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ESTIMATE FOR p - VALENTLY FUNCTIONS AT THE BOUNDARY

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Abstract. In this paper, a boundary version of the Schwarz lemma for classes $\mathcal{M}(p)$ is investigated. For the function $f(z) = z^p + a_{p+1}z^{p+1} + c_{p+2}z^{p+2} + ...$ defined in the unit disc $D = \{z : |z| < 1\}$ such that $f(z) \in \mathcal{M}(p)$, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point b with $f'(b) = 0$. The sharpness of these inequalities is also proved.

1. Introduction

The Schwarz lemma on the unit disc $D = \{z : |z| < 1\}$ says that if $f : D \to D$ be a holomorphic function with $f(0) = 0$, then $|f(z)| \leqslant |z|$ for all $z \in D$, and $|f'(0)| \leqslant 1$. In addition, if equality ever occurs in either of the two preceding inequalities, then $f(z)$ is of the form $f(z) = ze^{i\theta}$, θ real ([**5**], p. 329). Needless to say, this result has been generalized to many different situations. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [**1**], [**20**]).

To derive our main results, we have to recall here the following lemma due to Jack [**6**].

Lemma 1.1 (Jack's Lemma). *Let $f(z)$ be holomorphic function in the unit disc D with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then*

$$z_0 f'(z_0) = kf(z_0),$$

where $k \geqslant 1$ is a real number.

Let $\mathcal{N}(p)$ be the class of functions of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + ...,\ p \in \mathbb{N} = \{1, 2, ...\},$$

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which are holomorphic in the unit disc D. Also, let $\mathcal{M}(p)$ be the subclass of $\mathcal{N}(p)$ consisting of all functions f which satisfy

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(1+\frac{1}{4p}\right)\right\}\right|>0,\ z\in D \tag{1.1}$$

where $0<\alpha\leqslant 1$.

Let $f\in\mathcal{M}(p)$ and consider the function

$$h(z)=\frac{1}{p}\left(\frac{zf'(z)}{f(z)}-p\right).$$

The function h is a holomorphic function in D and $h(0)=0$. From the definition of h, we take

$$\frac{zf'(z)}{f(z)}=p\left(1+h(z)\right). \tag{1.2}$$

Differentiating both sides of (1.2) logarithmically, we obtain

$$1+\frac{zf''(z)}{f'(z)}=p\left(1+h(z)\right)+\frac{zh'(z)}{1+h(z)},$$

and

$$\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)=1+\frac{zh'(z)}{p\left(1+h(z)\right)^2}.$$

Now, let us show that the function $|h(z)|<1$ in D. We suppose that there exists a point $z_0\in D$ such that

$$\max_{|z|\leqslant|z_0|}|h(z)|=|h(z_0)|=1.$$

From Jack's lemma, we have

$$h(z_0)=e^{i\theta}(\theta\neq\pi)\text{ and }z_0h'(z_0)=kh(z_0).$$

Thus, we have

$$\begin{aligned}\frac{f(z_0)}{z_0f'(z_0)}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)&=1+\frac{z_0h'(z_0)}{p\left(1+h(z_0)\right)^2}=1+\frac{ke^{i\theta}}{p\left(1+e^{i\theta}\right)^2}\\&=1+\frac{k}{p}\left(\frac{e^{i\theta}}{1+2e^{i\theta}+e^{2i\theta}}\right)=1+\frac{k}{p}\left(\frac{1}{\frac{1}{e^{i\theta}}+2+e^{i\theta}}\right)\\&=1+\frac{k}{p}\left(\frac{1}{\frac{1}{\cos\theta+i\sin\theta}+2+\cos\theta+i\sin\theta}\right)\\&=1+\frac{k}{2p\left(1+\cos\theta\right)}\geqslant 1+\frac{1}{4p},\end{aligned}$$

that is,

$$\frac{f(z_0)}{z_0f'(z_0)}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\geqslant 1+\frac{1}{4p}. \tag{1.3}$$

Note that condition (1.1) implies

$$\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\neq\gamma,\ z\in D,$$

where $\gamma\geqslant 1+\frac{1}{4p}$. Therefore, (1.3) contradicts our condition (1.1) .This shows that $|h(z)|<1$ for $|z|<1$. By the Schwarz lemma, we obtain

$$|h'(0)|\leqslant 1$$

and

$$|a_{p+1}|\leqslant p. \tag{1.4}$$

The equality in (1.4) holds if and only if $h(z)=ze^{i\theta}$ (see, [**16**]) that is,

$$f(z)=z^p e^{pze^{i\theta}}.$$

That proves

Lemma 1.2. *If $f(z)\in\mathcal{M}(p)$, then we have*

$$|a_{p+1}|\leqslant p. \tag{1.5}$$

The equality in (1.5) *holds if and only if*

$$f(z)=z^p e^{pze^{i\theta}},$$

where θ is a real number.

The boundary version of Schwarz lemma is simply known as below:

Let $f(z)$ be a holomorphic function in the unit disc D, $f(0)=0$ and $|f(z)|<1$ for $|z|<1$. Assume that, there is a $b\in\partial D=\{z:|z|=1\}$ so that f extends continuously to b, $|f(b)|=1$ and $f'(b)$ exists. Therefore, the inequality $|f'(b)|\geqslant 1$, that is, known as Schwarz lemma at the boundary from the classic Schwarz lemma, is obtained. The equality in $|f'(b)|\geqslant 1$ holds if and only if $f(z)=ze^{i\theta}$, θ real.

This result of Schwarz lemma and its generalization are described as Schwarz lemma at the boundary in the literature. This improvement was obtained in [**22**] by Helmut Unkelbach, and rediscovered by R. Osserman in [**16**] 60 years later.

In the last 15 years, there have been tremendous studies on Schwarz lemma at the boundary (see, [**1**], [**2**], [**3**], [**4**], [**7**], [**8**], [**10**], [**11**], [**16**], [**17**], [**18**], [**20**] and references therein). Some of them are about the estimation from lower of the modulus of the derivative of the holomorphic function which satisfies the condition $|f(b)|=1$ for $b\in\partial D$.

In [**16**], R. Osserman gave the following boundary refinement of the classical Schwarz lemma.

Lemma 1.3. *Let $f:D\to D$ be holomorphic function with $f(0)=0$. Assume that there is a $b\in\partial D$ so that f extends continuously to b, $|f(b)|=1$ and $f'(b)$ exists. Then*

$$|f'(b)|\geqslant\frac{2}{1+|f'(0)|}. \tag{1.6}$$

Inequality (1.6) *is sharp, with equality possible for each value of $|f'(0)|$.*

COROLLARY 1.1. *Under the hypotheses of Lemma 1.3, we have*

$$|f'(b)| \geqslant 1 \tag{1.7}$$

and

$$|f'(b)| > 1 \text{ unless } \quad f(z) = ze^{i\theta},\ \theta \text{ real.}$$

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [**19**]).

LEMMA 1.4 (Julia-Wolff lemma). *Let f be a holomorphic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leqslant |f'(b)| \leqslant \infty$.*

COROLLARY 1.2. *The holomorphic function f has a finite angular derivative $f'(b)$ if and only if f' has the finite angular limit $f'(b)$ at $b \in \partial D$.*

D. M. Burns and S. G. Krantz [**9**] and D. Chelst [**2**] studied the uniqueness part of the Schwarz lemma. The similar types of results which are related with the subject of the paper can be found in ([**12**], [**13**] and [**14**]). In addition, the concerning results in more general aspects is discussed by M. Mateljević in [**15**] where was announced on ResearchGate.

X. Tang, T. Liu and J. Lu [**11**] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Taishun Liu, Jianfei Wang, Xiaomin Tang [**21**] established a new type of the classical boundary Schwarz lemma for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$. They then applied their new Schwarz lemma to study problems from the geometric function theory in several complex variables.

In addition, M. Jeong [**7**] showed some inequalities at a boundary point for different form of holomorphic functions. He also found the condition for equality. In [**6**], a holomorphic self map was defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main Results

In this section, for holomorphic function $f(z)$ belong to the class of $\mathcal{M}(p)$, it will be estimated from below the modulus of the angular derivative of the function on a boundary point of the unit disc. These results are sharp.

THEOREM 2.1. *Let $f \in \mathcal{M}(p)$. Suppose that, for some $b \in \partial D$, f' has a non-tangential limit $f'(b)$ at b and $f'(b) = 0$. Then f has the second non-tangential derivative and*

$$\left| \frac{f''(b)}{f(b)} \right| \geqslant p. \tag{2.1}$$

Moreover, the inequality (2.1) *is sharp with extremal function*

$$f(z) = z^p e^{pz}.$$

PROOF. Let

$$h(z)=\frac{1}{p}\left(\frac{zf'(z)}{f(z)}-p\right).$$

Then $h(z)$ is holomorphic function in the unit disc D, $h(0)=0$ and $|h(z)|<1$ for $|z|<1$. Also, since $f'(b)=0$, we have $|h(b)|=1$ for $b\in\partial D$.

From (1.7), we obtain

$$1\leqslant|h'(b)|=\frac{1}{p}\left|\frac{(f'(b)+bf''(b))\,f(b)-f'(b)bf'(b)}{f^2(b)}\right|$$

Since $f'(b)=0$, we have

$$1\leqslant|h'(b)|\leqslant\frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|$$

and hence

$$\left|\frac{f''(b)}{f(b)}\right|\geqslant p.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z)=z^pe^{pz}.$$

Then by logarithmic differentiation

$$\frac{f'(z)}{f(z)}=\frac{p}{z}+p$$

and then by the quotient rade,

$$\frac{f''(z)f(z)-f'(z)f'(z)}{(f(z))^2}=-\frac{p}{z^2}.$$

From the hypothesis, since $f'(-1)=0$ for $-1\in\partial D$, we obtain

$$\left|\frac{f''(-1)}{f(-1)}\right|=p.$$

□

THEOREM 2.2. *Let $f\in\mathcal{M}(p)$. Suppose that, for some $b\in\partial D$, f' has a non-tangential limit $f'(b)$ at b and $f'(b)=0$. Then f has the second non-tangential derivative and*

$$\left|\frac{f''(b)}{f(b)}\right|\geqslant\frac{2p^2}{p+|a_{p+1}|}. \tag{2.2}$$

The inequality (2.2) is sharp with equality for the function

$$f(z)=z^pe^{pz}.$$

PROOF. Let h be the same as in the proof of Theorem 2.1. By the proof of Theorem 2.1, f has the second non-tangential derivative. From (1.6), we obtain

$$\frac{2}{1+|h'(0)|}\leqslant|h'(b)|\leqslant\frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|.$$

Since

$$|h'(0)| = \frac{|a_{p+1}|}{p},$$

we have

$$\frac{2}{1+\frac{|a_{p+1}|}{p}} \leqslant \frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|$$

and hence

$$\left|\frac{f''(b)}{f(b)}\right| \geqslant \frac{2p^2}{p+|a_{p+1}|}.$$

Thus, we obtain the inequality (2.2) with an obvious equality case. $\square$

The inequality (2.2) can be strengthened as below by taking into account a_{p+2}, which is the third coefficient in the expansion of the function $f(z)$.

THEOREM 2.3. *Let $f \in \mathcal{M}(p)$. Suppose that, for some $b \in \partial D$, f' has a non-tangential limit $f'(b)$ at b and $f'(b) = 0$. Then f has the second non-tangential derivative and*

$$\left|\frac{f''(b)}{f(b)}\right| \geqslant p\left(1+\frac{2\left[p-|a_{p+1}|\right]^2}{p^2-|a_{p+1}|^2+p\left|2a_{p+2}-a_{p+1}^2\right|}\right). \tag{2.3}$$

The equality in (2.3) *occurs for the function*

$$f(z) = z^p e^{pz}.$$

PROOF. Let h be the same as in the proof of Theorem 2.1 and $\varkappa(z) = z$. By the maximum principle, for each $z \in D$, we have

$$|h(z)| \leqslant |\varkappa(z)|.$$

Therefore

$$\varpi(z) = \frac{h(z)}{\varkappa(z)} = \frac{h(z)}{z}$$

is a holomorphic function in D and $|\varkappa(z)| < 1$ for $|z| < 1$. By Schwarz lemma, in particular, we have

$$|\varpi(0)| = \frac{|a_{p+1}|}{p} \leqslant 1 \tag{2.4}$$

and

$$|\varpi'(0)| = \frac{1}{p}\left|2a_{p+2}-a_{p+1}^2\right|.$$

Furthermore, the geometric meaning of the derivative and the inequality $|h(z)| \leqslant |\varkappa(z)|$ imply the inequality

$$\frac{bh'(b)}{h(b)} \geqslant |h'(b)| \geqslant |\varkappa'(b)| = \frac{b\varkappa'(b)}{\varkappa(b)}.$$

That is, since the expression $\frac{bh'(b)}{h(b)}$ is a real number greater than or equal to 1 (see,[**1**]) and $f'(b)=0$ yields $|h(b)|=1$, we get

$$\frac{bh'(b)}{h(b)}=\left|\frac{bh'(b)}{h(b)}\right|=|h'(b)|.$$

Also, $|h(z)|\leqslant|\varkappa(z)|$, we take

$$\frac{1-|h(z)|}{1-|z|}\geqslant\frac{1-|\varkappa(z)|}{1-|z|}.$$

Passing to the angular limit in the last inequality yields

$$|h'(b)|\geqslant|\varkappa'(z)|.$$

The function

$$\Theta(z)=\frac{\varpi(z)-\varpi(0)}{1-\varpi(z)\overline{\varpi(0)}}$$

is holomorphic function in D, $|\Theta(z)|<1$ for $|z|<1$, $\Theta(0)=0$ and $|\Theta(b)|=1$ for all $b\in\partial D$.

From (1.6), we obtain

$$\begin{aligned}\frac{2}{1+|\Theta'(0)|} &\leqslant |\Theta'(b)|\leqslant\frac{1+|\varpi(0)|}{1-|\varpi(0)|}\left|\frac{h'(b)}{\varkappa(b)}-\frac{h(b)\varkappa'(b)}{\varkappa^2(b)}\right|\\ &= \frac{1+|\varpi(0)|}{1-|\varpi(0)|}\left|\frac{h(b)}{b\varkappa(b)}\right|\left|\frac{bh'(b)}{h(b)}-\frac{b\varkappa'(b)}{\varkappa(b)}\right|\\ &= \frac{1+|\varpi(0)|}{1-|\varpi(0)|}\left(|h'(b)|-|\varkappa'(b)|\right)\\ &= \frac{1+|\varpi(0)|}{1-|\varpi(0)|}\left\{|h'(b)|-1\right\}.\end{aligned}$$

Since

$$\Theta'(z)=\frac{1-|\varpi(0)|^2}{\left(1-\overline{\varpi(0)}\varpi(z)\right)^2}\varpi'(z)$$

and

$$|\Theta'(0)|=\frac{|\varpi'(0)|}{1-|\varpi(0)|^2}=\frac{\frac{1}{p}\left|2a_{p+2}-a_{p+1}^2\right|}{1-\left(\frac{|a_{p+1}|}{p}\right)^2}=p\frac{\left|2a_{p+2}-a_{p+1}^2\right|}{p^2-|a_{p+1}|^2}$$

we have

$$\frac{2}{1+p\frac{\left|2a_{p+2}-a_{p+1}^2\right|}{p^2-|a_{p+1}|^2}}\leqslant\frac{p+|a_{p+1}|}{p-|a_{p+1}|}\left\{\frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|-1\right\},$$

$$\frac{2\left[p^2-|a_{p+1}|^2\right]}{p^2-|a_{p+1}|^2+p\left|2a_{p+2}-a_{p+1}^2\right|}\frac{p-|a_{p+1}|}{p+|a_{p+1}|}\leqslant\frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|-1,$$

$$\frac{2\left[p-\left|a_{p+1}\right|\right]^2}{p^2-\left|a_{p+1}\right|^2+p\left|2a_{p+2}-a_{p+1}^2\right|}\leqslant\frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|-1,$$

$$1+\frac{2\left[p-\left|a_{p+1}\right|\right]^2}{p^2-\left|a_{p+1}\right|^2+p\left|2a_{p+2}-a_{p+1}^2\right|}\leqslant\frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|$$

and

$$\left|\frac{f''(b)}{f(b)}\right|\geqslant p\left(1+\frac{2\left[p-\left|a_{p+1}\right|\right]^2}{p^2-\left|a_{p+1}\right|^2+p\left|2a_{p+2}-a_{p+1}^2\right|}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z)=z^pe^{pz}.$$

Then as in the proof of Theorem 2.1,

$$\frac{f''(z)f(z)-f'(z)f'(z)}{f^2(z)}=p\left(-\frac{1}{z^2}\right).$$

By the $f(z)=z^pe^{pz}$, we have

$$\begin{aligned} z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+... &= z^pe^{pz} \\ &= z^p\left(1+pz+\frac{p^2z^2}{2!}+...\right) \\ &= z^p+pz^{p+1}+\frac{p^2z^{p+2}}{2!}+..., \end{aligned}$$

and

$$a_{p+1}+a_{p+2}z+...=p+\frac{p^2z}{2!}+...$$

Passing to the limit in the last equality yields

$$\left|a_{p+1}\right|=p.$$

Since $f'(-1)=0$ and $p=\left|a_{p+1}\right|$, (2.3) is satisfied with equality. □

If $f(z)-z^p$ has no zeros different from $z=0$ in Theorem 2.2, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{M}(p)$, $f(z)-z^p$ has no zeros in D except $z=0$ and $a_{p+1}>0$. Suppose that, for some $b\in\partial D$, f' has a non-tangential limit $f'(b)$ at b, $f'(b)=0$. Then f has the second non-tangential derivative and*

$$\left|\frac{f''(b)}{f(b)}\right|\geqslant p\left(1-\frac{2\left|a_{p+1}\right|\left(\ln\left(\frac{a_{p+1}}{p}\right)\right)^2}{2\left|a_{p+1}\right|\ln\left(\frac{a_{p+1}}{p}\right)-\left|2a_{p+2}-a_{p+1}^2\right|}\right). \tag{2.5}$$

The equality in (2.5) *occurs for the function*

$$f(z)=z^pe^{pz}.$$

Proof. Let $a_{p+1}>0$ in the expression of the function $f(z)$. Besides, let $h(z)$, $\kappa(z)$ and $\varphi(z)$ be as in the proof of Theorem 2.3 and the function $f(z)-z^p$ has no zeros point in D except $z=0$. Having in mind inequality (2.4), we denote by $\ln\varpi(z)$ the holomorphic branch of the logarithm normalized by the condition

$$\ln\varpi(0)=\ln\left(\frac{a_{p+1}}{p}\right)<0\,,\ \frac{a_{p+1}}{p}\leqslant 1.$$

The quotient function

$$H(z)=\frac{\ln\varpi(z)-\ln\varpi(0)}{\ln\varpi(z)+\ln\varpi(0)}$$

is holomorphic in the unit disc D, $|H(z)|<1$ for $|z|<1$, $H(0)=0$ and $|H(b)|=1$ for all $b\in\partial D$. That is,

$$|H(b)|=\left|\frac{\ln\varpi(b)-\ln\varpi(0)}{\ln\varpi(b)+\ln\varpi(0)}\right|=1.$$

Since $f'(b)=0$ and

$$|\varpi(b)|=\left|\frac{h(b)}{b}\right|=|h(b)|=\left|\frac{1}{p}\left(\frac{bf'(b)}{f(b)}-p\right)\right|=1,\ |b|=1,\ b\in\partial D,$$

from (1.6), we obtain

$$\begin{aligned}\frac{2}{1+|H'(0)|} &\leqslant |H'(b)|=\frac{|2\ln\varpi(0)|}{|\ln\varpi(b)+\ln\varpi(0)|^2}\left|\frac{\varpi'(b)}{\varpi(b)}\right|=\frac{|2\ln\varpi(0)|}{|\ln\varpi(b)+\ln\varpi(0)|^2}\,|\varpi'(b)|\\ &= \frac{|2\ln\varpi(0)|}{|\ln\varpi(b)+\ln\varpi(0)|^2}\left|\frac{h'(b)}{\varkappa(b)}-\frac{h(b)\varkappa'(b)}{\varkappa^2(b)}\right|\\ &= \frac{|2\ln\varpi(0)|}{|\ln\varpi(b)+\ln\varpi(0)|^2}\left|\frac{h(b)}{b\varkappa(b)}\right|\left|\frac{bh'(b)}{h(b)}-\frac{b\varkappa'(b)}{\varkappa(b)}\right|\\ &= \frac{-2\ln\varphi(0)}{\ln^2\varpi(0)+\arg^2\varpi(b)}\left\{|h'(b)|-|\varkappa'(b)|\right\}.\end{aligned}$$

Moreover, it can be seen that

$$H'(z)=\frac{2\ln\varpi(0)}{(\ln\varpi(z)+\ln\varpi(0))^2}\frac{\varpi'(z)}{\varpi(z)},$$

and

$$\begin{aligned}|H'(0)| &= \frac{1}{|2\ln\varpi(0)|}\left|\frac{\varpi'(0)}{\varpi(0)}\right|=\frac{1}{-2\ln\left(\frac{a_{p+1}}{p}\right)}\frac{\frac{1}{p}\left|2a_{p+2}-a_{p+1}^2\right|}{\frac{|a_{p+1}|}{p}}\\ &= \frac{1}{-2\ln\left(\frac{a_{p+1}}{p}\right)}\frac{\left|2a_{p+2}-a_{p+1}^2\right|}{|a_{p+1}|}\end{aligned}$$

We are trying to enlarge the expression $\frac{-2\ln\varphi(0)}{\ln^2\varpi(0)+\arg^2\varpi(b)}\left\{|h'(b)|-|\varkappa'(b)|\right\}$, so we take $\arg^2\varpi(b)$=0.

Therefore, replacing $\arg^2 \varpi(b)$ by zero, we have

$$\frac{2}{1-\frac{1}{2\ln\left(\frac{a_{p+1}}{p}\right)}\frac{\left|2a_{p+2}-a_{p+1}^2\right|}{\left|a_{p+1}\right|}} \leqslant \frac{-2}{\ln\left(\frac{a_{p+1}}{p}\right)}\left\{\frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|-1\right\},$$

$$-\frac{2\left|a_{p+1}\right|\left(\ln\left(\frac{a_{p+1}}{p}\right)\right)^2}{2\left|a_{p+1}\right|\ln\left(\frac{a_{p+1}}{p}\right)-\left|2a_{p+2}-a_{p+1}^2\right|} \leqslant \frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|-1,$$

$$1-\frac{2\left|a_{p+1}\right|\left(\ln\left(\frac{a_{p+1}}{p}\right)\right)^2}{2\left|a_{p+1}\right|\ln\left(\frac{a_{p+1}}{p}\right)-\left|2a_{p+2}-a_{p+1}^2\right|} \leqslant \frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|.$$

and we obtain (2.5) with an obvious equality case. □

The following inequality $(2,6)$ is weaker, but is simpler than $(2,5)$ and does not contain the coefficient a_{p+2}. It is formulated in the following theorem.

Theorem 2.5. *Under the hypotheses of Theorem 2.4, we have*

$$\left|\frac{f''(b)}{f(b)}\right| \geqslant p\left(1-\frac{1}{2}\ln\left(\frac{a_{p+1}}{p}\right)\right). \tag{2.6}$$

The equality in (2.6) *occurs for the function*

$$f(z)=z^p e^{pz}.$$

Proof. From Theorem 2.4, using the inequality (1.4) for the function $H(z)$, we obtain

$$\begin{aligned} 1 \quad &\leqslant \quad |H'(b)| = \frac{|2\ln\varpi(0)|}{|\ln\varpi(b)+\ln\varpi(0)|^2}\left|\frac{\varpi'(b)}{\varpi(b)}\right| = \frac{|2\ln\varpi(0)|}{|\ln\varpi(b)+\ln\varpi(0)|^2}\left|\varpi'(b)\right| \\ &= \quad \frac{-2\ln\varpi(0)}{\ln^2\varpi(0)+\arg^2\varpi(b)}\left\{|h'(b)|-|\varkappa'(b)|\right\}. \end{aligned}$$

Replacing $\arg^2 \varpi(b)$ by zero, we have

$$1 \leqslant \frac{-2}{\ln\left(\frac{a_{p+1}}{p}\right)}\left\{\frac{1}{p}\left|\frac{f''(b)}{f(b)}\right|-1\right\}$$

and we obtain (2.6) with an obvious equality case. □

3. Example

Example 3.1. Let us consider the function $f \in \mathcal{M}(p)$ given by

$$f(z)=z^p e^{pz}.$$

By logaritmic differentiation

$$f'(z)=pz^{p-1}e^{pz}\left(1+z\right),$$

we get

$$f''(z) = p(p-1)z^{p-2}e^{pz} + 2p^2 z^{p-1}e^{pz} + p^2 z^p e^{pz}.$$

So, we have that

$$\begin{aligned}\frac{zf''(z)}{f'(z)} &= \frac{(p-1)+2pz+pz^2}{1+z} \\ &= p-1+pz+\frac{z}{1+z}\end{aligned}$$

and

$$\frac{f(z)}{zf'(z)} = \frac{1}{p(1+z)}.$$

Therefore, we obtain

$$\begin{aligned}\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(1+\frac{1}{4p}\right) &= \frac{1}{p(1+z)}\left(p\left(1+z\right)+\frac{z}{1+z}\right) \\ &\quad -\left(1+\frac{1}{4p}\right) \\ &= -\frac{1}{4p}\left[\frac{1-z}{1+z}\right]^2\end{aligned}$$

and

$$\arg\left\{\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(1+\frac{1}{4p}\right)\right\} = \arg\left(-\frac{(z-1)^2}{4p(1+z)^2}\right).$$

Note that

$$\begin{aligned}\arg\left(-\frac{(z-1)^2}{4p(1+z)^2}\right) &= \arg(-1)+\arg\left(\frac{(z-1)^2}{4p(1+z)^2}\right) \\ &= \arg(-1)+2\arg(z-1)-\arg 4p(1+z)^2 \\ &= \pi+2\arg(z-1)-2\arg(1+z).\end{aligned}$$

By letting $\arg(z-1)=\varphi$, $\arg(z+1)=\theta$, we obtain $\varphi=\theta+\frac{\pi}{2}$ and

$$\arg\left(-\frac{(z-1)^2}{4p(1+z)^2}\right) = \pi+2\varphi-2\theta = 2\pi.$$

In addition, for some $-1=b\in\partial D$, since

$$f'(z) = pz^{p-1}e^{pz}\left(z+1\right),$$

we have

$$f'(-1)=0.$$

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