \tag{3.1.12}$$

The above equation can be rewritten as

$$\int_{\alpha}^{\theta}\frac{H(u)\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du=F(\theta)\tan^{m}\frac{\theta}{2}-\frac{1}{2\pi}\int_{0}^{\alpha}\frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\int_{\alpha}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx,\qquad \alpha<\theta<\beta$$

(3.1.13)

where, $H(u)=\frac{1}{2\pi}\int_{u}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx,$ (3.1.14)

In view of the equations (2.6) and (2.8), equation (3.1.13) gives

$$H(u)=\frac{cot^{2m}\frac{u}{2}}{\pi}\frac{d}{du}\int_{\alpha}^{u}\frac{sin\theta d\theta}{(cos\theta-cosu)^{1/2}}\Bigg\{\tan^{m}\frac{1}{2}\theta F(\theta)$$

$$-\frac{1}{2\pi}\int_{0}^{\alpha}\frac{\tan^{2m}\frac{\theta}{2}}{(cosv-cos\theta)^{1/2}}d\vartheta\int_{\alpha}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosv-cosx)^{1/2}}dx\Bigg\}\alpha<\theta<\beta \tag{3.1.15}$$

Changing the order of integration of the above equation, we get

$$H(u)=F_{1}(u)-\frac{cot^{2m}\frac{u}{2}}{2\pi^{2}}\Bigg[\int_{0}^{\alpha}\tan^{2m}\frac{\theta}{2d\theta}\times\frac{d}{du}\int_{\alpha}^{u}\frac{sin\theta d\theta}{(cos\theta-cos\theta)^{1/2}(cos\theta-cosu)^{1/2}}$$

$$\times\int_{\alpha}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cos\vartheta-cosx)^{1/2}}dx\Bigg],\times\int_{\alpha}^{\beta}\frac{h(x)cot^{m}\frac{x}{2}.sinx}{(cosv-cosx)^{1/2}}dx\Bigg],\quad \alpha<\theta<\beta \tag{3.1.16}$$

where, $F_{1}(u)=\frac{cot^{2m}\frac{u}{2}}{\pi}\frac{d}{du}\int_{\alpha}^{u}-\frac{F(\theta).\tan^{m}\frac{\theta}{2}.sin\theta}{(cos\theta-cosu)^{1/2}}d\theta$ (3.1.17)

Using the result given by

$$\frac{d}{du}\int_{\alpha}^{u}\frac{sin\theta d\theta}{(cosv-cos\theta)^{1/2}(cos\theta-cosu)^{1/2}}=\frac{sinu(cos\vartheta-cos\alpha)^{1/2}}{(cos\alpha-cosu)^{1/2}(cosv-cosu)} \tag{3.1.18}$$

Equation (3.1.16) becomes

$$H(u)=F_1(u)-\frac{cot^{2m}\frac{u}{2}.sinu}{2\pi^2(cos\alpha-cosu)^{1/2}}\left[\int_0^{\alpha}\frac{\tan^{2m}\vartheta(cos\vartheta-cos\alpha)^{1/2}}{(cos\vartheta-cosu)}d\vartheta\times\int_{\alpha}^{\beta}\frac{h(x)cot^m\frac{x}{2}.sinx}{(cos\vartheta-cosx)^{1/2}}dx\right],\ \alpha<\theta<\beta \tag{3.1.19}$$

Now, in view of the equations (3.1.6) and (2.9), the solution of the equation (3.1.14) is given by

$$h(x)cot^m\frac{1}{2}xsinx=-2\frac{d}{dx}\int_x^{\beta}\frac{H(u)sinu}{(cosx-cosu)^{1/2}}du \tag{3.1.20}$$

Using equation (3.1.20), we obtain

$$\int_{\alpha}^{\beta}\frac{h(x)cot^m\frac{1}{2}x.sinx}{(cos\vartheta-cosx)^{1/2}}dx=\frac{2}{(cos\vartheta-coss)^{1/2}}\times\int_{\alpha}^{\beta}\frac{H(s)sinsds}{(cos\alpha-coss)^{1/2}(cos\vartheta-coss)}du \tag{3.1.21}$$

From equations (3.1.1) and (3.1.19), we get

$$H(u)=F_1(u)-\frac{cot^{2m}\frac{1}{2}u.sinu}{\pi^2(cos\alpha-cosu)^{1/2}}\int_0^{\alpha}\frac{\tan^{2m}\frac{1}{2}\theta(cos\theta-cos\alpha)^{1/2}}{(cos\theta-cosu)}d\theta$$

$$\times\frac{1}{(cos\theta-cos\alpha)^{1/2}}\int_{\alpha}^{\beta}\frac{H(s)sinsds}{(cos\alpha-coss)^{1/2}(cos\theta-coss)},\ \alpha<\theta<\beta \tag{3.1.22}$$

Changing the order of integration in equation (3.1.22), we obtain

$$H(u)=F_1(u)-\frac{cot^{2m}\frac{1}{2}u.sinu}{\pi^2}\int_{\alpha}^{\beta}\frac{H(s).\sin ssinuds}{(cos\alpha-coss)^{1/2}(cos\alpha-cosu)^{1/2}}$$

$$\times\int_0^{\alpha}\frac{\tan^{2m}\frac{1}{2}\theta d\theta}{(cos\theta-coss)(cos\theta-cosu)},\quad \alpha<\theta<\beta \tag{3.1.23}$$

Equation (3.1.23) can now be rewritten as

$$H(u)+\frac{1}{\pi^2}\int_{\alpha}^{\beta}H(s)A(s,\ \mathrm{u})ds=F_1(u),\quad \alpha<\theta<\beta \tag{3.1.24}$$

where $$A(s,\ u)=\frac{cot^{2m}\frac{1}{2}u.\sin usins}{(cos\alpha-coss)^{1/2}(cos\alpha-cosu)^{1/2}}B(s,\ \mathrm{u}) \tag{3.1.25}$$

and $$B(s,\ \mathrm{u})=\int_0^{\alpha}\frac{\tan^{2m}\frac{1}{2}\theta}{(cos\theta-coss)(cos\theta-cosu)}d\theta \tag{3.1.26}$$

Equation (3.1.24) is a Fredholm integral equation of the second kind which determines H(u). h(x) can be found from equation (3.1.20). Finally, the value of unknown coefficients A_n can be computed with the help of the equation (3.1.2) which satisfies the equations (1.1), (1.2) and 1.3).

PARTICULAR CASE

If we put α=0 in equations (1.1), (1.2) and (1.3), they reduce to the dual series equations and the solution obtained agrees with the solution obtained for dual equations.

3.2 Equations of the Second kind

To solve the triple series equations (1.4), (1.5) and (1.6), let us suppose

$$\sum_{n=0}^{\infty}(2n+2m+1)B_n T_{m+n}^{-m}(cos\theta)=\begin{cases} k_1(\theta), & 0\le\theta<\alpha \qquad (3.2.1)\\ k_2(\theta), & \beta<\theta<\pi \qquad (3.2.2)\end{cases}$$

Applying the inversion theorem (2.2) in equations (1.2), (3.2.1) and (3.2.2) we get

$$B_n=\frac{1}{2}(-1)^m\left[\int_0^{\alpha}k_1(x)+\int_{\alpha}^{\beta}g_2(x)+\int_{\beta}^{\pi}k_2(x)\right]T_{m+n}^{-m}(cosx)sinxdx \qquad (3.2.3)$$

Substituting this expression for B_n in equation (1.4) and (2.6), we get

$$\frac{1}{2}(-1)^m\sum T_{m+n}^{-m}(cos\theta)\left[\int_0^{\alpha}k_1(x)+\int_{\alpha}^{\beta}g_2(x)+\int_{\beta}^{\pi}k_2(x)\right]$$

$$\times T_{m+n}^{m}(cosx)sindx=\begin{cases} g_1(\theta), & 0\le\theta<\alpha \qquad (3.2.4)\\ g_3(\theta), & \beta<\theta<\pi \qquad (3.2.5)\end{cases}$$

Interchanging the order of summation and integration, we get

$$\frac{1}{2}(-1)^m\left[\int_0^{\alpha}k_1(x)+\int_{\alpha}^{\beta}g_2(x)+\int_{\beta}^{\pi}k_2(x)\right]\sum_{n=0}^{\infty}T_{m+n}^{-m}(cos\theta)$$

$$T_{m+n}^{m}(cosx)\times sindx=\begin{cases} g_1(\theta), & 0\le\theta<\alpha \qquad (3.2.6)\\ g_3(\theta), & \beta<\theta<\pi \qquad (3.2.7)\end{cases}$$

Using the result (2.3) in equations (3.2.6) and (3.2.7), we have

$$\frac{1}{2}(-1)^m\left\{\int_0^{\alpha}k_1(x)+\int_{\beta}^{\pi}k_2(x)\right\}S_m(\theta,\ x)sinxdx=\begin{cases} M(\theta), & 0\le\theta<\alpha \qquad (3.2.8)\\ N(\theta), & \beta<\theta<\pi \qquad (3.2.9)\end{cases}$$

where, $M(\theta)=g_1(\theta)-\frac{1}{2}(-1)^m\int_{\alpha}^{\beta}g_2(x)S_m(\theta,\ x)sinxdx$ (3.2.10)

and $N(\theta)=g_3(\theta)-\frac{1}{2}(-1)^m\int_{\alpha}^{\beta}g_2(x)S_m(\theta,\ x)sinxdx$ (3.2.11)

Now using the summation result (2.5) in terms of integral in equation (3.2.8), we obtain

$$\frac{2.\frac{1}{2}(-1)^{2m}}{\pi(S_1S_2)^m}\left[\int_0^{\alpha}k_1(x)\int_0^{\min(S_1,\,S_2)}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2^2-S^2\right)\right]^{1/2}}.sindx\right.$$

$$\left.+\int_{\beta}^{\pi}k_1(x)\int_0^{\min(S_1,\,S_2)}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2^2-S^2\right)\right]^{1/2}}sindx\right]=M(\theta),\quad 0\le\theta<\alpha \tag{3.2.12}$$

$$\frac{sin^{-m}\theta}{\pi}\left[\int_0^{\theta}k_1(x)sin^{1-m}xdx\int_0^{s_2}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2^2-S^2\right)\right]^{1/2}}\right.$$

$$+\int_0^{\alpha}k_1(x)sin^{1-m}xdx\int_0^{s_1}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2^2-S^2\right)\right]^{1/2}}$$

$$+\int_{\beta}^{\pi}k_2(x)sin^{1-m}xdx\int_0^{s_1}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2^2-S^2\right)\right]^{1/2}}=M(\theta),\qquad 0\le\theta<\alpha \tag{3.2.13}$$

Now changing the variables by substituting the values of S_1, S_2 and S in equation (3.2.14), we get

$$\frac{sin^{-m}\theta}{\pi}\left[\int_0^{\theta}k_1(x)sin^{1-m}xdx\int_0^{x}\frac{\left(2cos\frac{\theta}{2}cos\frac{x}{2}\tan\frac{u}{2}\right)^{2m}}{\left[\left(cosu-cos\theta\right)\left(cosu-cosx\right)\right]^{1/2}}du.\right.$$

$$+\int_0^{\alpha}k_1(x)sin^{1-m}xdx\frac{1}{2}\int_0^{\theta}\frac{\left(2cos\frac{\theta}{2}cos\frac{x}{2}\tan\frac{u}{2}\right)^{2m}}{\left[\left(cosu-cos\theta\right)\left(cosu-cosx\right)\right]^{1/2}}du$$

$$\left.+\int_{\beta}^{x}k_2(x)sin^{1-m}xdx\frac{1}{2}\int_0^{\theta}\frac{\left(2cos\frac{\theta}{2}cos\frac{x}{2}\tan\frac{u}{2}\right)^{2m}}{\left[\left(cosu-cos\theta\right)\left(cosu-cosx\right)\right]^{1/2}}du\right]=M(\theta),\qquad 0\le\theta<\alpha \tag{3.2.14}$$

Inverting the order of integration, we have

$$\frac{2^{-m}sin^{-m}\frac{\theta}{2}cos^{-m}\frac{\theta}{2}.2^{2m}cos^{2m}\frac{\theta}{2}}{2\pi}\left[\int_0^{\theta}\frac{\tan^{2m}\frac{u}{2}}{\left(cosu-cos\theta\right)^{1/2}}du\right.$$

$$\times\int_{\theta}^{\alpha}\frac{2^{-m}k_1(x)cot^m\frac{x}{2}.sinx}{\left(cosu-cosx\right)^{1/2}}dx+\int_0^{\theta}\frac{\tan^{2m}\frac{u}{2}}{\left(cosu-cos\theta\right)^{1/2}}du$$

$$\left.\times\int_{\beta}^{x}\frac{2^{-m}k_2(x)cot^m\frac{x}{2}.sinx}{\left(cosu-cosx\right)^{1/2}}dx\right]=M(\theta),\qquad 0\le\theta<\alpha \tag{3.2.15}$$

$$\frac{2^{2m}\tan^{-m}\frac{\theta}{2}cot^{2m}\frac{\theta}{2}}{2\pi}\left[\int_0^\theta \frac{\tan^{2m}\frac{u}{2}du}{(cosu-cos\theta)^{1/2}}\int_u^\alpha \frac{k_1(x)cot^m\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx+\int_0^\theta \frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\right.$$

$$\left.\int_\beta^x \frac{k_2(x)cot^m\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx\right]=M(\theta), \qquad 0\le\theta<\alpha$$

The above equation can be written as

$$\frac{1}{2\pi}\int_0^\theta \frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\left[\int_u^\alpha \frac{k_1(x)cot^m\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx+\int_\beta^x \frac{k_2(x)cot^m\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx\right]=\tan^m\frac{1}{2}\theta.M(\theta),\ 0\le\theta<\alpha$$

(3.2.16)

Equation (3.2.16) is now reduced to the following form

$$\int_0^\theta \frac{\tan^{2m}\frac{u}{2}}{(cosu-cos\theta)^{1/2}}du\left[J(u)+\frac{1}{2\pi}\int_\beta^x \frac{k_2(x)cot^m\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx\right]=\tan^m\frac{\theta}{2}M(\theta),\ 0\le\theta<\alpha \qquad (3.2.17)$$

where $J(u)=\frac{1}{2}\pi\int_u^\alpha \frac{k_1(x)cot^m\frac{x}{2}.sinx}{(cosu-cosx)^{1/2}}dx$ (3.2.18)

Solution of the equation (3.2.17) can be given, with the help of the results (2.6) and (2.8), as

$$J(u)+\frac{1}{2\pi}\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx=\frac{cot^{2m}\frac{u}{2}}{\pi}\frac{d}{du}\int_0^u \frac{M(\theta)\tan^{2m}\frac{1}{2}\theta.sin\theta}{(cos\theta-cosu)^{1/2}}d\theta \qquad 0\le u<\alpha$$

(3.2.19)

Now equation (3.2.19) becomes

$$M_1(u)=J(u)+\frac{1}{2\pi}\int_\beta^\pi \frac{k_2(x)cot^m\frac{x}{2}sinx}{(cosu-cosx)^{1/2}}dx, \qquad 0\le u<\alpha \qquad (3.2.20)$$

where $M_1(u)=\frac{cot^{2m}\frac{u}{2}}{\pi}\frac{d}{du}\int_0^u \frac{M(\theta)\tan^m\frac{1}{2}\theta.sin\theta}{(cos\theta-cosu)^{1/2}}d\theta$ (3.2.21)

Again from equation (3.2.9), we have, $\frac{1}{2}(-1)^m\left[\int_0^\alpha k_1(x)+\int_\beta^\pi k_2(x)\right]S_m(\theta,\ x)sinxdx=N(\theta),\quad \beta<\theta<\pi$

Using the summation result in terms of integral by equation (2.5), we get

$$\frac{2.\frac{1}{2}(-1)^{2m}}{\pi(S_1S_2)^m}\left[\int_0^\alpha k_1(x)\int_0^{\min(S_1,S_2)}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2^2-S^2\right)\right]^{1/2}}.sindx\right.$$

$$\left.+\int_\beta^\pi k_2(x)\int_0^{\min(S_1,S_2)}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2-S^2\right)\right]^{1/2}}sindx\right]=N(\theta),\quad \beta\le\theta<\alpha$$

$$\frac{sin^{-m}\theta}{\pi}\left[\int_0^\alpha k_1(x)\,sin^{1-m}xdx\int_0^{s_2}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2^2-S^2\right)\right]^{1/2}}.\right.$$

$$\left.+\int_\theta^x k_2(x)sin^{1-m}xdx\int_0^{s_1}\frac{S^{2m}ds}{\left[\left(S_1^2-S^2\right)\left(S_2^2-S^2\right)\right]^{1/2}}\right]=N(\theta),\quad \beta\le\theta<\pi \tag{3.2.22}$$

Now making change of variable in equation (3.2.22)

$$\frac{sin^{-m}\theta}{\pi}\left[\int_0^\alpha k_1(x)sin^{1-m}xdx.\frac{1}{2}\int_0^x\frac{\left(\frac{2cos\theta}{2cosx}\frac{2\tan u}{2}\right)^{2m}}{\left[\left(cosu-cos\theta\right)\left(cosu-cosx\right)\right]^{1/2}}du.\right.$$

$$+\int_\beta^\theta k_2(x)sin^{1-m}xdx.\frac{1}{2}\int_0^x\frac{\left(\frac{2cos\theta}{2.cosx}\frac{2.\tan u}{2}\right)^{2m}}{\left[\left(cosu-cos\theta\right)\left(cosu-cosx\right)\right]^{1/2}}du$$

$$\left.+\int_\theta^x k_2(x)sin^{1-m}xdx.\frac{1}{2}\int_0^\theta\frac{\left(\frac{2cos\theta}{2.cosx}\frac{2.\tan u}{2}\right)^{2m}}{\left[\left(cosu-cos\theta\right)\left(cosu-cosx\right)\right]^{1/2}}du\right]=N(\theta),\ \beta<\theta<\pi \tag{3.2.23}$$

Inverting the order of integration in equation (3.2.23), we get

$$\frac{\frac{2^{-m}sin^{-m}\theta}{2cos^{-m}\theta}\frac{2.2^{2m}cos^{2m}\theta}{2}}{2\pi}\left[\int_0^\alpha\frac{\tan^{2m}\frac{u}{2}}{\left(cosu-cos\theta\right)^{1/2}}du\times\int_u^\alpha\frac{2^{-m}k_1(x)\frac{cot^m x}{2.sinx}}{(cosu-cosx)^{1/2}}dx+\int_0^\beta\frac{\tan^{2m}\frac{u}{2}}{\left(cosu-cos\theta\right)^{1/2}}du\right.$$

$$\left.\times\int_\beta^x\frac{2^{-m}k_2(x)\frac{cot^m x}{2.sinx}}{(cosu-cosx)^{1/2}}dx+\int_\beta^\theta\frac{\tan^{2m}\frac{u}{2}}{\left(cosu-cos\theta\right)^{1/2}}du\times\int_u^x\frac{2^{-m}k_2(x)\frac{cot^m x}{2.sinx}}{(cosu-cosx)^{1/2}}dx\right]=N(\theta),\ \beta<\theta<\pi \tag{3.2.24}$$

$$\frac{1}{2\pi}\int_\beta^\theta\frac{\tan^{2m}\frac{1}{2u}du}{\left(cosu-cos\theta\right)^{1/2}}\int_u^\pi\frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx=\tan^m\frac{1}{2}\theta.N(\theta)-\frac{1}{2\pi}\int_0^\alpha\frac{\tan^{2m}\frac{1}{2u}}{\left(cosu-cos\theta\right)^{1/2}}du$$

$$\times\int_u^x \frac{k_1(x)cot^m \frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx-\frac{1}{2\pi}\int_0^\beta \frac{\tan^{2m}\frac{1}{2u}}{\left(cosu-cos\theta\right)^{1/2}}du\times\int_\beta^x \frac{k_2(x)cot^m \frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx,\ \beta<\theta<\pi \tag{3.2.25}$$

Equation (3.2.25) now becomes

$$\int_\beta^\theta \frac{k(u)\tan^{2m}\frac{1}{2}u}{(cosu-cos\theta)^{1/2}}du=\tan^m\frac{1}{2}\theta.N(\theta)-\int_0^\alpha \frac{J(u)\tan^{2m}\frac{1}{2}u}{\left(cosu-cos\theta\right)^{1/2}}du$$

$$-\frac{1}{2\pi}\int_0^\beta \frac{\tan^{2m}\frac{1}{2}u}{\left(cosu-cos\theta\right)^{1/2}}du\times\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx,\ \beta<\theta<\pi \tag{3.2.26}$$

where, $$K(u)=\frac{1}{2\pi}\int_u^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx, \tag{3.2.27}$$

Substituting the value of J(u) from (3.2.26), we get

$$\int_\beta^\theta \frac{k(u)\tan^{2m}\frac{1}{2}u}{(cosu-cos\theta)^{1/2}}du=\tan^m\frac{1}{2}\theta.N(\theta)-\int_0^\alpha \frac{\tan^{2m}\frac{1}{2}u}{\left(cosu-cos\theta\right)^{1/2}}du$$

$$\left\{M_1(u)-\frac{1}{2\pi}\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx,\right\}-\frac{1}{2\pi}\int_0^\beta \frac{\tan^{2m}\frac{1}{2}u}{\left(cosu-cos\theta\right)^{1/2}}du\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx,\ \beta<\theta<\pi \tag{3.2.28}$$

$$\int_\beta^\theta \frac{k(u)\tan^{2m}\frac{1}{2}u}{(cosu-cos\theta)^{1/2}}du=\tan^m\frac{1}{2}\theta.N(\theta)-\int_0^\alpha \frac{M_1(u)\tan^{2m}\frac{1}{2}u}{\left(cosu-cos\theta\right)^{1/2}}du$$

$$-\frac{1}{2\pi}\int_\alpha^\beta \frac{\tan^{2m}\frac{1}{2}u}{\left(cosu-cos\theta\right)^{1/2}}du\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cosu-cosx)^{1/2}}dx,\ \beta<\theta<\pi \tag{3.2.29}$$

In view of (2.6) , (2.8) and equation (3.2.29), we have

$$K(u)=\frac{cot^{2m}\frac{1}{2}u}{\pi}\frac{d}{du}\int_\beta^u \frac{sin\theta d\theta}{(cos\theta-cosu)^{1/2}}\Big[\tan^m\frac{1}{2}\theta.N(\theta)-\int_0^\alpha \frac{M_1(\theta)\tan^{2m}\frac{1}{2}\theta}{\left(cos\nu-cos\theta\right)^{1/2}}d\theta-\int_\alpha^\beta \frac{\tan^{2m}\frac{1}{2}\theta}{(cos\nu-cos\theta)^{1/2}}d\theta,$$

$$\left.\times\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cos\nu-cosx)^{1/2}}dx\right],\qquad \beta<u<\pi \tag{3.2.30}$$

Equation (3.2.30) is now reduced to

$$K(u)=N_1(u)+M_2(u)-\frac{cot^{2m}\frac{1}{2}u}{\pi}\frac{d}{du}\int_\beta^u \frac{sin\theta d\theta}{(cos\theta-cosu)^{1/2}}\times\int_\alpha^\beta \frac{\tan^{2m}\frac{1}{2}\theta d\theta}{(cos\vartheta-cosx)^{1/2}}\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cos\theta-cosx)^{1/2}}dx,\ \beta<u<\pi \tag{3.2.31}$$

$$N_1(u)=\frac{cot^{2m}\frac{1}{2}u}{\pi}\frac{d}{du}\int_\beta^u \frac{N(\theta)\tan^m\frac{1}{2}\theta.sin\theta}{(cos\theta-cosu)^{1/2}}d\theta \tag{3.2.32}$$

and

$$M_2(u)=\frac{-cot^{2m}\frac{1}{2}u}{\pi}\frac{d}{du}\int_\beta^u \frac{sin\theta d\theta}{(cos\theta-cosu)^{1/2}}\int_0^\alpha \frac{M_1(\theta)\tan^{2m}\frac{1}{2}\theta}{(cos\theta-cosx)^{1/2}}d\theta \tag{3.2.33}$$

Changing the order of integration in the equation (3.2.31), we get

$$K(u)=N_1(u)+M_2(u)-\frac{cot^{2m}\frac{1}{2}u}{2\pi^2}\left[\int_\alpha^\beta \tan^{2m}\frac{1}{2}\theta d\theta\right.$$

$$\left.\frac{d}{du}\int_\beta^u \frac{sin\theta d\theta}{(cos\theta-cosu)^{1/2}(cos\theta-cosu)^{1/2}}\int_\beta^\pi \frac{K_2(x)cot^m\frac{1}{2}x.sinx}{(cos\theta-cosx)^{1/2}}dx\right],\beta<u<\pi \tag{3.2.34}$$

We know that, $\frac{d}{du}\int_\beta^u \frac{sin\theta d\theta}{(cos\vartheta-cos\theta)^{1/2}(cos\theta-cosu)^{1/2}}=\frac{sinu(cos\theta-cos\beta)^{1/2}}{(cos\beta-cosu)^{1/2}(cos\theta-cosu)^{1/2}}$

Using the above result in equation (3.2.34), we get

$$K(u)=N_1(u)+M_2(u)-\frac{cot^{2m}\frac{1}{2}usinu}{2\pi^2(cos\beta-cosu)^{1/2}}\left[\int_\alpha^\beta \frac{\tan^{2m}\frac{1}{2}\theta(cos\theta-cos\beta)^{1/2}}{(cos\theta-cosu)}d\theta\right.$$

$$\left.\times\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cos\theta-cosx)^{1/2}}dx,\right],\qquad \beta<u<\pi \tag{3.2.36}$$

Now with the help of the results (2.7) and (2.9), the solution of the equation (3.2.27) is given by

$$K_2(x)cot^m\frac{1}{2}x.sinx=-2\frac{d}{dx}\int_x^\pi \frac{K(u)sinu}{(cosx-cosu)^{1/2}}du \tag{3.2.37}$$

From (3.2.37) we obtain

$$\int_\beta^\pi \frac{k_2(x)cot^m\frac{1}{2}x.sinx}{(cos\theta-cosx)^{1/2}}dy=\frac{2}{(cos\theta-cos\beta)^{1/2}}\times\int_\beta^\pi \frac{K(s)sinsds}{(cos\beta-coss)^{1/2}(cos\theta-coss)} \tag{3.2.38}$$

Putting the value from (3.2.36), we get

$$K(u)=N_1(u)+M_2(u)-\frac{cot^{2m}\frac{1}{2}usinu}{\pi^2(cos\beta-cosu)^{1/2}}\int_\alpha^\beta \frac{\tan^{2m}\frac{1}{2}\theta(cos\theta-cos\beta)^{1/2}}{(cos\theta-cosu)}$$

$$\times\frac{1}{(cos\theta-cos\beta)^{1/2}}\int_\beta^\pi \frac{k(s)sinsds}{(cos\beta-cosu)^{1/2}(cos\theta-cosx)},\ \beta<u<\pi \tag{3.2.39}$$

Changing the order of integration in equation (3.2.39), we get

$$K(u)=N_1(u)+M_2(u)-\frac{cot^{2m}\frac{1}{2}u}{\pi^2}\int_\beta^\pi \frac{k(s).sinsds}{(cos\beta-cosu)^{1/2}(cos\beta-coss)^{1/2}}$$

$$\times\int_\alpha^\beta \frac{\tan^{2m}\frac{1}{2}\theta d\theta}{(cos\theta-coss)(cos\theta-coss)},\qquad \beta<u<\pi \tag{3.2.40}$$

Equation (3.2.40) can now be rewritten as

$$K(u)+\frac{1}{\pi^2}\int_\beta^\pi K(s)R(s,\ u)ds=N_1(u)+M_2(u)\qquad \beta<u<\pi \tag{3.2.41}$$

where R(s, u) is symmetric kernel, $R(s,\ u)=-\frac{cot^{2m}\frac{1}{2}u.sins.sinu}{(cos\beta-coss)^{1/2}(cos\beta-cosu)^{1/2}}S(s,\ u)$ (3.2.42)

and $S(s,\ u)=\int_\alpha^\beta \frac{\tan^{2m}\frac{1}{2}\theta d\theta}{(cos\theta-coss)(cos\theta-cosu)}$ (3.2.43)

Equation (3.2.41) is a Fredholm integral equation of the second kind which determines K(u) and from equation (3.2.37) $k_2(x)$ can be found. After that, we can calculate the value of J(u) from equation (3.2.20) and consequently $k_1(x)$ can be found with the help of equation (3.2.18). Finally, the unknown coefficients B_n can be computed with the help of the equation (2.2) which satisfies the equations (2.4), (2.5) and (2.6).

PARTICULAR CASE

Let $\beta=\pi$. Then the equations from (2.4) to (2.6) reduce to the dual series equations and the solution obtained agrees with the one that was obtained earlier by Collins [9].

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