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Wiener Index and Some Hamiltonian Properties of Graphs

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Abstract

The Wiener index of a connected graph is defined as the sum of distances between all pairs of vertices in the graph. Yang presented a sufficient condition in terms of the Wiener index for a graph to be traceable. Motivated by Yang's result, we present sufficient conditions based on the Wiener index for a graph to be Hamiltonian or Hamilton-connected in this note.

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. For two vertices u and v in a graph G, we use $d_G(u, v)$ to denote the distance between them. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. A graph G is called Hamilton-connected if for each pair of vertices in G there is a Hamiltonian path between them. If G and H are two vertex-disjoint graphs, we use $G \vee H$ to denote the join of G and H. We use $C(n, r)$ to denote the number of r - combinations of a set with n elements.

For a connected graph G, its Wiener index [8], denoted by $W(G)$, is defined as

$$W(G) = \sum_{\{u,\, v\} \subseteq V(G)} d_G(u, v).$$

If we use $\widehat{D}_G(v)$ to denote $\sum_{u \in V(G)} d_G(u, v)$, then $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} \widehat{D}_G(v)$. It can be easily verified that $\widehat{D}_G(v) \geq d(v) + 2(n - 1 - d(v))$.

For a nontrivial connected graph G, its Harary index [5, 7] is defined as $\sum\limits_{\{u,\, v\} \subseteq V(G)} \frac{1}{d_G(u,v)}$.

In [4], Hua and Wang presented a sufficient condition for a graph to be traceable by using Harary index. Li [6] presented sufficient conditions in terms of the Harary index for a graph to be Hamiltonian or Hamilton-connected using some proof ideas in [4].

In [9], Yang presented the following sufficient condition for a graph to be traceable by using Wiener index.

Theorem 1.1. [9]. Let G be a connected graph of order $n \geq 4$. If $W(G) \leq \frac{(n+5)(n-2)}{2}$, then G is traceable, unless $G = K_1 \vee (K_{n-3} \cup 2K_1)$ or $K_2 \vee (3K_1 \cup K_2)$ or $K_4 \vee 6K_1$.

In this paper, we combine the ideas in [9] and [6] to present the following sufficient conditions in terms of the Wiener index for a graph to be Hamiltonian or Hamilton-connected.

Theorem 1.2. Let G be a connected graph of order $n \geq 3$. If $W(G) \leq \frac{n^2+n-4}{2}$, then G is Hamiltonian, unless $G = K_1 \vee (K_1 \cup K_{n-2})$ or $K_2 \vee (K_2^c \cup K_1)$.

Theorem 1.3. Let G be a connected graph of order $n \geq 4$. If $W(G) \leq \frac{n^2+n-6}{2}$, then G is Hamilton-connected, unless $G = K_2 \vee (K_1 \cup K_{n-3})$ or $K_3 \vee (3K_1)$.

Theorem 1.4. Let $G = (X, Y; E)$, where $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$ and $n \geq 2$ be a connected bipartite graph. If $W(G) \leq 3n^2 - 2n + 2$, then G is Hamiltonian, unless $G = P_4$, a path having four vertices and three edges.

Theorem 1.5. Let G be a 2-connected graph of order $n \geq 12$. If $W(G) \leq \frac{n^2+3n-13}{2}$, then G is Hamiltonian, unless $G = K_2 \vee ((2K_1) \cup K_{n-4})$.

Theorem 1.6. Let G be a 3-connected graph of order $n \geq 18$. If $W(G) \leq \frac{n^2+5n-29}{2}$, then G is Hamiltonian, unless $G = K_3 \vee ((3K_1) \cup K_{n-6})$.

Theorem 1.7. Let G be a k-connected graph of order n. If $W(G) \leq \frac{n(n-1)+(k+1)(n-k-1)-1}{2}$, then G is Hamiltonian.

2 Preliminary Results

Lemma 2.1. Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $d_k \leq k < \frac{n}{2} \Longrightarrow d_{n-k} \geq n-k$, then G is Hamiltonian.

Lemma 2.2. Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $2 \leq k \leq \frac{n}{2}$, $d_{k-1} \leq k \Longrightarrow d_{n-k} \geq n-k+1$, then G is Hamilton-connected.

Lemma 2.3. Let $G = (X, Y; E)$ be a bipartite graph such that $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$, $n \geq 2$, and $d_G(x_1) \leq d_G(x_2) \leq \cdots \leq d_G(x_n)$, $d_G(y_1) \leq d_G(y_2) \leq \cdots \leq d_G(y_n)$. If $d_G(x_k) \leq k < n \Longrightarrow d_G(y_{n-k}) \geq n-k+1$, then G is Hamiltonian.

Lemma 2.4. [3] Let G be a 2-connected graph of order $n \geq 12$. If $e(G) \geq C(n-2, 2) + 4$, then G is Hamiltonian or $G = K_2 \vee ((2K_1) \cup K_{n-4})$.

Lemma 2.5. [3] Let G be a 3-connected graph of order $n \geq 18$. If $e(G) \geq C(n-3, 2) + 9$, then G is Hamiltonian or $G = K_3 \vee ((3K_1) \cup K_{n-6})$.

Lemma 2.6. [3] Let G be a k-connected graph of order n. If $e(G) \geq C(n,2)-(k+1)(n-k-1)/2+1$, then G is Hamiltonian.

Note that Lemma 2.1 is Corollary 3 on Page 209 in [1], Lemma 2.2 is Theorem 12 on Page 218 in [1], Lemma 2.3 is Corollary 5 on Page 210 in [1], and Lemmas 2.4, 2.5, and 2.6 can be found in [3].

3 Main Results

Proof of Theorem 1.2. Let G be a graph satisfying the conditions in Theorem 1.2. Suppose that G is not Hamiltonian. Then, from Lemma 2.1, there exists an integer $k < \frac{n}{2}$ such that $d_k \leq k$ and $d_{n-k} \leq n-k-1$. Obviously, $k \geq 1$.
Therefore,

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{v\in V(G)} \widehat{D}_G(v) \geq \frac{1}{2}\sum_{v\in V(G)} (d_G(v)+2(n-1-d_G(v))) \\ &= \frac{1}{2}\sum_{v\in V(G)} (2(n-1)-d_G(v)) = n(n-1)-\frac{1}{2}\sum_{v\in V(G)} d_G(v) \\ &\geq n(n-1)-\frac{1}{2}\left(k^2+(n-2k)(n-k-1)+k(n-1)\right) \\ &= \frac{n^2+n-4}{2}+\frac{(k-1)(k-2)}{2}+(k-1)(n-2k-1). \end{aligned}$$

From $W(G) \leq \frac{n^2+n-4}{2}$, $k \geq 1$ and $n > 2k$, we have that $W(G) = \frac{n^2+n-4}{2}$, $k = 1$ or ($k = 2$ and $n = 2k+1$), $d_1 = \cdots = d_k = k$, $d_{k+1} = \cdots = d_{n-k} = n-k-1$ and $d_{n-k+1} = \cdots = d_n = n-1$.

If $k = 1$, then $d_1 = 1$, $d_2 = d_3 = \cdots = d_{n-1} = n-2$ and $d_n = n-1$. Thus $G = K_1 \vee (K_1 \cup K_{n-2})$, which is not Hamiltonian.

If $k = 2$ and $n = 2k+1$, then we have $n = 5$. Therefore $d_1 = 2$, $d_2 = 2$, $d_3 = 2$, $d_4 = 4$ and $d_5 = 4$. Hence $G = K_2 \vee (K_2^c \cup K_1)$, which is not Hamiltonian.
This completes the proof of Theorem 1.2. ■

Proof of Theorem 1.3. Let G be a graph satisfying the conditions in Theorem 1.3. Suppose that G is not Hamilton-connected. Then, from Lemma 2.2, there exists an integer k with $2 \leq k \leq \frac{n}{2}$ such that $d_{k-1} \leq k$ and $d_{n-k} \leq n-k$.
Therefore,

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{v\in V(G)} \widehat{D}_G(v) \geq \frac{1}{2}\sum_{v\in V(G)} (d_G(v)+2(n-1-d_G(v))) \\ &= \frac{1}{2}\sum_{v\in V(G)} (2(n-1)-d_G(v)) = n(n-1)-\frac{1}{2}\sum_{v\in V(G)} d_G(v) \\ &\geq n(n-1)-\frac{1}{2}\left(k(k-1)+(n-2k+1)(n-k)+k(n-1)\right) \end{aligned}$$

$$= \quad \frac{n^2+n-6}{2} + \frac{(k-2)(k-3)}{2} + (k-2)(n-2k).$$

From $W(G) \leq \frac{n^2+n-6}{2}$, $k \geq 2$, and $n \geq 2k$, we have that $W(G) = \frac{n^2+n-6}{2}$, $k = 2$ or ($k = 3$ and $n = 2k$), $d_1 = \cdots = d_{k-1} = k$, $d_k = \cdots = d_{n-k} = n-k$ and $d_{n-k+1} = \cdots = d_n = n-1$.

If $k = 2$, then $d_1 = 2$, $d_2 = d_3 = \cdots = d_{n-2} = n-2$ and $d_{n-1} = d_n = n-1$. Thus $G = K_2 \vee (K_1 \cup K_{n-3})$, which is not Hamilton-connected.

If $k = 3$ and $n = 2k$, then we have that $n = 6$. Therefore $d_1 = 3$, $d_2 = 3$, $d_3 = 3$, $d_4 = 5$, $d_5 = 5$ and $d_6 = 5$. Hence $G = K_3 \vee (3K_1)$, which is not Hamilton-connected.
This completes the proof of Theorem 1.3. ■

Proof of Theorem 1.4. Let G be a graph satisfying the conditions in Theorem 1.4. Suppose that G is not Hamiltonian. Then, from Lemma 2.3, there exists an integer $k < n$ such that $d_G(x_k) \leq k$ and $d_G(y_{n-k}) \leq n-k$. Next we find an upper bound for $\widehat{D}_G(x_1)$. Let $N_G(x_1) := \{z_1, z_2, ..., z_s\}$ be the neighbors of x_1, where $s = d_G(x_1)$. Then $d_G(x_1, z_i) = 1$ for each $z_i \in N_G(x_1)$, $d_G(x_1, x_i) \geq 2$ for each x_i with $2 \leq i \leq n$, and $d_G(x_1, y_i) \geq 3$ for each $y_i \in Y - N_G(x_1)$. Thus

$$\widehat{D}_G(x_1) \geq d_G(x_1) + 2(n-1) + 3(n - d_G(x_1)) = 5n - 2 - 2d_G(x_1).$$

Similarly, we have that for each i with $2 \leq i \leq n$ and each j with $1 \leq j \leq n$,

$$\widehat{D}_G(x_i) \geq d_G(x_i) + 2(n-1) + 3(n - d_G(x_1)) = 5n - 2 - 2d_G(x_i),$$

$$\widehat{D}_G(y_j) \geq d_G(y_j) + 2(n-1) + 3(n - d_G(y_j)) = 5n - 2 - 2d_G(y_j).$$

Therefore,

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{v\in V(G)} \widehat{D}_G(v) \geq \frac{1}{2}\left(10n^2 - 4n - 2\sum_{i=1}^{n}(d_G(x_i) + d_G(y_i))\right) \\ &\geq \frac{1}{2}\left(10n^2 - 4n - 2(k^2 + (n-k)n + (n-k)^2 + kn)\right) \\ &= \frac{1}{2}\left(10n^2 - 4n - 2((k + (n-k))^2 - 2k(n-k) + n^2)\right) \\ &= \frac{1}{2}\left(10n^2 - 4n - 2(2n^2 - 2k(n-k))\right) = 3n^2 - 2n + 2k(n-k) \\ &\geq 3n^2 - 2n + 2 * 1 * 1 = 3n^2 - 2n + 2. \end{aligned}$$

From $W(G) \leq 3n^2 - 2n + 2$, $1 \leq k < n$, we have that $k = 1$, $n - k = 1$, $d_G(x_1) = 1$, $d_G(x_2) = 2$, $d_G(y_1) = 1$ and $d_G(y_2) = 2$. Thus $G = P_4$, which is not Hamiltonian.
This completes the proof of Theorem 1.4. ■

Proof of Theorem 1.5. Let G be a graph satisfying the conditions in Theorem 1.5. Note that if $G = K_2 \vee ((2K_1) \cup K_{n-4})$, then $W(G) = \frac{n^2+3n-14}{2}$. Suppose that G is not Hamiltonian and G is not $K_2 \vee ((2K_1) \cup K_{n-4})$. Then, from Lemma 2.4, we have that $e(G) \leq C(n-2,2)+3$. Therefore,

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{v\in V(G)} \widehat{D}_G(v) \geq \frac{1}{2}\sum_{v\in V(G)} (d_G(v)+2(n-1-d_G(v))) \\ &= \frac{1}{2}\sum_{v\in V(G)} (2(n-1)-d_G(v)) = n(n-1) - \frac{1}{2}\sum_{v\in V(G)} d_G(v) \\ &= n(n-1)-e(G) \geq n(n-1) - C(n-2,2) - 3 = \frac{n^2+3n-12}{2}, \end{aligned}$$

which is a contradiction.
This completes the proof of Theorem 1.5. ■

Proof of Theorem 1.6. Let G be a graph satisfying the conditions in Theorem 1.6. Note that if $G = K_3 \vee ((3K_1) \cup K_{n-6})$, then $W(G) = \frac{n^2+5n-30}{2}$. Suppose that G is not Hamiltonian and G is not $K_3 \vee ((3K_1) \cup K_{n-6})$. Then, from Lemma 2.5, we have that $e(G) \leq C(n-3,2)+8$. Therefore,

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{v\in V(G)} \widehat{D}_G(v) \geq \frac{1}{2}\sum_{v\in V(G)} (d_G(v)+2(n-1-d_G(v))) \\ &= \frac{1}{2}\sum_{v\in V(G)} (2(n-1)-d_G(v)) = n(n-1) - \frac{1}{2}\sum_{v\in V(G)} d_G(v) \\ &= n(n-1)-e(G) \geq n(n-1) - C(n-3,2) - 8 = \frac{n^2+5n-28}{2}, \end{aligned}$$

which is a contradiction.
This completes the proof of Theorem 1.6. ■

Proof of Theorem 1.7. Let G be a graph satisfying the conditions in Theorem 1.7. Suppose that G is not Hamiltonian. Then, from Lemma 2.6, we have that $e(G) \leq C(n,2) - (k+1)(n-k-1)/2$. Therefore,

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{v\in V(G)} \widehat{D}_G(v) \geq \frac{1}{2}\sum_{v\in V(G)} (d_G(v)+2(n-1-d_G(v))) \\ &= \frac{1}{2}\sum_{v\in V(G)} (2(n-1)-d_G(v)) = n(n-1) - \frac{1}{2}\sum_{v\in V(G)} d_G(v) \\ &= n(n-1)-e(G) \geq n(n-1) - C(n,2) + (k+1)(n-k-1)/2 \\ &= \frac{n(n-1)+(k+1)(n-k-1)}{2}, \quad \text{which is a contradiction.} \end{aligned}$$

This completes the proof of Theorem 1.7. ■

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