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THE MULTIPLICATIVE VERSIONS OF THE RECIPROCAL DEGREE DISTANCE AND RECIPROCAL GUTMAN INDEX OF SOME GRAPH PRODUCTS

R. Muruganandam, R.S. Manikandan, and M.Aruvi

Abstract. In this paper, we provide exact value of the multiplicative version of the reciprocal degree distance and the multiplicative version of the reciprocal Gutman index of Cartesian product of complete graphs. Also, we establish sharp upper bounds for the multiplicative version of the reciprocal degree distance and multiplicative version of the reciprocal Gutman index of strong product of graphs.

1. Introduction

In this paper, all graphs considered are simple and connected graphs. We denote the vertex and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. $d_G(v)$ denotes the degree of a vertex v in G. The number of elements in the vertex set of a graph G is called the order of G and is denoted by $v(G)$. The number of elements in the edge set of a graph G is called the size of G and is denoted by $e(G)$. A graph with order n and size m edges is called a (n, m)-graph. For any $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G.

A topological index of a graph is a parameter related to the graph, it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacological, toxicological, biological and other properties of chemical compounds [7]. Several types of such indices exist, expecially those based on vertex and edge distance. One of the most intensively studied

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topological indices is the Wiener index. The Wiener index [16] is one of the oldest molecular graph based structure descriptors [15]. Its chemical applications and Mathematical properties are well studied in [3].

The Cartesian product of the graph G_1 and G_2, denoted by $G_1 \Box G_2$ has the vertex set $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and $(u,x)(v,y)$ is an edge of $G_1 \Box G_2$ is $u = v$ and $xy \in E(G_2)$ or $uv \in E(G_1)$ and $x = y$. For two simple graphs G_1 and G_2, their strong product, denoted by $G_1 \boxtimes G_2$, has vertex set $V(G_1) \times V(G_2) = \{(u,v) : u \in V(G_1), v \in V(G_2)\}$ and $(u,x)(v,y)$ is an edge whenever $(i) u = v$ and $xy \in E(G_2)$, or $(ii) uv \in E(G_1)$ and $x = y$, or $(iii) uv \in E(G_1)$ and $xy \in E(G_2)$.

Dobrynin and Kochetova [4] and Gutman [6] independently presented a vertex degree weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)] = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u) + d_G(v)]$$

with the summation runs over all ordered pairs of vertices of G.

The Gutman index of a connected graph G, denoted by $Gut(G)$, is defined as

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) d_G(u) d_G(v) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v) d_G(u) d_G(v).$$

The degree distance is a degree-weight version of the Wiener index. Hua and Zhang [9] introduced a new graph invariant named reciprocal degree distance, that is

$$RDD(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{[d_G(u) + d_G(v)]}{d_G(u,v)}.$$

Also, the reciprocal Gutman index of a connected graph G, denoted by $RGut(G)$ is defined as

$$RGut(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{d_G(u) d_G(v)}{d_G(u,v)}.$$

Hua and Zhang [9] have acquired lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariant including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices , independence number, chromatic number and vertex and edge-connectivity. The upper bounds for the product version of reciprocal degree distance of the composition, Cartesian product and double of a graph [13].

In this progression, the multiplicative version of reciprocal degree distance and the multiplicative version of reciprocal Gutman index are defined as

$$[RDD^*(G)]^2 = \prod_{u,v \in V(G), u \neq v} \frac{[d_G(u) + d_G(v)]}{d_G(u,v)}$$

$$[RGut^*(G)]^2 = \prod_{u,v \in V(G), u \neq v} \frac{d_G(u)d_G(v)}{d_G(u,v)},$$

respectively.

The first Zagreb index $M_1(G)$ of a graph G is defined as

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] = \sum_{v \in V(G)} d_G^2(v).$$

The second Zagreb index $M_2(G)$ of a graph G is defined as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v).$$

The Harary index of a graph G is defined as:

$$H(G) = \frac{1}{2} \sum_{u,v \in V(G), u \neq v} \frac{1}{d_G(u,v)}.$$

The Zagreb indices are observed to have applications in QSPR and QSAR studies as well, see[5]. There are many papers studying topological index, see $[1, 2, 8, 10-12, 14, 17, 18]$.

Therefore the study of this new topological index is important and we have obtained the exact value and sharp upper bounds for the graph products such as Cartesian and Strong of graphs.

2. The Multiplicative Reciprocal Degree Distance of Cartesian Product of Complete Graphs.

In this section, we compute the multiplicative reciprocal degree distance of the Cartesian product $K_m \Box K_n$, of the complete graphs K_m and K_n. Let $V(K_m) = \{u_0, u_1, ..., u_{m-1}\}$, $V(K_n) = \{v_0, v_1, ..., v_{n-1}\}$ and let w_{ij} denote the vertex (u_i, v_j) of $K_m \Box K_n$.

LEMMA 2.1. *Let K_m and K_n be two complete graphs. Let $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$ be in $V(K_m \Box K_n)$. Then $d_{K_m \Box K_n}(w_{ij}, w_{pq}) = d_{K_m}(u_i, u_p) + d_{K_n}(v_j, v_q)$ and $d_{K_m \Box K_n}(w_{ij}) = d_{K_m}(u_i) + d_{K_n}(u_j)$.*

THEOREM 2.1. *Let K_m and K_n be two complete graphs. Then*

$$[RDD^*(K_m \Box K_n)]^2 = 2^{nm(m+n-2)} \times (m+n-2)^{nm(nm-1)}$$

PROOF.

$$[RDD^*(K_m \Box K_n)]^2 = \prod_{w_{ij}, w_{pq} \in V(K_m \Box K_n)} \left[\frac{d_{(K_m \Box K_n)}(w_{ij}) + d_{(K_m \Box K_n)}(w_{pq})}{d_{(K_m \Box K_n)}(w_{ij}, w_{pq})} \right]$$

$$\begin{aligned}
&= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{(K_m\Box K_n)}(w_{ij}) + d_{(K_m\Box K_n)}(w_{pj})}{d_{(K_m\Box K_n)}(w_{ij}, w_{pj})}\Big] \\
&\times \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} \Big[\frac{d_{(K_m\Box K_n)}(w_{ij}) + d_{(K_m\Box K_n)}(w_{iq})}{d_{(K_m\Box K_n)}(w_{ij}, w_{iq})}\Big] \\
&\times \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{(K_m\Box K_n)}(w_{ij}) + d_{(K_m\Box K_n)}(w_{pq})}{d_{(K_m\Box K_n)}(w_{ij}, w_{pq})}\Big] \\
&= A \times B \times C, \text{ where } A, B, C \text{ are terms of above product taken in order.}
\end{aligned}$$

Next we calculate A, B and C separtely one by one. Now,

$$\begin{aligned}
A &= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{(K_m\Box K_n)}(w_{ij}) + d_{(K_m\Box K_n)}(w_{pj})}{d_{(K_m\Box K_n)}(w_{ij}, w_{pj})}\Big] \\
&= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{K_m}(u_i) + d_{K_n)}(v_j) + d_{K_m}(u_p) + d_{K_n)}(v_j)}{d_{K_m}(u_i, u_p)}\Big] \text{Lemma 2.1} \\
&= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{(m-1) + (n-1) + (m-1) + (n-1)}{1}\Big] \\
&= (2m + 2n - 4)^{nm(m-1)} \\
B &= \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} \Big[\frac{d_{(K_m\Box K_n)}(w_{ij}) + d_{(K_m\Box K_n)}(w_{iq})}{d_{(K_m\Box K_n)}(w_{ij}, w_{iq})}\Big] \\
&= \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} \Big[\frac{d_{K_m}(u_i) + d_{K_n)}(v_j) + d_{K_m}(u_i) + d_{K_n)}(v_j)}{d_{K_n}(v_i, v_q)}\Big] \text{Lemma 2.1} \\
&= \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} \Big[\frac{(m-1) + (n-1) + (m-1) + (n-1)}{1}\Big] \\
&= (2m + 2n - 4)^{nm(n-1)} \\
C &= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{(K_m\Box K_n)}(w_{ij}) + d_{(K_m\Box K_n)}(w_{pq})}{d_{(K_m\Box K_n)}(w_{ij}, w_{pq})}\Big] \\
&= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{K_m}(u_i) + d_{K_n)}(v_j) + d_{K_m}(u_p) + d_{K_n)}(v_q)}{d_{K_m}(u_i, u_p) + d_{K_n}(v_j, v_q)}\Big] \text{Lemma 2.1} \\
&= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{(m-1) + (n-1) + (m-1) + (n-1)}{2}\Big] \\
&= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{2m + 2n - 4}{2}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[m+n-2\Big] \\
&= \Big[m+n-2\Big]^{nm(n-1)(m-1)}
\end{aligned}
$$

By multiplying A, B and C , the desired result follows after simple calculation. □

3. The Multiplicative Reciprocal Gutman Index of Cartesian Product of Complete Graphs.

In this section, we compute the Multiplicative reciprocal Gutman index of the cartesian product, $K_m \Box K_n$ of the complete graphs K_m and K_n.

THEOREM 3.1. *If K_m and K_n are two complete graphs, then*

$$
[RGut^*(K_m \Box K_n)]^2 = \frac{\Big(m+n-2\Big)^{2nm(mn-1)}}{2^{nm(m-1)(n-1)}}
$$

PROOF.

$$
\begin{aligned}
[RGut^*(K_m \Box K_n)]^2 &= \prod_{w_{ij}, w_{pq} \in V(K_m \Box K_n)} \Big[\frac{d_{(K_m \Box K_n)}(w_{ij}) d_{(K_m \Box K_n)}(w_{pq})}{d_{(K_m \Box K_n)}(w_{ij}, w_{pq})}\Big] \\
&= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{(K_m \Box K_n)}(w_{ij}) d_{(K_m \Box K_n)}(w_{pj})}{d_{(K_m \Box K_n)}(w_{ij}, w_{pj})}\Big] \\
&\times \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} \Big[\frac{d_{(K_m \Box K_n)}(w_{ij}) d_{(K_m \Box K_n)}(w_{iq})}{d_{(K_m \Box K_n)}(w_{ij}, w_{iq})}\Big] \\
&\times \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{(K_m \Box K_n)}(w_{ij}) d_{(K_m \Box K_n)}(w_{pq})}{d_{(K_m \Box K_n)}(w_{ij}, w_{pq})}\Big] \\
&= S_1 \times S_2 \times S_3,
\end{aligned}
$$

where S_1, S_2, S_3 are terms of above product taken in order.
Next we caculate S_1, S_2 and S_3 separtely.

$$
\begin{aligned}
S_1 &= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{(K_m \Box K_n)}(w_{ij}) d_{(K_m \Box K_n)}(w_{pj})}{d_{(K_m \Box K_n)}(w_{ij}, w_{pj})}\Big] \\
&= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{\Big(d_{K_m}(u_i) + d_{K_n}(v_j)\Big)\Big(d_{K_m}(u_p) + d_{K_n}(v_j)\Big)}{d_{K_m}(u_i, u_p)}\Big] \text{Lemma 2.1} \\
&= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{(m-1+n-1)(m-1+n-1)}{1}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{j=0}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} (m+n-2)^2 \\
&= \Big[(m+n-2)^2\Big]^{nm(m-1)} \\
S_2 &= \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} \Big[\frac{d_{(K_m\Box K_n)}(w_{ij})d_{(K_m\Box K_n)}(w_{iq})}{d_{(K_m\Box K_n)}(w_{ij},w_{iq})}\Big] \\
&= \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} \Big[\frac{\Big(d_{K_m}(u_i)+d_{K_n}(v_j)\Big)\Big(d_{K_m}(u_i)+d_{K_n}(v_q)\Big)}{d_{K_n}(v_i,v_q)}\Big] \text{Lemma 2.1} \\
&= \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} \Big[\frac{\big((m-1)+(n-1)\big)\big((m-1)+(n-1)\big)}{1}\Big] \\
&= \prod_{i=0}^{m-1} \prod_{j,q=0,j\neq q}^{n-1} (m+n-2)^2 \\
&= \Big[(m+n-2)^2\Big]^{nm(n-1)} \\
S_3 &= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{d_{(K_m\Box K_n)}(w_{ij})d_{(K_m\Box K_n)}(w_{pq})}{d_{(K_m\Box K_n)}(w_{ij},w_{pq})}\Big] \\
&= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{\Big(d_{K_m}(u_i)+d_{K_n}(v_j)\Big)\Big(d_{K_m}(u_p)+d_{K_n}(v_q)\Big)}{d_{K_m}(u_i,u_p)}\Big] \text{Lemma 2.1} \\
&= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \Big[\frac{\Big((m-1)+(n-1)\Big)\Big((m-1)+(n-1)\Big)}{2}\Big] \\
&= \prod_{j,q=0,j\neq q}^{n-1} \prod_{i,p=0,i\neq p}^{m-1} \frac{(m+n-2)^2}{2} \\
&= \Big[\frac{(m+n-2)^2}{2}\Big]^{nm(n-1)(m-1)} \\
&= \frac{\Big[(m+n-2)^2\Big]^{nm(n-1)(m-1)}}{2^{nm(n-1)(m-1)}}
\end{aligned}
$$

By multiplying S_1, S_2 andS_3 , we get the desired result. □

4. The Multiplicative reciprocal degree distance of Strong product of graphs.

LEMMA 4.1. ***(Arithmetic-Geometric inequality)*** *Let $x_1, x_2, ..., x_n$ be non-negative numbers. Then* $\frac{x_1+x_2+...+x_n}{n} \geqslant \sqrt[n]{x_1x_2...x_n}$

LEMMA 4.2. *Let G be a connected a graph with n vertices and m edges. Let $V(K_r) = \{v_1, v_2, v_3, ..., v_r\}$. Let x_{ij} denote the vertex (u_i, u_j) of $G \boxtimes K_r$. Now $d_{G\boxtimes K_r}(x_{ij}) = rd_G(u_i) + (r-1)$ and*

$$d_{G\boxtimes K_r}(x_{ij}, x_{kp}) = \begin{cases} 1, & i = k, j \neq p \\ d_G(u_i, u_k), & i \neq k, j = p \\ d_G(u_i, u_k), & i \neq k, j \neq p \end{cases}$$

THEOREM 4.1. *Let G be a $(n, m)-$ graph, then*

$$[RDD^*(G \boxtimes K_r)]^2 \leqslant \Big[\frac{4rm + 2n(r-1)}{n}\Big]^{nr(r-1)} \times \Big[\frac{2rRDD^+(G) + 4(r-1)H^+(G)}{n(n-1)}\Big]^{nr(nr-r)}$$

PROOF.

$$[RDD^*(G \boxtimes K_r)]^2 = \prod_{w_{ij}, w_{pq} \in V(G)} \Big[\frac{d_{(G\boxtimes K_r)}(w_{ij}) + d_{(G\boxtimes K_r)}(w_{pq})}{d_{(G\boxtimes K_r)}(w_{ij}, w_{pq})}\Big]$$

$$\begin{aligned}
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} \Big[\frac{d_{(G\boxtimes K_r)}(w_{ij}) + d_{(G\boxtimes K_r)}(w_{iq})}{d_{(G\boxtimes K_r)}(w_{ij}, w_{iq})}\Big] \\
&\times \prod_{j=0}^{r-1} \prod_{i,p=0, i\neq p}^{n-1} \Big[\frac{d_{(G\boxtimes K_r)}(w_{ij}) + d_{(G\boxtimes K_r)}(w_{pj})}{d_{(G\boxtimes K_r)}(w_{ij}, w_{pj})}\Big] \\
&\times \prod_{j,q=0, j\neq q}^{r-1} \prod_{i,p=0, i\neq p}^{n-1} \Big[\frac{d_{(G\boxtimes K_r)}(w_{ij}) + d_{(G\boxtimes K_r)}(w_{pq})}{d_{(G\boxtimes K_r)}(w_{ij}, w_{pq})}\Big] \\
&= C_1 \times C_2 \times C_3,
\end{aligned}$$

where C_1, C_2, C_3 are terms of above product taken in order.

Next we calculate C_1, C_2 and C_3 separtely one by one. Now,

$$\begin{aligned}
C_1 &= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} \Big[\frac{d_{(G\boxtimes K_r)}(w_{ij}) + d_{(G\boxtimes K_r)}(w_{iq})}{d_{(G\boxtimes K_r)}(w_{ij}, w_{iq})}\Big] \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} \Big[\frac{rd(u_i) + (r-1) + rd(u_i) + (r-1)}{1}\Big] \text{Lemma 4.2} \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} \Big[\frac{2rd(u_i) + 2(r-1)}{1}\Big] \\
&\leqslant \Big[\sum_{i=0}^{n-1} \sum_{j,q=0, j\neq q}^{r-1} \frac{2rd(u_i) + 2(r-1)}{nr(r-1)}\Big]^{nr(r-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\Big(\sum_{i=0}^{n-1}\frac{2rd(u_i)+2(r-1)}{nr(r-1)}\Big)\Big(\sum_{j,q=0,j\neq q}^{r-1}1\Big)\Big]^{nr(r-1)}\\
&= \Big[\frac{r(r-1)}{nr(r-1)}\Big\{2r\sum_{i=0}^{n-1}d(u_i)+2(r-1)\sum_{i=0}^{n-1}1\Big\}\Big]^{nr(r-1)}\\
&= \Big[\frac{1}{n}\Big\{4mr+2n(r-1)\Big\}\Big]^{nr(r-1)}\\
&\leqslant \Big[\frac{4mr+2n(r-1)}{n}\Big]^{nr(r-1)}\\
C_2 &= \prod_{j=0}^{r-1}\prod_{i,p=0,i\neq p}^{n-1}\Big[\frac{d_{(G\boxtimes K_r)}(w_{ij})+d_{(G\boxtimes K_r)}(w_{pj})}{d_{(G\boxtimes K_r)}(w_{ij},w_{pj})}\Big]\\
&= \prod_{j=0}^{r-1}\prod_{i,p=0,i\neq p}^{n-1}\Big[\frac{rd(u_i)+(r-1)+rd(u_p)+(r-1)}{d(u_i,u_p)}\Big]\text{Lemma 4.2}\\
&= \prod_{j=0}^{r-1}\prod_{i,p=0,i\neq p}^{n-1}\Big[\frac{r[d(u_i)+d(u_p)]}{d(u_i,u_p)}+2\frac{(r-1)}{d(u_i,u_p)}\Big]\\
&\leqslant \Big[\frac{\sum_{j=0}^{r-1}\sum_{i,p=0,i\neq p}^{n-1}\Big(\frac{r[d(u_i)+d(u_p)]}{d(u_i,u_p)}+2\frac{(r-1)}{d(u_i,u_p)}\Big)}{rn(n-1)}\Big]^{nr(n-1)}\\
&= \Big[\frac{1}{rn(n-1)}\sum_{j=0}^{r-1}\Big\{r\sum_{i,p=0,i\neq p}^{n-1}\frac{[d(u_i)+d(u_p)]}{d(u_i,u_p)}\\
&\quad +2(r-1)\sum_{i,p=0,i\neq p}^{n-1}\frac{1}{d(u_i,u_p)}\Big\}\Big]^{nr(n-1)}\\
&= \Big[\frac{1}{rn(n-1)}\sum_{j=0}^{r-1}\Big\{2rRDD^+(G)+2(r-1)2H^+(G)\Big\}\Big]^{nr(n-1)}\\
&= \Big[\frac{2rRDD^+(G)+4(r-1)H^+(G)}{rn(n-1)}\sum_{j=0}^{r-1}1\Big]^{nr(n-1)}\\
&= \Big[\frac{2rRDD^+(G)+4(r-1)H^+(G)}{rn(n-1)}r\Big]^{nr(n-1)}\\
&\leqslant \Big[\frac{2rRDD^+(G)+4(r-1)H^+(G)}{n(n-1)}\Big]^{nr(n-1)}\\
C_3 &= \prod_{i,p=0,i\neq p}^{n-1}\prod_{j,q=0,j\neq q}^{r-1}\Big[\frac{d_{(G\boxtimes K_r)}(w_{ij})+d_{(G\boxtimes K_r)}(w_{pq})}{d_{(G\boxtimes K_r)}(w_{ij},w_{pq})}\Big]\\
&= \prod_{i,p=0,i\neq p}^{n-1}\prod_{j,q=0,j\neq q}^{r-1}\Big[\frac{rd(u_i)+(r-1)+rd(u_p)+(r-1)}{d(u_i,u_p)}\Big]\text{Lemma 4.2}
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{i,p=0,i\neq p}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} \Big[\frac{r[d(u_i)+d(u_p)]}{d(u_i,u_p)} + 2\frac{(r-1)}{d(u_i,u_p)}\Big] \\
&\leqslant \Big[\frac{\sum\limits_{i,p=0,i\neq p}^{n-1} \sum\limits_{j,q=0,j\neq q}^{r-1} \Big(\frac{r[d(u_i)+d(u_p)]}{d(u_i,u_p)} + 2\frac{(r-1)}{d(u_i,u_p)}\Big)}{rn(n-1)(r-1)}\Big]^{nr(r-1)(n-1)} \\
&= \Big[\frac{r(r-1)}{rn(n-1)(r-1)}\Big\{r \sum_{i,p=0,i\neq p}^{n-1} \frac{[d(u_i)+d(u_p)]}{d(u_i,u_p)} \\
&\quad +2(r-1)\sum_{i,p=0,i\neq p}^{n-1} \frac{1}{d(u_i,u_p)}\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \Big[\frac{1}{n(n-1)}\Big\{2rRDD^+(G) + 2(r-1)2H^+(G)\Big\}\Big]^{nr(n-1)(r-1)} \\
C_3 &\leqslant \Big[\frac{2rRDD^+(G)+4(r-1)H^+(G)}{n(n-1)}\Big]^{nr(n-1)(r-1)}
\end{aligned}
$$

By multiplying C_1, C_2 and C_3 the desired result follows after simple calculation. □

Lemma 4.3.

$$[RDD^* K_n \boxtimes K_r] = (2nr-2)^{\frac{rn(rn-1)}{2}}$$

Proof. In previous Theorem $G = K_n$.
The degree of every vertex in $K_n \boxtimes K_r$ is $r(n-1)+r-1 = (rn-1)$.
$\therefore K_n \boxtimes K_r$ is a complete graph.
Hence

$$\therefore [RDD^* K_n \boxtimes K_r] = \Big(\frac{2nr-2}{1}\Big)^{\frac{rn(rn-1)}{2}}$$

$$[RDD^* K_n \boxtimes K_r] = (2nr-2)^{\frac{rn(rn-1)}{2}} \tag{4.1}$$

□

Remark 4.1. *Clearly* $RDD^+(K_n) = (2n-2) \times \frac{n(n-1)}{2} = n(n-1)^2$ *and* $H^+(K_n) = \frac{n(n-1)}{2}$
When $G = K_n$ *in Theoerm 4.1, we get*

$$
\begin{aligned}
[RDD^* K_n \boxtimes K_r] &\leqslant \Big[\frac{4rm+2n(r-1)}{n}\Big]^{nr(r-1)} \\
&\quad \times \Big[\frac{2rRDD^+(G)+4(r-1)H^+(G)}{n(n-1)}\Big]^{nr(n-1)} \\
&\quad \times \Big[\frac{2rRDD^+(G)+4(r-1)H^+(G)}{n(n-1)}\Big]^{nr(n-1)(r-1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\frac{4rn(n-1)+4n(r-1)}{2n}\Big]^{\frac{nr(r-1)}{2}}\\
&\times\Big[\frac{2rn(n-1)^2+4(r-1)\frac{n(n-1)}{2}}{n(n-1)}\Big]^{\frac{nr(n-1)}{2}}\\
&\times\Big[\frac{2rn(n-1)^2+4(r-1)\frac{n(n-1)}{2}}{n(n-1)}\Big]^{nr(n-1)(r-1)}\\
&= \Big[2r(n-1)+2(r-1)\Big]^{\frac{nr(r-1)}{2}}\times\Big[2r(n-1)+2(r-1)\Big]^{\frac{nr(n-1)}{2}}\\
&\times\Big[2r(n-1)+2(r-1)\Big]^{\frac{nr(r-1)(n-1)}{2}}
\end{aligned}
$$

$$
[RDD^*K_n\boxtimes K_r] \leqslant (2nr-2)^{\frac{rn(rn-1)}{2}} \tag{4.2}
$$

$\therefore$ (4.1) *and* (4.2) *our bound is tight.*

5. The Multiplicative Reciprocal Gutman Index of Strong Product of Graphs.

THEOREM 5.1. *Let G be a $(m,n)-$graph. Then*

$$
\begin{aligned}
[RGut^*(G\boxtimes K_r)]^2 &\leqslant \Big[\frac{r^2M_1(G)+4r(r-1)m+n(r-1)^2}{n}\Big]^{nr(r-1)}\\
&\times \Big[\frac{2r^2RGut^+(G)+2r(r-1)RDD^+(G)+2(r-1)^2H^+(G)}{n(n-1)}\Big]^{nr(nr-r)}
\end{aligned}
$$

PROOF.

$$
\begin{aligned}
[RGut^*(G\boxtimes K_r)]^2 &= \prod_{w_{ij},w_{pq}\in V(G_1\boxtimes G_2)}\Big[\frac{d_{(G\boxtimes K_r)}(w_{ij})d_{(G\boxtimes K_r)}(w_{pq})}{d_{(G\boxtimes K_r)}(w_{ij},w_{pq})}\Big]\\
&= \prod_{i=0}^{n-1}\prod_{j,q=0,j\neq q}^{r-1}\Big[\frac{d_{(G\boxtimes K_r)}(w_{ij})d_{(G\boxtimes K_r)}(w_{iq})}{d_{(G\boxtimes K_r)}(w_{ij},w_{iq})}\Big]\\
&\times \prod_{j=0}^{r-1}\prod_{i,p=0,i\neq p}^{n-1}\Big[\frac{d_{(G\boxtimes K_r)}(w_{ij})d_{(G\boxtimes K_r)}(w_{pj})}{d_{(G\boxtimes K_r)}(w_{ij},w_{pj})}\Big]\\
&\times \prod_{j,q=0,j\neq q}^{r-1}\prod_{i,p=0,i\neq p}^{n-1}\Big[\frac{d_{(G\boxtimes K_r)}(w_{ij})d_{(G\boxtimes K_r)}(w_{pq})}{d_{(G\boxtimes K_r)}(w_{ij},w_{pq})}\Big]\\
&= J_1\times J_2\times J_3,
\end{aligned}
$$

where J_1, J_2, J_3 are terms of above product taken in order.

Next we calculate J_1, J_2 and J_3 separtely one by one. Now,

$$\begin{aligned}
J_1 &= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} \Big[\frac{d_{(G\boxtimes K_r)}(w_{ij})d_{(G\boxtimes K_r)}(w_{iq})}{d_{(G\boxtimes K_r)}(w_{ij},w_{iq})}\Big]\\
&= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} \Big[\frac{\Big(rd(u_i)+(r-1)\Big)\Big(rd(u_i)+(r-1)\Big)}{1}\Big] \text{Lemma 4.2}\\
&= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} \Big[r^2d^2(u_i)+2r(r-1)d(u_i)+(r-1)^2\Big]\\
&\leqslant \Big[\frac{1}{nr(r-1)}\sum_{i=0}^{n-1}\sum_{j,q=0,j\neq q}^{r-1}\Big\{r^2d^2(u_i)+2r(r-1)d(u_i)+(r-1)^2\Big\}\Big]^{nr(r-1)}\\
&= \Big[\frac{r(r-1)}{nr(r-1)}\Big\{r^2\sum_{i=0}^{n-1}d^2(u_i)+2r(r-1)\sum_{i=0}^{n-1}d(u_i)+(r-1)^2\sum_{i=0}^{n-1}1\Big\}\Big]^{nr(r-1)}\\
&\leqslant \Big[\frac{r^2M_1(G)+4r(r-1)m+n(r-1)^2}{n}\Big]^{nr(r-1)}\\
J_2 &= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} \Big[\frac{d_{(G\boxtimes K_r)}(w_{ij})d_{(G\boxtimes K_r)}(w_{pj})}{d_{(G\boxtimes K_r)}(w_{ij},w_{pj})}\Big]\\
&= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} \Big[\frac{\Big(rd(u_i)+(r-1)\Big)\Big(rd(u_p)+(r-1)\Big)}{d(u_i,u_p)}\Big] \text{Lemma 4.2}\\
&= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} \Big[\frac{r^2d(u_i)d(u_p)+r(r-1)d(u_i)+r(r-1)d(u_p)+(r-1)^2}{d(u_i,u_p)}\Big]\\
&\leqslant \Big[\frac{1}{rn(n-1)}\sum_{j=0}^{r-1}\sum_{i,p=0,i\neq p}^{n-1}\Big\{\frac{1}{d(u_i,u_p)}\Big(r^2d(u_i)d(u_p)+r(r-1)d(u_i)\\
&\quad +r(r-1)d(u_p)+(r-1)^2\Big)\Big\}\Big]^{nr(n-1)}\\
&= \Big[\frac{1}{rn(n-1)}\sum_{j=0}^{r-1}\Big\{r^2\sum_{i,p=0,i\neq p}^{n-1}\frac{[d(u_i)d(u_p)]}{d(u_i,u_p)}\\
&\quad +r(r-1)\sum_{i,p=0,i\neq p}^{n-1}\frac{d(u_i)+d(u_p)}{d(u_i,u_p)}+(r-1)^2\sum_{i,p=0,i\neq p}^{n-1}\frac{1}{d(u_i,u_p)}\Big\}\Big]^{nr(n-1)}\\
&= \Big[\frac{1}{rn(n-1)}\sum_{j=0}^{r-1}\Big\{2r^2RGut^+(G)+2r(r-1)RDD^+(G)\\
&\quad +2(r-1)^2H^+(G)\Big\}\Big]^{nr(n-1)}
\end{aligned}$$

$$= \Big[\frac{1}{rn(n-1)} r\Big\{2r^2 RGut^+(G) + 2r(r-1)RDD^+(G) + 2(r-1)^2 H^+(G)\Big\}\Big]^{nr(n-1)}$$

$$\leqslant \Big[\frac{2r^2 RGut^+(G) + 2r(r-1)RDD^+(G) + 2(r-1)^2 H^+(G)}{n(n-1)}\Big]^{nr(n-1)}$$

$$J_3 = \prod_{j,q=0,j\neq q}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} \Big[\frac{d_{(G\boxtimes K_r)}(w_{ij}) d_{(G\boxtimes K_r)}(w_{pq})}{d_{(G\boxtimes K_r)}(w_{ij}, w_{pq})}\Big]$$

$$= \prod_{j,q=0,j\neq q}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} \Big[\frac{\Big(rd(u_i) + (r-1)\Big)\Big(rd(u_p) + (r-1)\Big)}{d(u_i, u_p)}\Big] \text{Lemma 4.2}$$

$$= \prod_{j,q=0,j\neq q}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} \frac{1}{d(u_i, u_p)}\Big[r^2 d(u_i)d(u_p) + r(r-1)d(u_i) + r(r-1)d(u_p) + (r-1)^2\Big]$$

$$\leqslant \Big[\frac{1}{rn(r-1)(n-1)} \sum_{j,q=0,j\neq q}^{r-1} \sum_{i,p=0,i\neq p}^{n-1} \frac{1}{d(u_i, u_p)}\Big\{r^2 d(u_i)d(u_p) + r(r-1)d(u_i) + r(r-1)d(u_p) + (r-1)^2\Big\}\Big]^{nr(r-1)(n-1)}$$

$$= \Big[\frac{1}{rn(r-1)(n-1)} \sum_{j,q=0,j\neq q}^{r-1} \Big\{r^2 \sum_{i,p=0,i\neq p}^{n-1} \frac{[d(u_i)d(u_p)]}{d(u_i, u_p)} + r(r-1) \sum_{i,p=0,i\neq p}^{n-1} \frac{d(u_i) + d(u_p)}{d(u_i, u_p)} + (r-1)^2 \sum_{i,p=0,i\neq p}^{n-1} \frac{1}{d(u_i, u_p)}\Big\}\Big]^{nr(r-1)(n-1)}$$

$$= \Big[\frac{r(r-1)}{rn(r-1)(n-1)}\Big\{2r^2 RGut^+(G) + 2r(r-1)RDD^+(G) + 2(r-1)^2 H^+(G)\Big\}\Big]^{nr(r-1)(n-1)}$$

$$= \Big[\frac{2r^2 RGut^+(G) + 2r(r-1)RDD^+(G) + 2(r-1)^2 H^+(G)}{n(n-1)}\Big]^{nr(r-1)(n-1)}$$

By multiplying J_1, J_2 and J_3 the desired result follows after simple calculation. □

Lemma 5.1.

$$[RGut^*(K_n \boxtimes K_r)] = (nr-1)^{nr(nr-1)}$$

Proof. When $G = K_n$ in Theoerm. Clearly $K_n \boxtimes K_r$ is a complete graph

$$[RGut^*(K_n \boxtimes K_r)] = \Big(\frac{(nr-1)(nr-1)}{1}\Big)^{\frac{nr(nr-1)}{2}}$$

$$\therefore [RGut^*(K_n \boxtimes K_r)] = (nr-1)^{nr(nr-1)} \tag{5.1}$$

□

Remark 5.1. *$RGut^+(K_n) = (n-1)^2 \times \frac{n(n-1)}{2} = \frac{n(n-1)^3}{2}$, $RDD^+(K_n) = n(n-1)^2$, $M_1(K_n) = n(n-1)^2$ and $H^+(K_n) = \frac{n(n-1)}{2}$. $K_n \boxtimes K_r$ is a complete graph.*

$\therefore$ By Theorem 5.1

$$[RGut^*(K_n \boxtimes K_r)] \leqslant (J_1)^{\frac{1}{2}} \times (J_2)^{\frac{1}{2}} \times (J_3)^{\frac{1}{2}}$$

$$= \Big[\frac{r^2M_1(G) + 4r(r-1)m + n(r-1)^2}{n}\Big]^{\frac{nr(r-1)}{2}}$$

$$\times \Big[\frac{2r^2RGut^+(G) + 2r(r-1)RDD^+(G) + 2(r-1)^2H^+(G)}{n(n-1)}\Big]^{\frac{nr(n-1)}{2}}$$

$$\times \Big[\frac{2r^2RGut^+(G) + 2r(r-1)RDD^+(G) + 2(r-1)^2H^+(G)}{n(n-1)}\Big]^{\frac{nr(r-1)(n-1)}{2}}$$

$$= \Big[\frac{r^2n(n-1)^2 + 4r(r-1)\ \frac{n(n-1)}{2} + n(r-1)^2}{n}\Big]^{\frac{nr(r-1)}{2}}$$

$$\times \Big[\frac{2r^2\ \frac{n(n-1)^3}{2} + 2r(r-1)n(n-1)^2 + 2(r-1)^2\ \frac{n(n-1)}{2}}{n(n-1)}\Big]^{\frac{nr(n-1)}{2}}$$

$$\times \Big[\frac{2r^2\ \frac{n(n-1)^3}{2} + 2r(r-1)n(n-1)^2 + 2(r-1)^2\ \frac{n(n-1)}{2}}{n(n-1)}\Big]^{\frac{nr(r-1)(n-1)}{2}}$$

$$= \Big[r^2(n-1)^2 + 2r(r-1)(n-1) + (r-1)^2\Big]^{\frac{nr(r-1)}{2}}$$

$$\times \Big[r^2(n-1)^2 + 2r(r-1)(n-1) + (r-1)^2\Big]^{\frac{nr(n-1)}{2}}$$

$$\times \Big[r^2(n-1)^2 + 2r(r-1)(n-1) + (r-1)^2\Big]^{\frac{nr(r-1)(n-1)}{2}}$$

$$= \Big[(nr-1)^2\Big]^{\frac{nr(nr-1)}{2}}$$

$$\therefore [RGut^*(K_n \boxtimes K_r)] \leqslant (nr-1)^{rn(rn-1)} \tag{5.2}$$

$\therefore$ (5.1) *and* (5.2) *our bound is tight.*

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Department of Mathematics, Government Arts College,Tiruchirappalli, India
E-mail address: muruganraju1980@gmail.com

Department of Mathematics, Bharathidasan University Constituent College, Lalgudi, Tiruchirappalli India
E-mail address: manirs2004@yahoo.co.in

Department of Mathematics, Anna University, Tiruchirappalli, India
E-mail address: aruvi.aut@gmail.com