

# SOME PARADOXES IN MATHEMATICS

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**Abstract :** We have explained some paradoxes in the set theory of mathematics.

**Keywords:** Set, paradox, series.

**Özet :** Matematiğin kümeler kuramındaki bazı paradoksları inceledik.

**Anahtar Kelimeler:** Küme, çeldirmeler, seriler.

An infinite series is simply an indicated sum of infinitely many terms. But with this seemingly simple statement, came many questions. What is meant by the sum of infinitely many numbers? Can we always add infinitely many numbers to get a sum? Can we ever add infinitely many numbers?

Actually, we can see sums of infinitely many numbers. For example, when we write

$$\frac{1}{3} = 0,333...,$$

we have an infinite sum. In our decimal notation, the symbol 0.333... means

$$0.333... = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$$

Thus we maintain that there is a sum of infinite set of numbers and sum is  $1/3$ .

The ancient Greeks thought that no infinite set of numbers could possibly have a finite sum. Because of this feeling, they were caught in some logical paradoxes. Perhaps the most famous of these was given by Zeno of Elea (c. 500 B.C.). He pointed out that it is logically impossible to walk from one place to another. He reasoned that, before a person could go the entire distance,  $d$ , he first has to walk half of  $d$ . Then, of the distance  $d/2$  that remained, he had to go half of that, leaving a distance  $d/4$  yet to be covered. But he would have to go half of that distance. Continuing in this way, he could never walk the entire distance  $d$ .

The members of the Electric School were famous for the difficulties they raised in connection with questions that require the use of infinite series, such, for example, as the well-known paradox of Achilles and the tortoise, enunciated by Zeno, one of their

most prominent members. Zeno was born in 495 B.C., and was executed at Elea in 435 B.C. in consequence of some conspiracy against the state; he was a pupil of Parmenides, with whom he visited Athens, c. 455-450 B.C.

Zeno argued that if Achilles ran ten times as fast as a tortoise, yet if the tortoise had (say) 1000 meters start it could never overtaken; for, when Achilles had gone the 1000 meters, the tortoise would still be 100 meters in front of him; by the time he had covered these 100 meters, it would still be 10 meters in front of him; and so on for ever: thus Achilles would get nearer and nearer to the tortoise, but never overtake it.

These paradoxes made the Greeks look with suspicion on the use of infinitesimal, and ultimately led to the invention of the method of exhaustions.

You will notice that, except for the very last sentence, there is no mention of time in this argument. It was simply assumed that, since each segment walked requires some time and there are infinitely many segments, infinite time is required to cover all of them. But let us suppose that a man is walking at a constant rate, and that he walks half of the desired distance in a half-hour. Then half of the remaining distance requires half of the first time, or a quarter-hour. Similarly the next segment requires an eighth of an hour, and so on. Thus the total time in hour to walk distance is the infinite sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

Now surely, if he walked half the distance in a half-hour and continued walking at a constant speed, he must walk the entire distance in one hour. This would imply that the infinite sum given above has the finite value 1. That is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1.$$

On the other hand, if the man were to walk progressively slower, so that each segment walked required the same time as any other segment, say a half-hour, then the total time is

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

This certainly cannot have a finite sum.

How can infinitely many numbers add up a number? Perhaps the simplest way of visualizing this is by looking at the following example.

Consider a line of length 2. Divide it into two equal segments of length 1 each. Leave the left segment alone, and divide the right one into two equal segments of length 1/2 each. Divide the right segment of length 1/2 into two equal segments of length 1/4 each. Continue this process indefinitely. We obtain a decomposition of the segment

of length 2 into segments of length

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

and so forth. Therefore,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 2. \quad (1)$$

The Greek philosopher Zeno objected to this argument. But Zeno certainly saw runners arriving at the finishing line, and precisely what he meant by this and other paradoxes is a matter of controversy. If he wanted to say that one cannot talk about adding infinitely many numbers as if this were a procedure equivalent to adding finitely many numbers, he was certainly right. If we were to compute the sum by carrying out all infinitely many additions, this would indeed take forever.

Yet we feel that the equation (1) is correct. We can compute, for any positive integer  $n$ , the sum of the first  $n$  terms on the left-hand side we obtain

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}} = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

If  $n$  goes to infinity, then we have the equation .

Zeno's well-known paradox of the race between Achilles and the tortoise may serve as our first example of how mathematicians use the idea of limits in their deductions.

Our second paradox was given by Dichotomy which was motion is impossible. Because, whatever moves must reach the middle of its course before it reaches the end; but before it has reached the middle it must have reached the quartermark, and so on, indefinitely. Hence the motion can never even start.

Now for the other side: The Arrow. A moving arrow at any instant is either at rest or not at rest, that is, moving. If the instant is indivisible, the arrow cannot move, for if it did the instant would immediately be divided. But time is made up of instants. As the arrow cannot move in any one instant, it cannot move in any time. Hence it always remains at rest.

Zeno is said to have been a self-taught country boy. Without attempting to decide what was his purpose in inventing his paradoxes, authorities hold widely divergent opinions, we shall merely state them. With these before us it will be fairly obvious that Zeno would have objected to our "infinitely continued" division of that two-meters line a moment ago.

Consider the problem of determining the velocity of an object which is dropped from

rest and allowed to fall freely to the surface of the earth. Then, neglecting air resistance, the distance  $s$  in feet that the object falls in  $t$  seconds is given by the formula

$$s= 16t^2 .$$

Since  $s$  increases as  $t$  increases, the formula implies that "down" has been chosen as the positive direction on the line of fall; a choice of "up" as positive would have led to the formula  $s = - 16t^2$ . As we see simply, this formula implies a constantly changing velocity. What is the velocity at a particular instant? What is it, for example, at the end of 2 seconds of fall, that is, when  $t=2$ ? This question has no immediately accessible answer, for we do not have a definition of velocity at a specific instant. Indeed, the Greek philosopher Zeno of Elea suggested that there was no such thing. If an arrow falls, he said, it exists at each instant of its fall, and at each instant it must therefore have a specific location in space. How then can it move? Hence, Zeno concluded, motion is impossible. We do not know whether Zeno really believed in the impossibility of motion; at least he succeeded in pointing out the difficulty of defining instantaneous velocity which is a special kind of a limit problem.

Let us come to the third paradox which is the Stadium. To prove that half of the time may be equal to double the time. Consider three rows of bodies.

First Position					Second Position				
(A)	0	0	0	0	(A)		0	0	0
(B)	0	0	0	0	(B)	0	0	0	0
(C)	0	0	0	0	(C)			0	0

(A) is at rest while the other two (B), (C) are moving with equal velocities in opposite directions. By the time they are all in the same part of the course (B) will have passed twice as many of the bodies in (C) as in (A). Therefore the time which it takes to pass (A) is twice as long as the time it takes to pass (C). But the time which (B) and (C) take to reach the position of (A) is the same. Therefore double the time is equal to half the time.

These, in non-mathematical language, are the sort of difficulties the early grapplers with continuity and infinity encountered. We need remark only that Zeno finally lost his head for the reason or something of the sort, and pass on to those who did not lose their heads over his arguments.

### Serving Up A Paradox :

Zeno confounded his fellow thinkers mightily by pointing out that the heroic Achilles, no matter how fast he ran, could not overtake a crawling tortoise with a head start, since when he reached the tortoise's starting point, A, the tortoise would have moved ahead to B. When he got to B the tortoise would have moved ahead to C. In this wise, Zeno argued, the tortoise would always be out in front, even if by a mere eyelash.

## Paradoxes in set theory

Although set theory is recognized to be the cornerstone of the new mathematics, there is nothing essentially new in the intuitive idea of a set. From the earliest times, mathematicians have been led to consider sets of objects of one kind or another, and the elementary notions of modern set theory are implicit in a great many classical arguments. However, it was not until the latter part of the nineteenth century, in the work of German mathematician Georg Cantor (1845-1918), that sets came into their own as the principal object of a mathematical theory.

A set must be well defined. It must be possible to tell whether a given element belongs to a set either by checking it against the list of elements of the set or by deciding whether it satisfies or does not satisfy the rule governing membership for that set. Rule governing set membership must make perfectly clear what elements are included in a particular set. The set of "all nice people" is not well defined as there is no reasonable criterion given for deciding whether a person is "nice". The elements in a set are distinct. If an object is listed as an element of a set it should not be listed a second time. The order of the elements in a list is not significant. The set containing the elements 1,2,3 is exactly the same as the set containing 2,3,1.

Between 1895 and 1910 a number of contradictions were discovered in various parts of set theory. At first mathematicians paid little attention to them; they were termed paradoxes and regarded as little more than mathematical curiosities. The earliest of the paradoxes was published in 1897 by Burali-Forti, but it had already been discovered, two years earlier, by Cantor himself. Since the Burali-Forti paradox appeared in a rather technical region of set theory, which we will see later, it was hoped, at first, that a slight alteration of the basic definitions would be sufficient to correct it. However, in 1902 Bertrand Russell gave a version of the paradox which involved the most elementary aspects of set theory, and therefore could not be ignored. In the ensuing years other contradictions were discovered, which seemed to challenge many of the safest notions of mathematics.

The theory of sets was first studied as a mathematical discipline by Georg Cantor in the latter part of the nineteenth century. Today, the theory of sets lies at the foundations of mathematics and has revolutionized almost every branch of mathematics. At about the same time that set theory began to influence other branches of mathematics, various contradictions, called paradoxes, were discovered.

### Set of all sets (CANTOR's paradox)

Let  $C$  be the set of all sets. Then every subset of  $C$  is also member of  $C$ ; hence the power set of  $C$  is a subset of  $C$ , i.e.,

$$2^C \subset C.$$

But  $2^C \subset C$  implies that

$$\#(2^C) \leq \#(C).$$

However, according to Cantor's theorem

$$\#(C) < \#(2^C).$$

Thus the concept of the set of all sets leads to a contradiction.

## **RUSSELL's paradox**

Let  $Z$  be the set of all sets which do not contain themselves as members, that is,

$$Z = \{X: X \notin X\}$$

The question is the following: Does  $Z$  belong to itself or not?

If  $Z$  does not belong to  $Z$  then, by definition of  $Z$ ,  $Z$  does belong to itself. Furthermore, if  $Z$  does belong to  $Z$  then, by definition of  $Z$ ,  $Z$  does not belong to itself. In either case we are led to a contradiction.

The above Russell's paradox is somewhat analogous to the following popular paradox: In a certain town, there is a barber who shaves only and all those men who do not shave themselves. The question is the following: Who shaves the barber?

Similarly, if with Zermelo and Russell, we consider the collection  $C$  of all classes which taken one by one are not members of themselves, then it appears that we can prove  $C$  both to be a member of itself. And it is solitary to remember that Philotes (340-285 B.C.) died because he could not resolve the paradox of the man who said "I am lying" for if this statement is true, it is false, and if it is false, it is true.

Indeed there are around dozen of genuinely different puzzles of this description, some of them involving only terms of mathematics or logic, such as class and number, and some referring to non-logical terms, such as thought, language or symbolism. Among the second class of Russell's paradoxes are the paradoxes of the "Liar", of the least integer not in fewer than nineteen syllables. These paradoxes may well be due to faulty ideas concerning thought or language and not to faulty logic or mathematics and we therefore do no more than make this brief reference to the semantic and linguistic antinomies.

According to Russell, all these paradoxes result from a certain kind of vicious circle which arises from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole, as, for example, if we speak of the cardinal number of all cardinal numbers.

A Greek says that "all Greeks are liar". Is the Greek a liar or not?

## **Set of all ordinal numbers (BURALI-FORTI paradox)**

Let  $D$  be the set of all ordinal numbers. We know that  $D$  is a well-ordered set, say  $a = \text{ord}(D)$ . Now consider  $s(a)$ , the set of all ordinal numbers less than  $a$ . Note:

- i. Since  $s(a)$  consists of all elements in  $D$  which precede  $a$ ,  $s(a)$  is an initial segment of  $D$ .
- ii. As it is known  $a = \text{ord}(s(a))$ ; hence  $\text{ord}(s(a)) = a = \text{ord}(D)$ .

Therefore  $D$  is similar to one of its initial segments. Thus the concept of the set of all ordinal numbers leads to a contradiction.

Set of all cardinal numbers, family of all sets equivalent to a set and family of all sets similar to a well-ordered set are contradictions.

### Paradoxes in arithmetics

When mathematicians realized that we can sometimes find the sum of infinitely many numbers, there was a tendency to expect infinite sums to have the same properties as finite sums. This again led to difficulties. For example, let us consider

$$1 - 1 + 1 - 1 + 1 - 1 + \dots$$

Suppose we group the terms as  $(1-1) + (1-1) + (1-1) + \dots$  We then have

$$(1-1) + (1-1) + (1-1) + \dots = 0 + 0 + 0 + \dots = 0.$$

But if we group them in another way, we obtain

$$1 + (-1+1) + (-1+1) + (-1+1) + \dots = 1 + 0 + 0 + 0 + \dots = 1.$$

Clearly there is something which is wrong here. The practice of grouping terms, which is always valid for a finite sum, cannot always be used for sums with infinitely many terms. Problems such as this demand that we define more carefully what we are talking about.

The system of complex numbers, denoted  $C$ , is the set  $R^2$  together with the usual rules of vector addition and scalar multiplication by a real number  $k$ :

$$(a,b) + (c,d) = (a+c, b+d)$$

$$k(a,b) = (ka, kb)$$

and the operation of complex multiplication defined by

$$(a,b) (c,d) = (ac-bd, ad + bc).$$

Rather than using  $(x,y)$  to represent a complex number, we will find it more convenient to return to more standard notation as  $(x,y) = x+iy$  where  $i = \sqrt{-1} = (0,1)$  or  $i^2 = -1$ . Note that  $i^2 = i \cdot i = (0,1) \cdot (0,1) = (0 \cdot 0 - 1 \cdot 1, (0 \cdot 1 + 1 \cdot 0)) = (-1, 0) = -1$ , so we do have the property we want

$$i^2 = -1$$

On the other hand we have

$$-1 = i^2 = i \cdot i = \sqrt{-1} \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

that  $-1=1$  is a contradiction. Because of the operation of complex multiplication the equality of

$$\sqrt{-1} \sqrt{-1} = \sqrt{(-1)(-1)}$$

is false.

Note that the definition of division does not allow for division by zero, since there is no inverse for zero under the operation of multiplication. This can be explained as follows. Assume that there is a real number  $c$  such that  $a/b=c$ . This is a proper assumption since division is defined in terms of multiplication (and multiplication assumed to be closed).

If we multiply both sides of the above equation by  $b$ , we get  $ba / b = bc$  or  $a = bc$

Zero can enter into a division problem in three ways:  $a=0$ ,  $b=0$ , or both  $a$  and  $b$  equal 0. The latter two cases involve division by zero.

Suppose  $b=0$ ,  $a \neq 0$ , then  $a/0=c$  means that  $a=0.c$ . Thus,  $c$  must be a number that we can multiply by 0 to get  $a$  (remember that  $a \neq 0$ ). We say that any number times 0 is equal to 0. There is no way to get a real number other than 0 when multiplying by 0. Hence, division by zero yields no answer. The required  $c$  does not exist.

Suppose  $a=b=0$ , then  $0/0=c$  or  $0=0.c$ . This is interpreted as saying that we must find a  $c$  that gives 0 for an answer when we multiply it by 0. But  $0=0.c$  no matter what  $c$  is. In other words,  $c$  can be anything at all in the set of real numbers. Therefore, the problem  $0/0$  has an infinite set of answers. This is also an undesirable situation. We rule out these latter two cases as producing unsatisfactory results. Both involved division by zero.

Let  $a = b$ , then we have the followings

$$\begin{aligned} a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a+b)(a-b) &= b(a-b) \\ a + b &= b \\ 2a &= a \\ 2 &= 1. \end{aligned}$$

Hence,  $1 = 2 = 3 = \dots$ . Then all natural numbers are equal that we are led to a contradiction. Here we have divided both sides by zero as  $a-b=0$ . This case is false. Then division or multiplying by zero are false. That is, division or multiplying by zero are undefined.



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